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On Curves in n-Dimensional Euclidean Spaces.

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Preface.

This work examines curves in n-dimensional spaces, as well as varieties contained in such spaces, with the main focus on curves and osculating linear and spherical varieties. Absolute differential calculus — a method almost exclusively used in n-dimensional differential geometry in recent times — is convenient in systematic terms because it enables the determination of all differential invariants using classical techniques. However, it is cumbersome and inconvenient for discovering purely geometric relations, as each quantity must be defined by multiple numbers, which, in turn, depend on the choice of coordinate system. The derived relations must then be translated back into geometric quantities. This interpretation, along with the preliminary coordinate calculations, requires fairly long and complex operations. For instance, in the below-referenced work of E. E. Levi, the first 43 pages (out of 98!) are devoted to preliminary calculations, and only thereafter does the author begin studying geometric quantities.

Since I have sought to examine purely geometric properties and their relationships, I have consistently used vector analysis — both independently and in conjunction with direct geometric reasoning, as well as with Cartesian coordinates.

In the literature, I could not find information on osculating spheres or on defining curvatures when the parameter is arbitrary. These and some other results are therefore presumably new; the others are mentioned for illustrating the methods used.

While preparing this work, in addition to classical courses on differential geometry, I have used the following references:

I. In Vector Theory:

- G. Bouligand, Géométrie vectorielle (Vuibert, 1924).
- G. Julia, Introduction mathématique aux théories quantiques (Cahiers scientifiques XVI, Gauthier-Villars, 1936).

II. In Differential Geometry of n-Dimensional Spaces:

- C. Jordan, Sur la théorie des courbes de l'espace à n dimensions (Comptes Rendus T. 79, 1874, p. 795).
- C. Guichard, Les courbes de l'espace à n dimensions (Mémorial des Sciences mathématiques, XXIX, Gauthier-Villars, 1928).

- K. Kommerell, Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von 4 Dimensionen (Dissertation, Tübingen, 1897).
- E. E. Levi, Saggio sulla teoria delle superficie a due dimensioni immerse in un iperspazio (Annali della R. Sc. normale super. di Pisa, Vol. X, 1905, Nr. 2, pp. 1–99).
- J. A. Schouten and D. J. Struik, Über Krümmungseigenschaften einer m-dimensionalen Mannigfaltigkeit, die in einer n-dimensionalen Mannigfaltigkeit mit beliebiger quadratischer Massbestimmung eingebettet ist (Rendiconti del Circo. mat. di Palermo, T. XLVI, 1922, pp. 165–184).
- Fr. Kämmerer, Zur Flächentheorie im n-fach ausgedehnten Raum (Mitt. des Math. Seminars Giessen, IX. Heft, 1922).

Chapter 1.

The Theory of Free Curves.

In this work, we will use the following notations: V_n — n-dimensional variety; R_n — linear V_n or space; C_n — spherical V_n or sphere, i.e., the geometric locus of points in some R_{n+1} that are equidistant (this distance being called the radius) from a given point in R_{n+1}, called the center. The fact that some V_p is contained in V_n will be expressed by the symbol $V_p \subseteq V_n (p < n)$.

Free curves are arbitrary curves contained in R_n, in contrast to curves on curved varieties. We will consider only real varieties, i.e., in this chapter, real free curves, and in the next chapter — curves on real V_p. Similarly, we will assume that the derivatives of the radius vector of the current point of the curve or variety with respect to the parameter(s) are finite (i.e., with finite absolute value) and continuous in all the orders we use in the calculations.

Normal Components.

1. If p vectors $X_j (j = 1, 2, \dots, p)$ are given in a specific order, then the normal component of vector X_i will be denoted by $\bar{X}_i$, which is orthogonal to all preceding $X_j (j = 1, 2, \dots, i-1)$. This component is uniquely determined if we take $\bar{X}_i = X_i$ when all X_j are zero.

The vectors $X_1, X_2, \dots, X_p$ are linearly independent if the equation:

$$\sum_{i=1}^{p} k_i X_i = 0 \quad (k_i \text{ - scalar constants}) \tag{1}$$

is satisfied only when all k_i are zero. Geometrically, this means that X_i cannot all lie within some R_q with $q < p$. This condition is equivalent to the following: none of the X_i can be zero. Indeed, if any X_i is zero, then X_i is a linear combination of the preceding X_j, and a relationship of type (1) exists among the first i vectors; conversely, if (1) is possible when not all k_i are zero, the last vector X_p, whose coefficient is non-zero, can be expressed as a linear combination of the preceding vectors, and thus its normal component is zero.

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2. The figure formed by p vectors $X_1, X_2, \dots, X_p$ given in a specific order will be called a p-vector, and the volume of this p-vector will be the recursively defined quantity:

$$T_p = T(X_1, X_2, \dots, X_p) = T_{p-1} \cdot |\bar{X}_p|, \tag{2}$$

i.e.,

$$T_p = |\bar{X}_1| \cdot |\bar{X}_2| \cdots |\bar{X}_p|. \tag{2_1}$$

The volume of a p-vector defined in this way is identical to the volume of the parallelepiped whose edges correspond to the vectors X_i [1].

Thus, the volume of a two-vector corresponds to the area of the parallelogram, and the volume of a three-vector corresponds to the volume of the parallelepiped. Formula (2_1) shows that T_p is zero when the vectors X_i are not linearly independent.

To calculate T_p, let us take a special Cartesian coordinate system where the x_1 axis lies along the direction of $\bar{X}_1$. This system will then be orthogonal. If the coordinates of the endpoint of vector X_i are $x_{i,j}$ $(j = 1, 2, \ldots, p)$:

$$|X_i| = |k_i| b \quad \text{and} \quad x_{i,j} = 0 \quad \text{if} \quad j > i.$$

Thus:

$$T_p = |x_{11} x_{22} x_{33} \ldots x_{pp}|,$$

or:

$$T_p = \pm \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{p1} & x_{p2} & \ldots & x_{pp} \end{vmatrix} = ||x_{ij}||,$$

since all elements above the main diagonal are zero. If we now square this determinant by combining rows and columns
, we obtain:

$$(T_p)^2 = ||A_{ij}||, \quad \text{where} \quad A_{ij} = A_{ji} = X_i \cdot X_j, \tag{3}$$

i.e., the scalar product of these two vectors. This formula is evidently independent of the chosen coordinate system because the quantities A_{ij} are intrinsically related to the vectors X_i. Relationship (3), though proven only for linearly independent X_i, is also valid in the general case: the value of determinant A_{ij} is zero when a relationship (1) is possible because the same relationship exists among the rows or columns of the determinant. (3) also shows that T_p is independent of the order of the vectors X_i. Denoting:

$$D_p = (T_p)^2 = ||A_{ij}||,$$

relationship (2) gives:

$$X_p^2 = \frac{D_p}{D_{p-1}}. \tag{4}$$

Indeed, for the contact to be at least of second order, the space must contain the point M_1, whose distance from point M' is at least of second order, i.e.,

$$\lim_{dt \to 0} \frac{M_1 - M'}{dt} = 0,$$

or, equivalently:

$$M_1 - M' = (M_1 - M) - (M' - M),$$

$$\lim_{dt \to 0} \frac{M_1 - M'}{dt} X_1 = 0.$$

Thus, the sought space contains a vector equivalent to the vector X_1. Similarly, the proof can be extended to the other vectors, and since, in the general case, X_{p+1} is not a linear combination of the preceding X_i, achieving p-th order contact is the highest order of contact possible.

6. If the first n vectors X_1 are not linearly independent, let us consider the vectors $X_1, X_2, \ldots, X_n, \overline{X}_{n+1}, \ldots$. Among these, the largest n are not zero. Let the indices of these vectors be $k_1 (= 1), k_2, \ldots, k_q (q < n)$. The first $k_{p+1} - 1$ vectors X_i are then contained in R_p, which is defined by the vectors $X_j, X_{k_j}, \ldots, X_{k_p}$.

[1]See, for example, P. H. Schoute, *Mehrdimensionale Geometrie* (Leipzig, 1905, II, Nr. 33), or Dr. W. Somerville, *An introduction to the geometry of n dimensions* (London, 1929, VIII, 3). While the final formulas given by these authors differ in appearance from ours, they can easily be reduced to one another.

This space will again be called the p-th osculating space; the order of contact of the curve (L) with R_p at point M, as shown by analogous reasoning to the previous cases, is $k_{p+1} - 1$.

