ON THE HARMONIC DISTRIBUTION OF ADDITION CHAINS

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Abstract. In this note, we study the harmonic distribution of addition chains of a given length. If $1, 2, \ldots, s_{\delta(n)-1}, s_{\delta(n)} = n$ is an addition chain producing n and of length $\delta(n)$, with associated sequence of generators

$1+1, s_2 = a_2 + r_2, \ldots, s_{\delta(n)-1} = a_{\delta(n)-1} + r_{\delta(n)-1}, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)} = n$

then

$$\sum_{l=1}^{\delta(n)} \frac{1}{s_l} = \frac{3}{2} + \frac{\delta(n)}{n+1} + \sum_{l=3}^{\delta(n)} \sum_{v=1}^{\infty} \frac{1}{(n+1)^{v+1}} \bigg(\sum_{j=l}^{\delta(n)} r_j\bigg)^v + O(\frac{1}{n})$$

where $\sum\limits_{j=l}^{\delta(n)} r_j < n-1$ is the sum of the regulators in the generator of the chain for each $3 \leq l \leq \delta(n)$.

1. Introduction

The notion of an addition chain producing $n \geq 3$, introduced by Arnold Scholz, is a sequence of numbers of the form

$$1, 2, \ldots, s_{k-1}, s_k = n$$

where each term in the sequence is generated by adding two earlier terms and the terms are allowed to be homogeneous. The number of terms in the sequence determines the length of the chain. The length of the smallest such chain producing n is the shortest length of the addition chain. It is a well-known problem to determine the length of the shortest addition chain producing numbers $2^n - 1$ of special forms. More formally the conjecture states

Conjecture 1.1. Let $\iota(n)$ for $n \geq 3$ be the shortest addition chain producing n, then the inequality is valid

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

The conjecture was studied fairly soon afterwards by Alfred Brauer, who obtained some weaker bounds [1]. There had also been amazing computational work to verify the conjecture [2]. In this paper, we extend the study of addition chains by analyzing their harmonic partial sums, defined as the reciprocals of the terms in the sequence. This perspective offers a novel approach to understanding the structural properties of addition chains. Specifically, we derive precise expressions for these sums, revealing intricate relationships between the terms in the chain and providing a deeper insight into their behavior. Our main results are as follows:

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2. **The regulators and determiners of an addition chain**

In this section, we recall the notion of an addition chain and introduce the notion of the generators of the chain and their accompanying *determiners* and *regulators*.

Definition 2.1. Let $n \geq 3$, then by an addition chain of length $k-1$ producing n, we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

where $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i^{th} **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiner** and r_i the **regulator** of the i^{th} generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$. We call the subsequence (s_{j_m}) for $2 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a truncated addition chain producing n.

At any rate, we do not expect the regulators to be a part of the chain, although the determiners must be the terms in the chain.

Lemma 2.2. *Let $1, 2, \ldots, s_{k-1}, s_k$ be an addition chain producing $n \geq 3$ with associated generators*

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then the following relation for the regulators

$$\sum_{j=2}^{k} r_j = n - 1$$

hold.

Proof. We notice that $r_k = n - a_k$. It follows that

$$\begin{aligned} r_k + r_{k-1} &= n - a_k + r_{k-1} \\ &= n - (a_{k-1} + r_{k-1}) + r_{k-1} \\ &= n - a_{k-1}. \end{aligned}$$

Again we obtain from the following iteration

$$\begin{aligned} r_k + r_{k-1} + r_{k-2} &= n - a_{k-1} + r_{k-2} \\ &= n - (a_{k-2} + r_{k-2}) + r_{k-2} \\ &= n - a_{k-2}. \end{aligned}$$

By iterating downwards in this manner the relation follows. $\square$

3. Main result

We derive an *asymptotic* formula for the harmonic partial sums of terms in an addition chain.

