The Metric Problem in Polar Coordinates: A Comprehensive Analysis and a Solution

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Abstract

The purpose of this article is to demonstrate that the common method for calculating the metric tensor in polar coordinates is incorrect. However, it does not lead to erroneous results due to the presence of another error that cancels out the effects of the first error in calculations.   Nerveless, these offsetting errors creates a problem in understanding the metric.

Generalization

In certain cases, there are errors in defining some physical quantities using mathematical formulas. However, these errors do not result in any inaccuracies in the outcomes because the incorrectly defined physical quantity is closely related to another quantity. By adjusting the definition of the second quantity, the effects of the initial error can be canceled out. This, however, is an unsuccessful remedy because it addresses only the computational aspect of the error while neglecting other significant effects. For example, both concepts lose the properties that distinguish real physical quantities, such as simplicity, generality, symmetry, dimensional consistency in their relations with other physical quantities, and so on. All of this ultimately leads to difficulties in forming a clear, stable, and convincing understanding of these quantities for the learner.

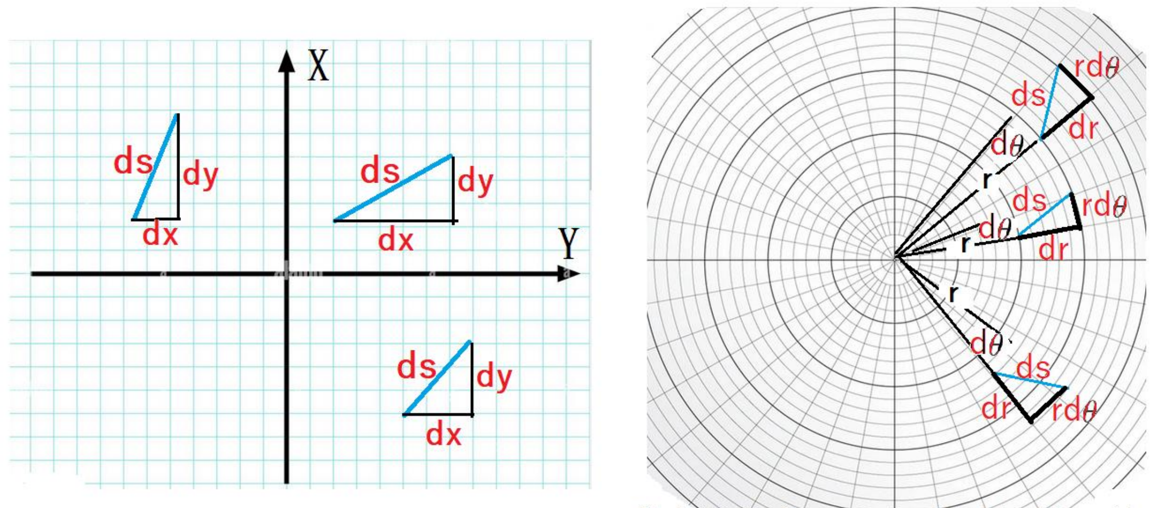
Let us clarify this with a simple and abstract example to establish this idea before applying it to the issue of the relationship between the metric tensor and the components of the line element in polar coordinate system:

Assume that the correct mathematical definition of the physical quantity   A is A = D(x), and the correct definition of another quantity B is B = F(y). If these two quantities always appear in equations as a product, then an error in defining both quantities like: A = D(x)/r and B = F(y). r will not affect the computations because the two error offset each other, leading to a correct result.

I want to argue that there are offsetting errors in the definition of the metric tensor in polar coordinates. I will explain what the error is, why it does not affect calculations, and finally highlight some of the negative consequences of this error.

What is the error, and what is the correct definition?

Let us start by the metric tensor in Cartesian coordinates and then transfer this definition to polar coordinates. For simplicity, we will work in two dimensions for now, with the understanding that the generalization to three or four dimensions is straightforward.



The Components of the Line Element in Cartisain Coordinates are (dx) and (dy) and they are related to it by the equation: $ds^2 = dx^2 + dy^2$

which can be written: $ds^2 = g_{xx}\ dx^2 + g_{yy}\ dy^2$

this leads to: $g_{xx} = 1$, $g_{yy} = 1$

Similarly, the Components of the Line Element in Polar Coordinates are (dr) and (r dθ) (Not dθ) and they are related to it by the equation: $ds^2 = dr^2 + (r\,d\theta)^2$

which can also be written: $ds^2 = g_{rr}\ dr^2 + g_{\theta\theta}\ (r\,d\theta)^2$

(Not $ds^2 = g_{rr}\ dr^2 + g_{\theta\theta}\ d\theta^2$) This leads to:

$g_{rr} = 1$, $g_{\theta\theta} = 1$ (Not $g_{rr} = 1$, $g_{\theta\theta} = r^2$)

This figure illustrates how the metric tensor represents the relationship between the line element and its components in the x- and y-directions in Cartesian coordinates or in in r-and Ө-directions in polar coordinates.