If $q < n$, R_q contains all X_i, and thus also the entire analytic arc (L) on which M lies, as demonstrated by expansion (5).

In the previous reasoning, we used the fact that at a point on (L), there exist linear relationships among the vectors X_i. Similar results can be obtained if there is only one such relationship, but it holds for all points of (L). For example, if $p < n$ is the smallest number for which the following relationship holds:

$$\sum_{i=1}^{p} f_i X_i = 0,$$

where f_i are some functions of t, it is sufficient for the vectors $X_1, X_2, \ldots, X_{p+1}$ to be continuous functions of t to conclude that the curve (L) is contained in a p-dimensional space. This can be most easily seen by taking an orthogonal Cartesian coordinate system. Let the projections of the radius vector X onto the axes be $x_1, x_2, \ldots, x_{p+1}, \ldots, x_n$, and let the projections of the vectors X_i be $x_{1,j}$ $(j = 1, 2, \ldots, n)$. Denote by A_i the minor determinant associated with the term a_i:

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_{p+1} \\ x_{11} & x_{21} & \cdots & x_{p+1,1} \\ x_{12} & x_{22} & \cdots & x_{p+1,2} \\ \vdots & \vdots & & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{p+1,p} \end{vmatrix} \tag{10}$$

and by A_i' the derivative of A_i with respect to t. If f_i are not identically zero, it is easy to see that:

$$A_i A_j' - A_i' A_j = 0.$$

Assuming that A_{p+1} is not identically zero (which can always be achieved by a suitable choice of axes), setting $i = p + 1$ and integrating the above relationship, we obtain:

$$A_j = c_j A_{p+1}, \tag{11}$$

where c_j are constants. Substituting $a_i = x_{i1}$ into determinant (10), we have:

$$A_i x_{i1} = 0,$$

which, substituting the value of A_i, dividing by A_{p+1}, and integrating, gives:

$$c_1 x_1 + c_2 x_2 + \cdots + c_p x_p + x_{p+1} = C, \tag{12}$$

where C is a constant. Replacing x_{p+1} in the column of determinant (10) with $x_{p+2}, \ldots, x_n$, we obtain $n - p$ relationships analogous to (12), where the coefficient of x_{p+i} is always one. Thus, (L) indeed lies in a p-dimensional space contained in R_n.

7. In the space R_n, $p + 2$ points generally determine one and only one C_p (the proof is analogous to that of a sphere C_2 in R_3). The $p + 2$ infinitely close points of (L) within the osculating R_{p+1} will generally determine one C_p, which we will call the p-th osculating sphere. The first osculating sphere will then be the osculating circle. In general, at a point M, the curve (L) touches the p-th osculating sphere at $p + 1$-th order, and this is the only C_p with such a property. For the order of contact to be $p + k$ $(k > 1)$, the C_p must contain $p + k + 1$ infinitely close points of (L). This condition is equivalent to the following: the osculating sphere C_p must touch (L) at an order greater than or equal to $p + k$ (i.e., among the first $p + k$ vectors, the highest $p + 1$-th vector is nonzero), and the first $k - 1$ derivatives of the radius c_p of C_p with respect to the parameter t must be zero.

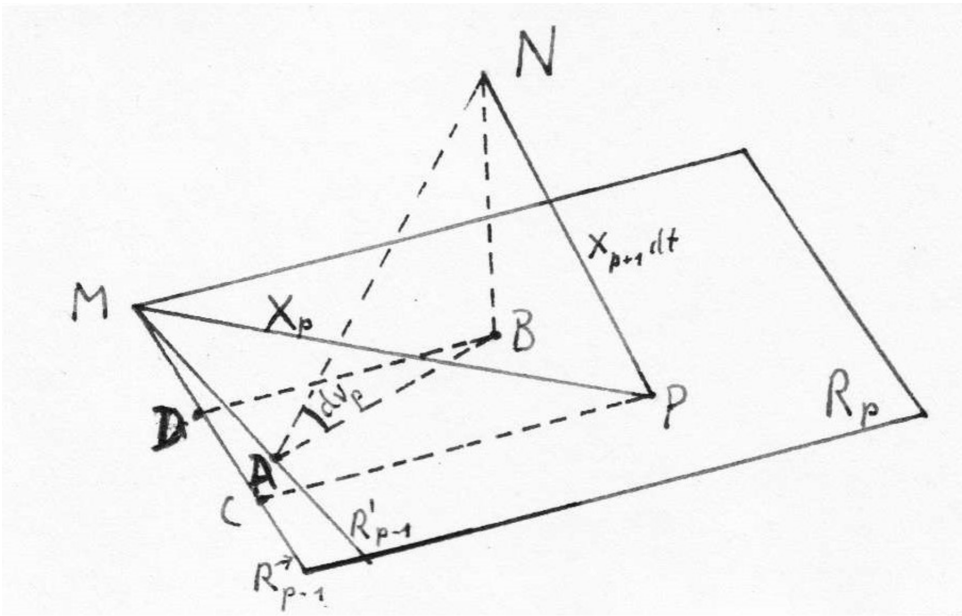
We will calculate the radius of osculating spheres later, after introducing curvatures and their values.

From now on, we will assume that the curve under consideration is indeed a curve in the space R_n, i.e., that it is not contained in any linear space of lower dimension.

Curvatures

8. In general, the osculating R_p spaces at points $M(t)$ and $M'(t+dt)$ intersect along R_{p-1} as $dt \to 0$. Indeed, the p-th osculating space at point M is determined by the vectors $X_1, X_2, \ldots, X_p$, while at point M' it is determined by the vectors $X_1+X_2dt, X_2+X_3dt, \ldots, X_p+X_{p+1}dt$. They share a common variety passing through point M', which is determined by the vectors $X_1+X_2dt, X_2+X_3dt, \ldots, X_{p-1}+X_pdt$, i.e., the $p-1$-th osculating space at point M', which becomes R_{p-1} as $dt \to 0$. The two successive p-th osculating spaces thus form a "biangular space,ānd the angle dv_p between them will be called the p-th curvature angle. The limiting ratio of this angle to the arc element ds will be the p-th curvature k_p, and its reciprocal will be the p-th radius of curvature r_p.

To determine dv_p, let R_{p-1} and R_p be the respective (L) osculating spaces at point M, R'_{p-1} and R'_p the spaces parallel to these osculating spaces through M', and P and N the points $M+X_p$ and $M+X_p+X_{p+1}dt$, respectively.



1. zīm.: Fig. 1

Project point N onto point A on R'_{p-1} and onto point B on R_p. In the right triangle NAB, angle A is dv_p, and its leading term is given by:

$$\frac{\mathrm{BN}}{\mathrm{AB}},$$

where BN is $|\bar{X}_p + 1\, dt|$. We only need to determine AB. We will show that the leading term of AB is $|X_p|$. To do so, project point P onto point C on R'_{p-1}; then $PC = |X_p|$. It remains to prove that the distances BP and AC are at least of the first order of infinitesimal smallness. If D is the projection of B onto R_{p-1}, BP and DC are projections of $X_{p+1}dt$ onto R_p and R'_{p-1}, respectively, and are thus at least of the first order of infinitesimal smallness.