Theorem 3.1. *Let* $1, 2, \ldots, s_{\delta(n)-1}, s_{\delta(n)} = n$ *be an addition chain producing* n *and of length* $\delta(n)$*, with associated sequence of generators*

$$1+1, s_2 = a_2 + r_2, \ldots, s_{\delta(n)-1} = a_{\delta(n)-1} + r_{\delta(n)-1}, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)} = n$$

then

$$\sum_{l=1}^{\delta(n)} \frac{1}{s_l} = \frac{3}{2} + \frac{\delta(n)}{n+1} + \sum_{l=3}^{\delta(n)} \sum_{v=1}^{\infty} \frac{1}{(n+1)^{v+1}} \bigg(\sum_{j=l}^{\delta(n)} r_j \bigg)^v + O(\frac{1}{n})$$

where $\sum\limits_{j=l}^{\delta(n)} r_j < n-1$ *is the sum of the regulators in the generator of the chain for each* $3 \leq l \leq \delta(n)$*.*

Proof. Consider an addition chain $1, 2, \ldots, s_{\delta(n)-1}, s_{\delta(n)} = n$ producing n and of length $\delta(n)$, with associated sequence of generators

$$1+1, s_2 = a_2 + r_2, \ldots, s_{\delta(n)-1} = a_{\delta(n)-1} + r_{\delta(n)-1}, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)} = n$$

and put (a_j) and (r_j) to be the sequence of determiners and regulators, respectively, in the chain. We make the following observations: $s_{\delta(n)-1} = a_{\delta(n)} = a_{\delta(n)-1} + r_{\delta(n)-1} = s_{\delta(n)-2} + r_{\delta(n)-1} = a_{\delta(n)-2} + r_{\delta(n)-2} + r_{\delta(n)-1} = \cdots = 1 + \sum\limits_{j=1}^{\delta(n)-1} r_j = n + 1 - r_{\delta(n)}$, where we have used Lemma 2.2. Similarly, we can write $a_{\delta(n)-1} = 1 + \sum\limits_{j=1}^{\delta(n)-2} = n + 1 - r_{\delta(n)} - r_{\delta(n)-1}$. Thus by induction, we can write $a_l = n + 1 - \sum\limits_{j=l}^{\delta(n)} r_j$ for each $3 \leq l \leq \delta(n)$. We observe that

$$\sum_{l=1}^{\delta(n)} \frac{1}{s_l} = \frac{3}{2} + \sum_{l=3}^{\delta(n)} \frac{1}{a_l} + \frac{1}{n}.$$

We now analyze the latter sum of the right-hand side involving the *determiners* of the addition chain. We can write

$$\sum_{l=3}^{\delta(n)} \frac{1}{a_l} = \sum_{l=3}^{\delta(n)} \frac{1}{(n+1) - \sum\limits_{i=l}^{\delta(n)} r_i}$$

which can be recast as

$$\sum_{l=3}^{\delta(n)} \frac{1}{a_l} = \sum_{l=3}^{\delta(n)} \frac{1}{n+1} + \sum_{l=3}^{\delta(n)} \sum_{v=1}^{\infty} \frac{1}{(n+1)^{v+1}} \bigg(\sum_{i=l}^{\delta(n)} r_i \bigg)^v$$

with $\sum\limits_{i=l}^{\delta(n)} r_i < n-1$ for each $3 \leq l \leq \delta(n)$ by Lemma 2.2. It follows that

$$\sum_{l=1}^{\delta(n)} \frac{1}{s_l} = \frac{3}{2} + \frac{\delta(n)}{n+1} + \sum_{l=3}^{\delta(n)} \sum_{v=1}^{\infty} \frac{1}{(n+1)^{v+1}} \bigg(\sum_{i=l}^{\delta(n)} r_i \bigg)^v + O(\frac{1}{n})$$

where $\sum\limits_{i=l}^{\delta(n)} r_i < n-1$ by Lemma 2.2 for each $3 \leq l \leq \delta(n)$. This completes the proof of the claimed formula. $\square$

4. **Further remarks**

This note investigated the harmonic partial sums of addition chains, providing a new perspective on their structure and behavior. By analyzing the reciprocals of the terms in the chain, we have derived explicit formulas that illuminate the intricate relationships within the sequences. Our results extend the work on addition chains, offering a deeper understanding of their properties and suggesting further directions for research, particularly in the context of conjecture on the minimal length of addition chains for special forms like 2^n-1. Future work could focus on refining the bounds for these sums and exploring other distributions such as the sums of powers of terms in an addition chain. The findings presented here contribute to a more comprehensive understanding of addition chains and their harmonic properties, opening the door to a more complete understanding on this topic.

[1].

References

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