Notice also that in Cartesian coordinates, the magnitudes of these components vary depending on the angle of inclination of the line element relative to the -axis. However, this variation does not affect the metric tensor because the relationship between the line element and its components remains unchanged despite the variations in their magnitudes.

Now, when we define the metric tensor in polar coordinates as shown in the right side of the figure, we find that the components of the differential of line element in the r-direction vary depending on the angle of inclination of the line element relative to the axis Ө = zero, while those in the Ө -direction vary depending on the distance from the center.

The variation in the value of the Ө-component of the line element as we move away from the center does not affect the relationship between the line element and its components (analogous to the variation of component magnitudes with the direction of the line element in Cartesian coordinates). Thus, the calculation of the metric tensor should be based on the relationship between the line element and its components, and not on the relationship between the line element and the grids of that form the fabric of the Cartesian or polar coordinate system.

Hence, the correct relation that define the metric in polar coordinates is:

$$\mathbf{ds^2 = \quad g_{rr} \quad dr^2 + g_{\theta\theta}(r\, d\theta)^2} \quad (1)$$

Which leads to: $\mathrm{g_{rr}} \quad = 1\,,\ \mathrm{g_{\theta\theta}} = 1$

Not:

$$\mathrm{ds^2 = \quad g_{rr} \quad dr^2 + g_{\theta\theta}}\, d\theta^2 \qquad (2)$$

Which leads to: $\mathrm{g_{rr}} \quad = 1\,,\ \mathrm{g_{\theta\theta}} = r^2$

That is because the components of the line element are (dr) and (r$d\theta$) not (dr) and (dθ)[1].

Why does the error not affect calculations?

We will see that all relationships involving the metric tensor remain unaffected whether we use the correct or the common incorrect definition. These relationships include: the line element, the geodesic equation, the volume element equation, the Riemann tensor (and related quantities Ricci tensor and Ricci scalar), and the Schwarzschild solution.

We can see that the geodesic equation represents the same reality whether we use the definition of metric according to equation (1) or (2) because the two equation are correct regarding the relation between the line element and the differentials of coordinates and the fact that the quantity r is a part of the components of the line element rather than a part of the metric will not affect this relation, similarly, the volume element equation remains unaffected, because if we use equation (1) then the determinant of the metric is equal to 1 and quantity $\sqrt{\mathrm{det(Metric)}}$ is equal to 1 but the length element related to θ

is r$d\theta$,and if we use equation (2) then $\sqrt{\det(\text{Metric})}$ is equal to r while the length element related to θ is d θ, so the result is the same.

As for the Riemann tensor, Ricci tensor, and Ricci scalar, they equal zero regardless of whether the correct or incorrect metric is used.

Regarding Schwarzschild solution, we have to rewrite it in the following form:

$$\boldsymbol{ds^2 = C^2\,dt^2 - dr^2 - (r\,d\theta)^2 - (r\sin\theta\,d\Omega)^2}$$

This leads to:

$$\boldsymbol{g_{tt} = C^2\,,\quad g_{rr} = -1\,,\ g_{\theta\theta} = -1\,,\ g_{\Omega\Omega} = -1}$$

Practically, this is the same as the widely accepted solution because the results remain the same when calculating the geodesics, as the definitions of the line element components also change in a way that cancels computational differences.

What are the problems caused by this error?

The method of the previous discussion should not lead us to believe that the difference between equation (1) and equation (2) is merely a matter of notation. No. The common method that relies on equation (2) leads to real conceptual problems.

The most significant issue with the commonly used definition of the metric in polar coordinates is the inconsistency in the dimensions of its components. While some components, like $\boldsymbol{g_{rr}}$, are dimensionless, others, like $\boldsymbol{g_{\theta\theta}}$ have dimensions corresponding to the square of length. This conceptual issue worsens when calculating physical quantities such as the Riemann tensor, whose definition involves Christoffel symbols that in turn include sums of derivatives of the metric. These derivatives will have different dimensions depending on the metric itself, resulting in physical equations involving sums of quantities with different dimensions—a meaningless operation in physics.

Another, less significant [2] issue with the incorrect method is the complexity arising from the idea that the metric tensor changes from point to point in a homogeneous and isotropic space, such as flat space. In contrast, the same space has a constant metric in Cartesian coordinates. This forces us to accept the idea that changing coordinate systems can make constants variable, a concept that causes considerable confusion, especially for bright young learners.

References

[1] The extension of Schwarzschild line element to include uniformly accelerated mass. Ranchhaigiri, Brahma.A.K.Sen. This is a good example of this misconception about the metric tensor.

[2] See: Space-time and Geometry, Sean Carroll. Page 71-72 The Metric.