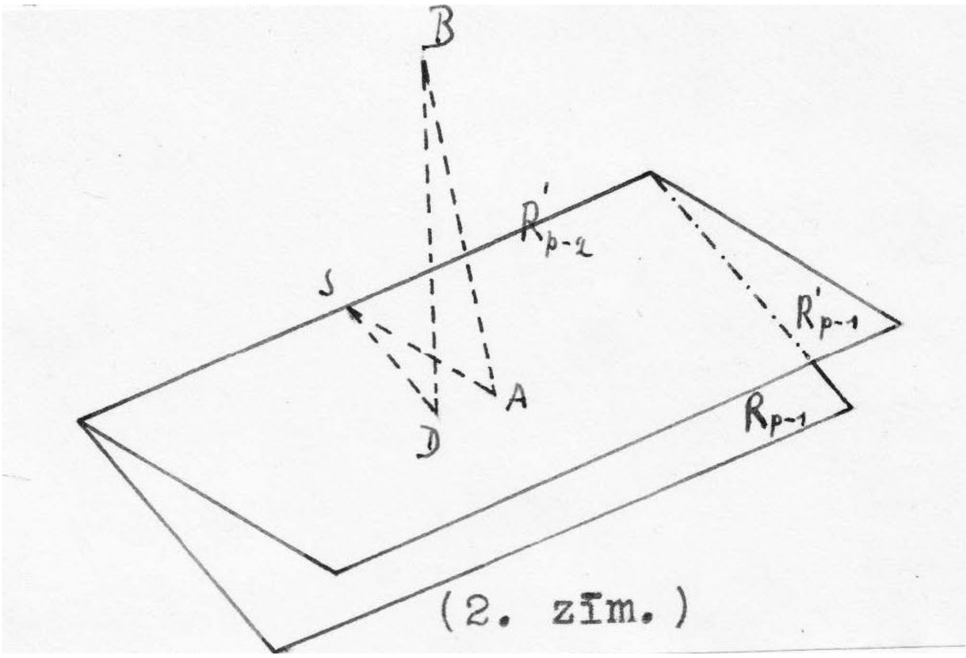
It remains to determine the magnitude of AD. Since R_p contains R_{p-1}, R'_{p-1}, and point B, it also contains the lines BA and BD; these, in turn, define the plane orthogonal to the intersection of R_{p-1} and R'_{p-1} in R_p. This plane intersects R_{p-2} at point S. The lines SA and SD form the angle dv_{p-1}, and points B, D, A, S lie on a circle with diameter BS. Thus:

$$AD = BS \sin(dv_{p-1}),$$

and AD is at least of the first order of infinitesimal smallness. Therefore, AB is equivalent to $|\bar{X}_p|$, and in the limiting case:

$$\left|\frac{dv_p}{dt}\right| = \frac{|\bar{X}_{p+1}|}{|\bar{X}_p|}.$$

Using the relationships (4) and (8) (on pages 7 and 8), we obtain:



2. zīm.: Fig. 2

$$k_p^2 = \frac{1}{r_p^2} = \frac{D_{p+1}D_{p-1}}{2D_pD_1}. \tag{13}$$

For this formula to also apply to k_1, we must assume $D_0 = 1$. Thus, for each k_p, we can determine the absolute value, while resolving the signs requires a specific convention. Formula (13) also reveals the interesting fact that, given M's Cartesian orthogonal coordinates as functions of t, only the second curvature of R_3 curves can be expressed as a rational function of the derivatives of the coordinates, while in all other cases, the curvature yields irrational expressions.

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0.1. The Serret–Frenet n-frame and Formulas

9. As shown by formula (7) (page 8), the lines containing the vectors X_i are independent of the choice of parameter. To orient them and thus obtain an orthogonal n-frame uniquely associated with L at point M—the Serret-Frenet n-frame—we can proceed in two ways:

1) At some point on L, we arbitrarily choose the directions of the axes and then translate the resulting n-frame along L. The curvatures k_1 will then be positive or negative, depending on whether X_{i+1} aligns with the corresponding axis direction or the opposite direction. This method is useful, for instance, when studying points on L where one or several k_i are zero.

2) We choose the direction of arc progression on L and then, taking s as the parameter, assign the first $n-1$ axes in the directions of the corresponding $\bar{X}_i$, with the last axis chosen such that the n-frame has the same orientation as a given fixed n-frame. Then, the sign of k_n, as in the above case, depends only on whether the direction of X_n matches the direction of the last axis. In other words, k_{n-1} will be positive or negative depending on whether the orientation of the n-frame defined by X_i vectors (or the n-frame oriented like the previous one) matches or opposes the orientation of the chosen reference n-frame.

We will choose the second method of determination. Let us clarify under which circumstances the sign of k_{n-1} depends on the direction of arc progression on L, and when it does not. Taking $t = -s$, formula (7) shows that X_i with odd indices reverse their direction, while those with even indices do not. To compare the orientation of two n-frames defined by axes a_i and a'_i, we denote the cosine of the angle between the positive directions of a_i and a'_i by c_{ij}. It suffices to consider the sign of the determinant $|c_{ij}|$: if it is $+$, the orientations of the two n-frames coincide; if it is $-$, they are opposite. In the current case, $c_{ii} = (-1)^j$, and all other c values are zero, so the determinant value is $(-1)^{\frac{n(n+1)}{2}}$. As we saw earlier, this is exactly the factor by which k_{n-1} must be multiplied when the positive direction on L is reversed. Thus, we can state the following:

- If n is of the form $4k+1$ or $4k+2$, reversing the positive direction on L changes the sign of k_{n-1}.
- If n is of the form $4k+3$ or $4k$, the sign of k_{n-1} is intrinsically associated with L, and we can, for example, refer to a curve as being twisted to the right or to the left.

It should be noted that our convention for the sign of k_{n-1} is opposite to that used by almost all authors for R_3 space: Hermite and Bianchi use the same convention as ours, and Darboux, although using the opposite, acknowledges this one as superior[2].

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10. To derive the Serret-Frenet formulas for R_n, we assign unit vectors t_i to the positive directions of the axes defined at the end of page 16. The formulas we seek will give the derivatives of t_i with respect to s, denoted by t_i'. Since:

$$t_i^2 = 1$$

and

$$t_i t_j = 0, \quad \text{if } i \neq j,$$

it follows that:

$$t_i t_i' = 0 \quad \text{and} \quad t_i t_j' + t_i' t_j = 0. \tag{14}$$

Furthermore:

$$t_i = \sum_{k=1}^{i} f_h X_h \quad (\text{where } f_h \text{ are scalar functions of } s).$$

Differentiating this (where X is parametrized by s), we get:

$$t_i' = \sum_{k=4}^{n} f_h' X_h + \sum_{k=4}^{n} f_h X_{h+1}.$$

Thus, $t_i' t_j = 0$ if $j - i \geq 2$. Formula (14) then also shows that $t_i t_j' = 0$, i.e., $t_i t_j' = 0$ if $|j-i| \geq 2$. Hence:

$$t_i' = a_i t_{i-1} + b_i t_{i+1}. \tag{15}$$

It remains to determine the coefficients a and b. Similarly, we write:

$$X_i = \sum_{k=h}^{i} g_h t_h,$$

(where g_h are again scalar functions of s). This expression, together with its derivative with respect to s and formula (15), gives:

$$\overline{X_i} = g_i t_i \quad \text{and} \quad \overline{X_{i+1}} = g_i b_i t_{i+1}.$$

For $1 < n$, due to the conventions for axis orientation, all g_i are positive, and b_i takes the sign of the corresponding k_i. Since the absolute value of k_i is the ratio of $|X_{i+1}|$ to $|X_i|$, it follows that:

$$b_i = k_i.$$

Finally, substituting $j = i-1$ into the second formula in (14), we get:

$$a_i + b_{i-1} = 0, \quad \text{i.e.,} \quad a_i = -k_{i-1}.$$

Thus:

$$t_i' = -k_{i-1} t_{i-1} + k_i t_{i+1}. \tag{16}$$

[2]Leçons sur la théor. gén. des surfaces, IV, p. 428

This formula is the general expression for the Serret-Frenet formula. To make it valid for the cases $i = 1$ and $i = n$, we define:

$$k_0 = k_n = 0.$$

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Formulas (16) are, of course, not new, but their method of derivation seems to be. Previously, they have been derived either by examining the determinant of the cosine angles of the t_i directions and determining its ṛotation"[3], or through tensor calculus[4].

C. Jordan appears to have been the first to provide these formulas (albeit without proof) in *Sur la théorie...*.

If the k_i are given as analytic functions, formulas (16) allow us to determine all X_i $(i = 1, 2, \ldots, n, n+1, \ldots)$ as linear combinations of t_i $(i = 1, 2, \ldots, n)$, with coefficients being polynomials composed of k_i and their derivatives. Thus, we can obtain an expansion of the type (5) (on page 7) for the radius vector. Assuming this expansion is convergent, we arrive at the following result:

Two curves with the same k_i are congruent; they can be transformed into each other by the same motion that maps one t_i n-frame to the other in two compatible but otherwise arbitrary points.

0.2. 11. Serret–Frenet Formulas

The Serret–Frenet formulas are particularly advantageous for questions involving repeated differentiation or cases where we are interested in the projections of a vector onto the axes of the Serret–Frenet n-frame, each individually. However, for other problems, they have the disadvantage of general coordinate methods: each geometric property is expressed through multiple relations. Thus, a purely geometric method is often more beneficial. To illustrate these cases, we will consider the following: the first terms of the expansion of X, the determination of osculating sphere radii, and evolutes. Throughout, we will take s as the parameter, and derivatives with respect to s will be denoted by accents.

