Proof of Beal's Conjecture

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Abstract

The 6 variable general equation of Beal's conjecture equation $x^a + y^b = z^c$, where x, y, z, a, b, and c are positive integers, and $a, b, c \geq 3$, is identified as an identity made by expansion of powers of binomials of integers x and y; where x, y and z have common prime factor. Here, a proof of the conjecture is presented in two folds. First, powers of binomials of integers x and y expand to all integer solutions of Beal's equation if they have common prime factor. Second, powers of binomials of coprime positive integers x and y expand to two terms such that if one of them is a perfect power the other one is not a perfect power.

Key words: Beal's conjecture, Binomial identity
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Introduction

Beal's conjecture states that if $x^a + y^b = z^c$, where *a*, *b*, *c*, *x*, *y* and *z* are positive integers and *a*, *b*, c > 2, then *x*, *y*, and *z* have a common prime factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. It is a generalization of Fermat's Last Theorem (FLT) which states that no three positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. FLT has been considered extensively in the literature [2-7] and was proved by Andrew Wiles [8]. Similar problems to Beal's conjecture have been suggested as early as the year 1914 [9] and the conjecture maybe referred to by different names in the literature [10-11]. So far a proof to the conjecture has been a challenge to the public as well as to mathematicians and no counterexample has been successfully presented to disprove it, i.e. Peter Norvig reported having conducted a series of numerical searches for counterexamples to Beal's conjecture. Among his results, he excluded all possible solutions having each of $a, b, c \leq 7$ and each of $x, y, z \leq 250{,}000$, as well as possible solutions having each of $a, b, c \leq 100$ and each of $x, y, z \leq 10{,}000$ [12]. In this paper, we prove Beal's conjecture by elementary approach.

Proof of the conjecture

Let's recall that a binomial identity describes the expansion of powers of a binomial as given in equation (1).

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^n = x^n + (\sum t + y^n) \qquad ,n > 2 \qquad (1.1)$$

$$(x+y)^n = (\sum t + x^n) + y^n \qquad ,n > 2 \qquad (1.2)$$

Lemma 1

For positive integers x, y, identity (1) produces three terms in Z+.

Proof

On the RHS of identity (1), leaving x^n as perfect power term, the term $\sum t + y^n$ is a positive integer and leaving y^n as perfect power term, the term $x^n + \sum t$ is a positive integer in Z+.

End of proof.

Lemma 2

For coprime positive integers x, y, the RHS of identity (1) produces a nonperfect power second term in Z+ if either x^n or y^n is held as perfect power of n.

Proof

For the case of $(\sum t + y^n)$ and $(\sum t + x^n)$ to be expressed in the form of $\lambda^n y^n$ and $\lambda^n x^n$ respectively, the identities (1.1) and (1.2) ensures that the terms $(\sum t + y^n)$ and $(\sum t + x^n)$ cannot be perfect power of n by FLT theorem, i.e. $(\sum t + y^n)$ cannot be reduced to $\lambda^n y^n$, neither $(\sum t + x^n)$ can be reduced to $\lambda^n x^n$, where λ^n is perfect power of n positive integer. Therefore, such λ does not exist.

For the case of $(\sum t + y^n)$ and $(\sum t + x^n)$ to be expressed in the form of $\lambda^n y^n$ and $\lambda^n x^n$ respectively, the identities (1.1) and (1.2) ensures that the terms $(\sum t + y^n)$, $(\sum t + x^n)$ cannot be reduced to $y^\lambda y^n$, $x^\lambda x^n$ respectively to form a perfect power term because $\sum t$ always reduces to a composite number for $n \geq 3$ of coprime factors. Let's expand the binomial $(x+y)^3$,

$$(x+y)^3 = x^3 + 3x^2y + 3y^2x + y^3 \qquad (2)$$

$$\sum t = 3x^2y + 3y^2x$$

$$\sum t = 3(x+y)xy \qquad (3)$$

The term $\sum t$ has coprime factors since the product of two coprime numbers is coprime with their sum and cannot reduce to $y^\lambda y^n$ or $x^\lambda x^n$, where λ is a positive integer. This is simply because equation (3) always gives a power of y or x that is less than n, and coefficients of composite numbers, i.e. $\sum t$ has highest power of 1 that is less than power 3 of the last term y^3, therefore, it cannot be combined to produce a perfect power term.

End of proof.

Theorem

Expansion of powers of binomials produces an identity of three terms that requires a common factor for all three terms to be perfect powers.

Proof

From Lemma 1 and 2, the two terms on the RHS of equation (1) cannot be both perfect power if x, y are coprime.

End of proof.

Examples

Let $x = 2$ and $y = 3$ in equation (2)

$$5^3 = 2^3 + 3 * 2^2 * 3 + 3 * 3^2 * 2 + 3^3$$

$(\sum t + y^n)$ produces the solution

$$5^3 = 2^3 + 117$$

We need to multiply the equation by the common factor k^3, where $k = 117$ with factors 1, 3, 9, 13, 39, 117, to produce all three perfect power terms.

$$(5k)^3 = (2k)^3 + 117k^3$$

Let $k = 117$, the solution with perfect power terms then is,

$$585^3 = 234^3 + 117^4$$

Let's set $y = x$ for common factor x. Equation (2) becomes,

$$(2x)^3 = x^3 + 7x^3$$

Taking the common factor as $x = 7$, the equation becomes,

$$7^3 + 7^4 = 14^3$$

Conclusion

We have proved Beal's conjecture by elementary means.

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