Expansion of X

12. By differentiating step by step, we obtain the following four first derivatives of X:

$$X_1 = t_1 \quad (\text{since } X_1^2 = 1),$$

$$X_2 = k_1 t_2,$$

$$X_3 = -k_1^2 t_1 + k_1' t_2 + k_1 k_2 t_3,$$

$$X_4 = -3k_1 k_1' t_1 + (k_1'' - k_1^3 - k_1 k_2^2) t_2 + (2k_1' k_2 + k_1 k_2') t_3 + k_1 k_2 k_3 t_4.$$

In general, it can be seen that

$$X_i t_i = k_1 k_2 k_3 \ldots k_{i-1}.$$

Taking

$$M = M(0), \quad X(0) = 0, \quad \text{and setting} \quad X t_i = x_i,$$

we obtain the expansion of the orthogonal coordinates of the arc s at the corresponding current point with respect to the Serret–Frenet n-frame:

$$x_1 = s - \frac{s^3}{6} k_1 + \frac{s^4}{8} k_1 k_1' + \ldots,$$

$$x_2 = -\frac{s^2}{2} k_1 + \frac{s^3}{6} k_1' - \frac{s^4}{24} (k_1'' - k_1^3 - k_1 k_2^2) + \ldots,$$

$$x_3 = -\frac{s^3}{6} k_1 k_2 + \frac{s^4}{24} (2k_1' k_2 + k_1 k_2') + \ldots,$$

$$x_4 = -\frac{s^4}{24} k_1 k_2 k_3 + \ldots,$$

[3] See, e.g., E. Cesàro, *Vorlesungen über natürliche Geometrie*, Chapter 16, or M. C. Guichard, *Les courbes de l'espace à n dimensions*, pp. 14–16.

[4] See, e.g., A. Duschek–W. Mayer, *Lehrbuch der Differentialgeometrie*, II, p. 74, or L. P. Eisenhart, *Riemannian Geometry*, pp. 103–106.

$$x_i = \frac{s^i}{i!} k_1 k_2 k_3 \dots k_{i-1} + \dots.$$

These expressions allow us to determine the leading term of various infinitesimally small quantities, as well as, for example, provide an alternative interpretation of curvatures and determine the appearance of the projection of L at point M when L is projected onto some R_k passing through M (particularly one defined by some of the t_j). Thus, we see that the infinitesimal arc and the corresponding chord are equivalent quantities, with their difference's leading term given by

$$\frac{s^3}{24} k_1^2.$$

For curvatures, we have:

$$k_{i-1} = \lim_{s \to 0} \frac{i \cdot x_i}{s \cdot x_{i-1}},$$

i.e.,

$$k'_{i-1}$$

is obtained by projecting a point M' infinitely close to M on L onto t_{i-1} and t_i in the space R_2, and taking i times the limit of the ratio of the angle between MM' and t_{i-1} to the corresponding arc length s on L.

Osculating Sphere Radii

13. The axes of the Serret–Frenet n-frame have a clear geometric interpretation: the axis containing t_1 is the oriented tangent, and the axis containing t_i is the normal in the i-th osculating space through M. We will use this perspective below.

Consider the p-th osculating sphere with center A_p; it lies in the osculating space R_{p+1}. The point A_p is equidistant from $p+2$ infinitely close points of L, all of which are contained in the osculating R_{p+1}. If we project A_p onto A_{p-1} in the osculating R_{p+1}, the line $A_p - A_{p-1}$ will be parallel to t_p, and A_{p-1} will be equidistant from $p+1$ infinitely close points of L in the osculating R_p; in other words, it is the center of the osculating sphere C_{p-1}. Thus, if

$$A_p = M + \sum_{i=1}^{p+1} d_i t_i, \quad \text{then}$$

$$A_{p-1} = M + \sum_{i=1}^{p} d_i t_i,$$

with d_i being the same in both cases and satisfying:

$$d_i = (A_p - M)t_i = (A_{p-1} - M)t_i, \quad (i \le p),$$

and iterating further, we get:

$$d_i = (A_{n-1} - M)t_i.$$

To determine A_{n-1}, which we will denote as A for brevity, we write:

$$(A - M)^2 = c_{n-1}^2, \tag{17}$$

and differentiate this relation n times, treating A and c_{n-1} as constants. The first derivative gives:

$$(A - M)t_1 = 0, \tag{18}$$

which indicates that A lies in the main normal plane at M, i.e., the space R_{n-1} passing through M and orthogonal to the tangent. Since (18) must be differentiated $n-1$ more times, A lies in n infinitely close normal planes.

Using a recurrence relation, we can compute all d_i and then:

$$c_p^2 = \sum_{i=1}^{p+1} d_i^2 \quad (\text{for } p \le n-1).$$

For a general parameter t, the second method shows that:

$$d_{i+1} = r_i \frac{c_{i-1}c'_{i-1}}{d_i} \left| \frac{dt}{ds} \right|. \tag{21}$$

We have already pointed out that if the curvatures are given as functions of the arc length s, the shape of the curve is completely determined (provided that the expansion of X in powers of s is convergent). If c_i values are given, they determine the absolute values of r_i, and therefore, the first $n-2$ k_i values are uniquely determined, while k_{n-1} may be either positive or negative. From the expansion of X, it is evident that X_n is an odd function of k_{n-1}, while all other x_j values are even functions. Two curves that share the same Serret–Frenet n-frame at a point but differ only in the sign of k_{n-1} are symmetric with respect to their common $n-1$-th osculating space.

Thus, two curves in R_n with the same c_i values can be made congruent by a transformation, which may include symmetry.

Polar Curve

14. The locus of the point A_{n-1}, which we will henceforth denote as A, represented by (L_1), can be interpreted in various ways. On the one hand, it is associated with the normal spaces of (L), which form the $n-1$-th osculating space family of (L_1). We will discuss this family briefly in the next section.

On the other hand, (L_1) has some interesting properties that can be conveniently studied, of which we will consider only those related to the Serret–Frenet n-frame and the curvatures, using the expression for the moving point A:

$$A = M + \sum_{i=1}^{n} d_i t_i. \tag{22}$$

Differentiating this with respect to s, we get:

$$A' = (d'_n + d_{n-1}k_{n-1})t_n.$$

This relation shows that the tangent to (L_1) at point A coincides with the line $A_{n-1}A_{n-2}$, which thereby forms a developable surface V_2 as M moves along (L).

If the positive directions of (L) and (L_1) correspond to each other, the derivative of the arc s_1 of (L_1) with respect to s is:

$$f(s) = \left| \frac{ds_1}{ds} \right| = |d'_n + d_{n-1}k_{n-1}|.$$

The unit tangent vector T_1 of the oriented curve (L_1) is:

$$T_1 = e_1 t_n, \tag{23}$$

where $|e_1| = 1$, and e_1 is the sign of $d'_n + d_{n-1}k_{n-1}$.

Denoting the curvatures of (L_1) as K_i, the unit vectors of its Serret–Frenet n-frame as T_i, and letting:

$$k_{n-1} = e|k_{n-1}|, \quad K_{n-1} = E|K_{n-1}|,$$

successive differentiation of (23) with respect to s yields the relationships:

$$f(s)|K_i| = |k_{n-i}|,$$

and

$$T_i = e_i t_{n+1-i}.$$

The coefficients e_i, starting from e_2, satisfy the relations:

$$e_i = (-1)^{i-1} e_1 e \quad (2 \leq i \leq n-1),$$

$$e_n = (-1)^{n-1} e_1 e E.$$

These relations allow the determination of all e_i, except e_n, as the latter involves the unknowns E and e_n. E is determined by the requirement that the n-frame T_i has the same orientation as the n-frame t_i. This means the determinant of the cosines on page 17 must equal $+1$. In this case, the elements of the determinant's second diagonal are e_i, while all others are zero, leading to:

$$E = (-1)^{1+\frac{n(n-1)}{2}} e_1^n e_n^{-1} \quad \text{and therefore,}$$

$$e_n = (-1)^{\frac{n(n-1)}{2}} e_1^{n-1} e_n.$$

Filarevolutes

15. To find the curve (L'), whose tangents are orthogonal trajectories of (L)—that is, the filarevolute of (L)—we need to determine the normal vectors (T) of (L) that generate a developable V_2 when their intersection point M with (L) moves along (L). Let N be the characteristic point of the normal (T), i.e., the moving point of (L'), U the unit vector along (T), and a the oriented distance $N - M$:

$$N = M + aU. \tag{24}$$

For the normal (T) to generate a developable V_2, it is necessary and sufficient that (T) coincides with the tangent of (L) at point N, i.e.,

$$t_1 + a'U + aU' = hU,$$

where h is an undetermined scalar function. Multiplying by U, we get:

$$a' = h,$$

which determines h in terms of a, leaving the condition:

$$t_1 + aU' = 0. \tag{25}$$

This equation shows that if U and W are the unit vectors of two sought normals, the angle between their positive directions is constant. Indeed, their cosine is UW, and due to (25), $UW' = WU' = 0$, so UW is constant. We will later show that it is possible to find $n-1$ normals (T_i) with linearly independent U_i. Each subsequent normal (T) will maintain a constant position relative to the $n-1$-frame formed by (T_i).

By solving (25), U can be expressed as a linear combination of the vectors t_i $(i = 2, 3, \ldots, n)$. The resulting system of differential equations shows that the general solution depends on $n-2$ parameters, allowing an arbitrary U to be chosen in the normal space at point M, thereby determining a and the corresponding U at every other point on (L)[5]. However, we will use a more direct geometric method, which will effortlessly yield other properties of (L').

The normal spaces R_{n-1} and R'_{n-1} of the curve (L) at two infinitely close points M and M' intersect along R_{n-2}, which we will call the polar space of (L) at the point M. This polar space is absolutely orthogonal to the osculating R_2 and intersects it at the point A_1. Indeed, disregarding second- and higher-order infinitesimals, R_{n-1} and R'_{n-1} intersect R_2 along straight lines, which are two infinitely close normals to the projection of (L) in the space R_2, and thus intersect at the point A_1. Since R_{n-1} and R'_{n-1} form the same angle $dv = dv_1$ as their orthogonal tangents at the points M and M', it is evident that R'_{n-1} is obtained by rotating R_{n-1} about the polar space R_{n-2} by the angle dv in the positive direction, i.e., the direction in which t_1 transitions to t_2 by a rotation of $+\frac{\pi}{2}$. Drawing vectors through A_1 that are equipollent to the vectors U and $U + dU$ corresponding to the points M and M', equation (25) gives:

$$a \cdot dU = -t_1 ds, \tag{26}$$

hence the vector dU is orthogonal to the polar space. Since U does not change its length, it is evident that $U + dU$ is obtained by allowing U to follow the aforementioned rotation. But then:

$$dU = -(Ut_2) \cdot t_1 dv.$$

[5] C. Guichard applied this method, though only for the specific case $n = 5$, and also derived the properties of the normals (T) mentioned earlier (loc. cit., pp. 20-22). The subsequent property of evolutes as geodesic lines of the polar V_{n-1} space appears to be new.

Comparing this relation with the previous one, we see that:

$$aUt_2 = r_1, \tag{27}$$

which expresses that the point $N = M + aU$ lies in the polar space. There are no further restrictions on the point N, and equations (26) and (27) provide a way to construct (L') point by point, starting from an arbitrary initial point N in the polar space.

Interpreting the above formulas, or directly using the fact that the point N lies in the polar space of the corresponding point M of (L), we arrive at the following purely geometric construction of (L') from its elements: On (L), we choose successive and infinitely close points $M, M_1, M_2, \ldots$. Their polar spaces are $P, P_1, P_2, \ldots$.

Viewing the normal spaces of (L) as the $n-1$-th osculating spaces of the polar curve (L_1), we see that the normal spaces of the points M and M_1 intersect along P_1.[6] Thus, the normal space at the point M contains P and P_1, while the normal space at M_i contains P_i and P_{i+1}.

To construct (L'), we arbitrarily choose a point $N \in P$. The line MN intersects P_1 at the point M_1, the line M_1N_1 intersects P_2 at the point N_2, and so on. The infinitesimal segments $NN_1, N_1N_2, \ldots$ form the successive elements of (L'). This construction shows that the osculating plane of (L') at point N, which is the limiting position of the plane NMM_1, contains the tangent to (L) at M, and is thus orthogonal to the normal space of (L).

If we consider the family of normal spaces of (L), we see that they wind around a polar variety V_{n-1} formed by the polar spaces, with each normal space being tangent to V_{n-1} at all points of the corresponding polar space. This V_{n-1}, which we will call the polar variety of (L), is developable, meaning that Euclidean metrics hold within it. Completely analogous to developable $V_2 \subset R_3$, the polar variety can be transformed into R_{n-1} (or a portion of such a space) by rotating each "planarēlement—i.e., the portion of R_{n-1} between two polar spaces—about the polar space it shares with the preceding element until both elements lie in a common R_{n-1}, on opposite sides of the shared polar space. As noted in the earlier footnote, the rotation that transforms U into $U + dU$ differs at most by second-order infinitesimals from the inverse rotation required to align the elements of V_{n-1} lying in the normal spaces of points M and M_1 into the same R_{n-1}. Unrolling the polar V_{n-1} onto R_{n-1}, the tangents to the transformed curve (L'') at two infinitely close points differ at most by second-order infinitesimals. Each point of (L'') is then an inflection point, which can only occur if (L'') is a straight line. Thus, we have proven that in a regular and sufficiently small region of the polar variety, the curves (L'') provide the shortest paths between any two of their points within V_{n-1}.

In the next chapter, we will call the geodesic lines of a variety V those curves whose osculating planes are orthogonal to the tangent to V. The curves (L') are therefore the geodesic lines of the polar variety. Furthermore, we have seen that they possess the shortest path property (assuming as proven the fact that in R_i, only straight lines possess the shortest path property); in general, proving the properties of geodesic lines requires the calculus of variations.

The normal spaces R_{n-1} and R'_{n-1} of the curve (L) at two infinitely close points M and M' intersect along R_{n-2}, which we will call the polar space of (L) at the point M. This polar space is absolutely orthogonal to the osculating R_2 and intersects it at the point A_1. Indeed, disregarding second- and higher-order infinitesimals, R_{n-1} and R'_{n-1} intersect R_2 along straight lines, which are two infinitely close normals to the projection of (L) in the space R_2, and thus intersect at the point A_1. Since R_{n-1} and R'_{n-1} form the same angle $dv = dv_1$ as their orthogonal tangents at the points M and M', it is evident that R'_{n-1} is obtained by rotating R_{n-1} about the polar space R_{n-2} by the angle dv in the positive direction, i.e., the direction in which t_1 transitions to t_2 by a rotation of $+\frac{\pi}{2}$. Drawing vectors through A_1 that are equipollent to the vectors U and $U + dU$ corresponding to the points M and M', equation (25) gives:

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hence the vector dU is orthogonal to the polar space. Since U does not change its length, it is evident that $U + dU$ is obtained by allowing U to follow the aforementioned rotation. But then:

$$dU = -(Ut_2) \cdot t_1 dv.$$

[6]The apparent contradiction with the conclusions that gave us (26) and (27) is entirely natural: differences in second-order infinitesimals when defining R'_{n-1} produce first-order differences when determining the intersecting spaces. These earlier conclusions need not be revised, as the rotation that transforms U into $U + dU$ differs at most by second-order infinitesimals from the inverse rotation required to bring the elemental parts of the V_{n-1} lying in the normal spaces of M and M_1 into alignment.

Comparing this relation with the previous one, we see that:

$$aUt_2 = r_1, \tag{27}$$

which expresses that the point $N = M + aU$ lies in the polar space. There are no further restrictions on the point N, and equations (26) and (27) provide a way to construct (L') point by point, starting from an arbitrary initial point N in the polar space.

Interpreting the above formulas, or directly using the fact that the point N lies in the polar space of the corresponding point M of (L), we arrive at the following purely geometric construction of (L') from its elements: On (L), we choose successive and infinitely close points $M, M_1, M_2, \ldots$. Their polar spaces are $P, P_1, P_2, \ldots$.

Viewing the normal spaces of (L) as the $n-1$-th osculating spaces of the polar curve (L_1), we see that the normal spaces of the points M and M_1 intersect along P_1.[7] Thus, the normal space at the point M contains P and P_1, while the normal space at M_i contains P_i and P_{i+1}.

To construct (L'), we arbitrarily choose a point $N \in P$. The line MN intersects P_1 at the point M_1, the line M_1N_1 intersects P_2 at the point N_2, and so on. The infinitesimal segments $NN_1, N_1N_2, \ldots$ form the successive elements of (L'). This construction shows that the osculating plane of (L') at point N, which is the limiting position of the plane NMM_1, contains the tangent to (L) at M, and is thus orthogonal to the normal space of (L).

If we consider the family of normal spaces of (L), we see that they wind around a polar variety V_{n-1} formed by the polar spaces, with each normal space being tangent to V_{n-1} at all points of the corresponding polar space. This V_{n-1}, which we will call the polar variety of (L), is developable, meaning that Euclidean metrics hold within it. Completely analogous to developable $V_2 \subset R_3$, the polar variety can be transformed into R_{n-1} (or a portion of such a space) by rotating each "planarēlement—i.e., the portion of R_{n-1} between two polar spaces—about the polar space it shares with the preceding element until both elements lie in a common R_{n-1}, on opposite sides of the shared polar space. As noted in the earlier footnote, the rotation that transforms U into $U + dU$ differs at most by second-order infinitesimals from the inverse rotation required to align the elements of V_{n-1} lying in the normal spaces of points M and M_1 into the same R_{n-1}. Unrolling the polar V_{n-1} onto R_{n-1}, the tangents to the transformed curve (L'') at two infinitely close points differ at most by second-order infinitesimals. Each point of (L'') is then an inflection point, which can only occur if (L'') is a straight line. Thus, we have proven that in a regular and sufficiently small region of the polar variety, the curves (L'') provide the shortest paths between any two of their points within V_{n-1}.

In the next chapter, we will call the geodesic lines of a variety V those curves whose osculating planes are orthogonal to the tangent to V. The curves (L') are therefore the geodesic lines of the polar variety. Furthermore, we have seen that they possess the shortest path property (assuming as proven the fact that in R_i, only straight lines possess the shortest path property); in general, proving the properties of geodesic lines requires the calculus of variations.

Meusnier's (Menjé) Theorem.

18. By setting $u_1 = t$, and assuming that all the first p derivatives of u_i with respect to t are zero, and the $p+1$-th derivative is a_i, we obtain a family of curves where all curves share the vectors $X_1, X_2, \ldots, X_p$. Hence, the first p Serret-Frenet vectors t_i, and the first $p-1$ curvatures and osculating radii are common. Any two curves in the family have at least p-th order contact at point M.

Conversely, given a family of curves with such a property in some V_m, if we take one of them as a 1-curve and set $t = u_1$, it follows that the first p derivatives of the remaining u at point M must be zero.

The unoriented collection of vectors $t_1, t_2, \ldots, t_p$ will be referred to as a p-direction. As seen, the family of curves considered above determines an associated p-direction. Conversely, a p-direction determines such a family, although the vectors t_i cannot be completely arbitrary if V_m is given. The t_i must be chosen in the tangent space, and consequently, the values of f_i' at this point are determined proportionally. To define f_i uniquely, a condition on the parameter t can be imposed, for example, requiring it to be the arc length of one of the sought family curves. As shown in (28), the axis t_2 must then be chosen in the space R_{m+1}, determined by the tangent space and the vector

[7] The apparent contradiction with the conclusions that gave us (26) and (27) is entirely natural: differences in second-order infinitesimals when defining R'_{n-1} produce first-order differences when determining the intersecting spaces. These earlier conclusions need not be revised, as the rotation that transforms U into $U + dU$ differs at most by second-order infinitesimals from the inverse rotation required to bring the elemental parts of the V_{n-1} lying in the normal spaces of M and M_1 into alignment.

$$A = \sum_{i=1,j=1}^{m} f_i' f_j' Y_{ij},$$

which gives f_i'', etc., so that all derivatives f_i up to order p inclusive are determined. The curves corresponding to a p-direction in V_m thus have at least p-th order contact, when taken in pairs.

19. Returning to the parameter mentioned at the beginning of the previous paragraph, denote by Z the $p+1$-th partial derivative of vector X with respect to u_1. Then,

$$X_{p+1} = Z + \sum_{i=1}^{m} a_i Y_i.$$

If $\overline{Z}$ and $\overline{Y_i}$ represent the components of the respective vectors orthogonal to the subspace R_p determined by $t_1, t_2, \ldots, t_p$, and these components are not zero and are linearly independent due to our normalization, then

$$\overline{X}_{p+1} = \overline{Z} + \sum_{i=1}^{m} a_i \overline{Y_i}.$$

This relation expresses that the endpoint of vector $\overline{X}_{p+1}$ can coincide with any finitely distant point (since a_i are arbitrary but finite), lying in the subspace R_{m-1}, which passes through the point $M+Z$ and contains vectors parallel to $\overline{Y_i}$. This space is evidently independent of the choice of the vectors Y_i or the respective curves. If N is the projection of M onto this space,

$$(N-M)\overline{Y_i} = 0$$

and

$$\overline{X}_{p+1}(N-M) = (N-M)^2.$$

Denoting by φ the angle between $N-M$ and $\overline{X}_{p+1}$ in positive directions, it follows that $0 \leq \varphi < \frac{\pi}{2}$, and

$$|\overline{X}_{p+1}| \cos\varphi = |N-M|.$$

If (L_0) is the curve in the family such that $X_{p+1} = N-M$, and for some other curve this vector is arbitrary, φ will be the angle between the $p-1$-th osculating subspaces of the two curves. On the other hand, for every curve in the family, the p-th curvature is proportional to $|\overline{X}_{p+1}|$, since $|\overline{X_p}|$ is the same for all. Thus, denoting by $\overline{k_p}$ and $\overline{r_p}$ the p-th curvature and curvature radius for the curve (L_0), for any other curve in the family, they will be

$$k_p = \frac{\overline{k_p}}{\cos\varphi}$$

and

$$r_p = \overline{r_p}\cos\varphi,$$

which is a generalization of Meusnier's theorem[8]. It shows the following: if we associate with each curve in the family the points

$$K = M + t_{p+1}k_p$$

and

$$P = M + t_{p+1}r_p,$$

the locus of K is an $m-1$-dimensional space homothetic to the locus of the endpoint of $\overline{X}_{p+1}$, and the locus of P is the inverse sphere of K's locus with respect to the center M and unit power.

Now, let $p=1$, and construct the remaining parameter curves so that Y_1 forms an orthogonal m-hedron. Then,

[8] This well-known relation is proven, e.g., by E. E. Levi in *Saggio sulla Teoria ...*, pp. 38–55.

$$X_1 = Y_1$$

and

$$\overline{Y_i} = \cdot Y_i \quad (i = 2, 3, \dots, m).$$

The curve (L_0) with the smallest first curvature will be the one for which t_2 is orthogonal to all Y_i, i.e., orthogonal to the tangent space. The curves V_m with this property at every point are called geodesic. Using the calculus of variations, it can be shown that under certain constraints, geodesics are the shortest curves in V_m connecting two points[9]. For developable $V_m \subset R_{m+1}$, complementing our reasoning on evolutes, this property of geodesics can also be demonstrated elementarily by postulating the shortest line property for straight lines in R_m.

By taking $t = u_1$ and stating that $\overline{X_2}$ is orthogonal to Y_i $(i = 2, 3, \dots, m)$, we obtain $m-1$ second-order differential equations for the functions f_i. The general integral of these contains $2(m-1)$ constants. In general, geodesics in a variety are determined by passing through two given points or by passing through a given point in a specified direction.

The curvature of a geodesic in a given direction at point M in V_m will be referred to as the geodesic curvature of V_m in that direction. The dependence of this curvature on the direction will be examined later.

The point $M + G$ thus lies in a plane that passes through the point $M + P$ and is parallel to the vectors S and Q. In this plane, it traces an ellipse (E) centered at $M + P$. Two orthogonal directions in the tangent plane V_2 correspond to diametrically opposite points on (E). We can therefore choose curves (L_1) and (L_2) such that they correspond to the endpoints of one axis of (E). Since the form of formula (33) remains the same, the directions and magnitudes of the vectors S and Q correspond to the directions and lengths of the semi-axes a and b of (E). If we take the previous bisectors as the base directions, (L_1) and (L_2) correspond to the endpoints of the second axis of (E), i.e., S and Q switch places.

The absolute value of G is extremal when G lies along one of the normals drawn from M to (E). The feet of these normals on (E) are also the feet of the normals drawn from the projection of $M' - M$ onto the plane of (E). The number of extrema of the geodesic curvature for real curves will thus be 4 if M' lies inside the evolute of (E), i.e., if:

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} - (a^2 - b^2)^{\frac{2}{3}} < 0, \tag{34}$$

and 2 if the expression on the left-hand side is positive or zero. Here, x and y are the orthogonal coordinates of M', measured along the axes of (E), which correspond to the semi-axes a and b. In our case, $ax = PS$ and $by = PQ$, so condition (34) becomes:

$$(PS)^{\frac{2}{3}} + (PQ)^{\frac{2}{3}} - (S^2 - Q^2)^{\frac{2}{3}} < 0.$$

As expected, the expression on the left-hand side is symmetric with respect to S and Q and thus independent of the choice of axis for (E), which defines (L_1) and (L_2). The only exceptional case occurs when (E) is a circle and M lies on its axis. In this case, $|G|$ is indfependent of φ; such points can be called umbilic points (ombilic, Nabelpunkt), as all curves passing through them have the same geodesic curvature properties.

Variety p-Directional Osculating Sphere

20. As we have already pointed out, for all families of curves (L) defined at point M in the p-direction, the first $p-1$ osculating spheres are identical. Let us determine the locus of the center A of the p-th osculating sphere. Substituting $i = j = p + 1$ in formula (20) on page 24, we obtain:

$$(A - M)X_{p+1} = f_{p+1}. \tag{29}$$

Here, f_{p+1} is the same for all (L) because it depends only on $\overline{X_i}$, where $i \leq p$. Denoting d_{p+1} by d, substituting A with:

$$A = A_{p-1} + dt_{p+1},$$

[9]See, e.g., T. Levi-Civita, *Der absolute Differentialkalkül* (Springer, 1928), pp. 56–61.

and X_{p+1} with its definition, while noting that $p+1<n$, and hence:

$$t_{p+1}X_{p+1}=|X_{p+1}|,$$

(29) becomes:

$$d|\overline{Z}+\sum_{i=2}^{m}a_i\overline{Y_i}|=b+\sum_{i=2}^{m}a_ib_i, \tag{30}$$

where:
$b=f_{p+1}\to(A_{p-1}-M)Z$, $b_1=-(A_{p-1}-M)Y_i$.

Formula (30) provides d as a parametric function. Let us now perform an inversion centered at A_{p-1} with a degree of $+1$. The inverse point A' is:

$$A'=A_{p-1}+\frac{\bar{Z}+\sum_{i=2}^{m}a_i\bar{Y}_i}{b+\sum_{i=2}^{m}a_ib_i}.$$

If A_0 corresponds to A' when $a_1=0$, then:

$$A'=A_0+\frac{\sum_{i=2}^{m}a_1(b\bar{Y}_i-b_i\bar{Z})}{b(b+\sum_{i=2}^{m}a_ib_i)}.$$

Thus, A' lies in the subspace R_{m-1}, drawn through A_0 parallel to the vectors $b\bar{Y}_i-b_i\bar{Z}$, and A resides on the inverse sphere c_{m-1} passing through A_{p-1}. If A'' is the point diametrically opposite A_{p-1}, and (L'') is the curve of the family for which $A=A''$, then $d=d''=(A''-A_{p-1})t_{p+1}$. For any other curve of the family:

$$d=d''\cos\psi,$$

where ψ is the angle between the corresponding tangent t_{p+1} and the line $A_{p-1}A''$. In other words, A is the projection of A'' onto the line parallel to t_{p+1} through A_{p-1}. For any arbitrary p-th osculating point s of a curve (L) in the family, the line AA'' is orthogonal to the corresponding $p+1$-dimensional osculating space, and:

$$(S-A'')^2=(S-A_{p-1})^2+(A_{p-1}-A)^2+(A-A'')^2=c_{p-1}^2+d''^2.$$

Thus, all p-th osculating points lie on $C_{m+p-1}\subset R_{m+p}$, determined by the vectors X_i $(i=1,2,\ldots,p)$, Z, and Y_j $(j=2,3,\ldots,m)$ passing through M. Conversely, intersecting this sphere with the $p+1$-dimensional osculating space of a particular curve (L), we obtain the corresponding p-th osculating sphere. For this reason, it is natural to name the sphere C_{m+p-1} the p-directional osculating sphere of the variety V_m.

Geodesic Curvatures

21. As seen on page 36, differentiation and solving systems of linear equations enable finding the family of curves associated with a given p-direction of the variety.

To determine geodesic curvatures, a two-direction system will be useful: in the tangent space, we take p mutually orthogonal vectors t_{1i}, and t_{2i} as their respective normals to the tangent space. For each two-direction family, we select a curve (L_i), and these m curves are chosen as the parameter curves with their loci as parameters. In this way, we obtain a locally orthogonal geodesic parameter line system at point M.

If (L) is an arbitrary curve passing through M, and t is its arc length, the corresponding f_i' values are the direction cosines c_i of its tangent relative to the m-hedron t_{1i}. To determine the geodesic curvature associated with the direction (c_i), formula (28) requires that a given vector X_2, orthogonal to X_1, must also be orthogonal to the tangent space of the variety, which implies that all $f''=0$. The geodesic curvature associated with direction (c_i) is therefore the absolute value of the vector:

$$G=\sum_{i,j=1}^{m}e_ic_jY_{1j}.$$

The geometric locus of the point $M + G$ for different values of G and for various m and n has been studied by several researchers mentioned in the introduction[10]. Hence, we will not delve into this question in the general case.

We will, however, briefly examine the special case $m = 2$ ($n \geq 5$), where we aim to resolve what seems to be an unresolved problem: determining the maximum number of extrema of geodesic curvatures for real curves in two-dimensional varieties.

If $m = 2$, and φ is the angle between the positive directions of curves (L_1) and (L):

$$c_1 = \cos\varphi, \quad c_2 = \sin\varphi,$$

then formula (31) becomes:

$$G = Y_{11}\cos^2\varphi + 2Y_{12}\cos\varphi\sin\varphi + Y_{22}\sin^2\varphi. \tag{32}$$

Substituting:

$$Y_{11} = P + S, \quad Y_{12} = Q, \quad Y_{22} = P - S,$$

formula (32) transforms into the following:

$$G(\varphi) = P + S\cos 2\varphi + Q\sin 2\varphi. \tag{33}$$

Thus, the point $M + G$ lies in a plane drawn through $M + P$, parallel to the vectors S and Q, and describes an ellipse (E) with its center at $M + P$. Two orthogonal directions V_2 in the tangent plane correspond to diametrically opposite points on (E). Consequently, we can choose (L_1) and (L_2) such that they correspond to the endpoints of one axis of (E). Since the form of formula (33) remains the same, the directions and magnitudes of the vectors S and Q will define the directions and lengths of the axes of (E) as a and b. If we use the bisectors of the previous axes as the basis directions, (L_1) and (L_2) will correspond to the endpoints of the second axis of (E), i.e., S and Q will swap roles.

The absolute value of G reaches an extremum when G lies along one of the principal normals drawn from M to (E). The projections of these normals onto (E) are also the projections of the normals drawn from the projection of $M' - M$ onto the plane of (E). The sought maximum number of extrema of geodesic curvatures for real curves is therefore 4 if M' lies inside the evolute of (E), i.e., if:

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} - (a^2 - b^2)^{\frac{2}{3}} < 0, \tag{34}$$

and 2 if the expression on the left-hand side is positive or zero. Here, x and y are the orthogonal coordinates of M', measured along the axes of (E) corresponding to the semi-axes a and b. In our case, $ax = PS$, $by = PQ$, so condition (34) becomes:

$$(PS)^{\frac{2}{3}} + (PQ)^{\frac{2}{3}} - (S^2 - Q^2)^{\frac{2}{3}} < 0.$$

As expected, the expression on the left-hand side is symmetric with respect to S and Q, and thus independent of the choice of the axis of (E) that defines (L_1) and (L_2). The only exceptional case is when (E) is a circle and M lies on its axis. In such cases, $|G|$ is independent of φ; such points are referred to as umbilical points (ombilic, Nabelpunkt) because all curves passing through them exhibit the same geodesic curvature properties.

22. Formulas (32) and (33) easily yield various properties of the geodesic curvature vector G. For example, letting $G_i = G(\frac{\pi}{4}i)$ $(i = 0, 1, 2, 3)$, the G_i vectors are the radius vectors to the vertices of (E), and any other G can be expressed in a symmetric form:

$$G(\varphi) = \sum_{i=0}^{3} G_i \left[\cos^2\left(\varphi - \frac{\pi}{4}i\right) - \frac{1}{4}\right]. \tag{35}$$

Since G_i are not linearly independent, one of them can be expressed in terms of the others, but this destroys the symmetry of formula (35). This formula seems more like a generalization of Euler's theorem than, for example, the formula:

$$\frac{1}{R^2} = \frac{\sin^4\varphi_1}{R_I^2} + \frac{\sin^4\varphi_2}{R_{II}^2} + C\sin^2\varphi_1\sin^2\varphi_2,$$

[10] See, in particular, J.A. Schouten and D.J. Struik, *Über Krümmungseigenschaften...*, especially the final paragraphs, where references to (G) and its connections to other configurations useful for studying geodesic curvatures are provided.

provided by F. Kemmerer.[11] In this formula, indices 1 and 2 refer to the directions of the affine axes of Dupin's indicatrix. In the next section, we will see that, in the general case, there are two such pairs of axes (and infinitely many for axial points), related by a complex relationship, so that the constants in F. Kemmerer's formula can have two (or infinitely many) different value sets. In contrast, formula (35) allows only permutations of G_i, and for axial points where two of the G_i coincide with P, it can be transformed into the classical Euler formula with an elementary transformation.

Incidentally, formula (33) directly shows that if φ is considered a linear function of time, the point M_G moves as if it were attracted by the center of (E) with a quasi-elastic force.[12]

Dupin's Indicatrix

23. The Dupin indicatrix (D) of a variety V_m at point M is obtained by marking the square root of the radius of geodesic curvature in each direction in the tangent space at M.

If r is the radius vector of the moving point on (D), the equation of (D) in polar coordinates is:

$$r^4 G^2 = 1,$$

where G is given by formula (31). Taking t_i along Cartesian coordinate axes x_1, x_2, ..., x_m, then $rc_i = x_i$, and the equation of (D) in these coordinates is:

$$\left(\sum_{i,j=1}^{m} x_i x_j Y_{ij} \right)^2 = 1. \tag{36}$$

The (D) can also be derived differently: by determining the locus of those points of V_m near M whose distance from the tangent space is $\frac{\epsilon^2}{2}$, projecting them onto the tangent space, performing a homothety centered at M with a ratio of $\frac{1}{\epsilon}$, and letting ϵ approach zero. This is easily achieved by taking coordinate axes $x_1, x_2, \ldots, x_m$ in the tangent space (e.g., along t_j), the remaining axes perpendicular to the tangent space, and x_j $(j = 1, 2, \ldots, m)$ as parameters u_j. At a regular point M of V_m, for small x_j, we have the expansion:

$$x_j = \frac{1}{2} f_j(x_1, \ldots, x_m) + g_j \quad (j = m+1, m+2, \ldots, n),$$

where f_j are homogeneous quadratic forms, and g_j are entire series starting with cubic terms. Performing the above-mentioned operations gives the equation of (D):

$$\sum_{j=m+1}^{n} f_j^2 = 1. \tag{37}$$

Since the coefficient of $x_i x_k$ in the form f_j is a component of Y_{ik} along the x_j-axis, equation (37) is equivalent to (36), just rewritten in scalar form.

Both representations of (D) easily reveal some of its properties. First, (D) is a fourth-degree V_{m-1}, embedded in the tangent space, with M as its center of symmetry. Every straight line in the tangent space passing through M intersects (D) at two real points. The real part of (D), corresponding to real curves on the base V_m, is therefore a closed V_m, though it is not always convex—for example, if $m = 2$, (D) may consist of two connected hyperbolas.

When drawing secants of (D) parallel to a given direction, the geometric locus of the centroid of the four intersection points is a real R_{m-1} passing through M. For varieties of the (D) type, as with second-degree varieties, one could develop theories of "diametersānd "diametral spaces."However, discussing such questions is beyond the scope of this work.

Affine Symmetry Axes of (D)

We will examine one more question, where some geometric reasoning allows us to avoid lengthy calculations: determining the affine symmetry axes of (D) when $m = 2$. In this case, (36) becomes, with $x = x_1$ and $y = x_2$:

$$(Y_{11}x^2 + 2Y_{12}xy + Y_{22}y^2)^2 = 1. \tag{38}$$

[11]P. Kämmerer, *Zur Flächentheorie*, p. 13.

[12]See F. Kümmerer, loc. cit., p. 22, where this fact is briefly explained.

If φ_1 and φ_2 are the angles between the positive directions of the affine coordinate axes (ξ) and (η) and the positive direction of the x-axis, then:

$$x = \xi \cos\varphi_1 + \eta \cos\varphi_2,$$
$$y = \xi \sin\varphi_1 + \eta \sin\varphi_2.$$

Substituting x and y with these expressions in (38), the equation becomes:

$$(A\xi^2 + 2B\xi\eta + C\eta^2)^2 = 1, \tag{39}$$

where:

$$A = G(\varphi_1), \quad C = G(\varphi_2),$$
$$B = Y_{11}\cos\varphi_1\cos\varphi_2 + Y_{12}(\cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2) + Y_{22}\sin\varphi_1\sin\varphi_2.$$

For (ξ) to be an affine symmetry axis corresponding to the direction (η), it is necessary and sufficient that the coefficients of $\xi\eta$ and ξ^3 in the expanded left-hand side of (39) vanish. If these conditions hold, (η) will also be a symmetry axis for (ξ), meaning (ξ) and (η) will form an affine symmetry axis pair. The conditions for these coefficients to vanish mean that B must be perpendicular to both A and C—that is, orthogonal to their plane—since, in the general case, B cannot be zero.

If we construct a cone (K) with its vertex at M and its directrix as (E), and draw the vectors $A, B,$ and C through M, A and C will lie along the generators (A) and (C) of (K), while B will lie along the polar plane of A and C with respect to (K). Indeed, if $F(\varphi)$ denotes the derivative of the vector G with respect to φ, all vectors in the tangent plane of (K) along its generator (G) can be obtained by linear combinations of G and F.

Since:

$$B = \cos(\varphi_2 - \varphi_1)G(\varphi_1)G(\varphi_2) + \frac{1}{2}\sin(\varphi_2 - \varphi_1)F(\varphi_1),$$

B lies in the tangent plane of (K) along (A). Similarly, B also lies in the tangent plane of (K) along (C), proving the claim.

B will be perpendicular to the plane of A and C only if it lies along one of the principal axes of (K). The corresponding endpoints of A and C are obtained by intersecting (E) with the principal planes of (K). In the general case, when (K) is not a cone of revolution, there are three such planes: two give real A and C, while the third gives complex ones. Since moving A and C results in the same pair (ξ), (η), a V_2 Dupin indicatrix generally has two real affine symmetry axis pairs and one complex pair. If (K) is a cone of revolution, i.e., M lies on the focal hyperbola of (E), there are infinitely many affine axes. In this case, (D) splits into two conics, which, as can be easily verified, are two connected ellipses. In addition to linear, planar, and umbilic points, a real V_2 may also have a fourth special type of point—those with a revolution cone (K), which generalize umbilic points.

Similar results are obtained for planar M. Exceptional cases are: M on (E), where (G) has only one pair of real affine axes, and M at (E)'s foci, where (G) again splits into two connected ellipses. Finally, for axial points, as expected, the results are the same as for surfaces in three-dimensional space.

The results of the last two paragraphs show that, at a general point M on real $V_2 \subset R_n$, there exist the following distinct real directions:

- Four directions corresponding to the vertices of (E) (each pair associated with an axis is orthogonal and bisects the other pair), - Four directions corresponding to the affine symmetry axes of (D) (these form a harmonic bundle), and - Four or two directions corresponding to extremal geodesic curvatures (in the latter case, an additional direction corresponding to a stationary geodesic curvature, which is not extremal, may join).

With this, we conclude this article, hoping that, despite some cumbersome proofs, awkward phrasing, and potential accidental errors, the examples presented sufficiently demonstrate the feasibility of replacing absolute differential calculations in n-dimensional differential geometry with the elementary methods employed here. These methods have provided us with both familiar properties and the opportunity to discover new ones.

For this reason, the author also intends to use these methods in a future systematic work to explore the questions briefly touched upon here, as well as others that arise from them.

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