The support

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Abstract

Let X be a complex projective manifold. First, we prove that for a real, singular cycle C of real codimension $2p$, if its current of integration is of bidegree (p,p) and positive, there exists an algebraic set A of complex codimension p that contains C. Secondly, we prove that for the singular cohomology $H^i(X;\mathbb{Q})$ of degree i with rational coefficients, a Hodge class $u \in H^{p,p}(X;\mathbb{Q})$ can be represented by the difference of two rational, singular cycles whose currents of integration are of bidegree (p,p) and positive. The combined result implies a cohomological assertion on the support:

$$u \in ker\left(H^{2p}(X;\mathbb{Q}) \to H^{2p}(X-A;\mathbb{Q})\right) \tag{0.1}$$

where " ker" denotes the kernel of the restriction map. Furthermore, the supportive assertion (0.1) implies that u is represented by an algebraic cycle.

1 Statements

Let X be a complex projective manifold. For any closed subset V, the subgroup

$$ker\left(H^i(X;\mathbb{Q}) \to H^i(X-V;\mathbb{Q})\right) \tag{1.1}$$

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will be denoted by $H^i_{(V)}(X;\mathbb{Q})$ where ker stands for the kernel of the restriction map. A Hodge class is a class in $H^{2p}(X;\mathbb{Q})$ whose bidegree is of (p,p) type in the Hodge decomposition over $\mathbb{C}$.

Main theorem 1.1. *For a Hodge class u of degree $2p$, there exists an algebraic set A of complex codimension p such that*

$$u \in H^{2p}_{(A)}(X;\mathbb{Q}). \tag{1.2}$$

Remark: We'll say that the u or its representative is class-supported on the algebraic set A if (1.2) is satisfied. The sub-cohomology $H^i_{(V)}(X;\mathbb{Q})$ was first introduced by Grothendieck in [4]. Indeed, the main theorem immediately implies the following (see [1], [9]),

Corollary 1.2. *Hodge classes are represented by algebraic cycles with rational coefficients.*

Proof of Corollary 1.2: Let u be a Hodge class of degree $2p$. The main theorem asserts u is class-supported on an algebraic set A of complex codimension p. Let

$$\tilde{A} \stackrel{J}{\rightarrow} A \stackrel{I}{\hookrightarrow} X$$

be the composite such that J is a smooth resolution and I is the inclusion. Since the codimension condition

$$deg(u) - 2cod(A) = 0 \geq 0$$

is satisfied, we apply Deligne's corollary 8.2.8, [2] which addresses the class-support. Precisely it states that the Gysin map

$$(I \circ J)_! : H^0(\tilde{A};\mathbb{Q}) \to H^{2p}_{(A)}(X;\mathbb{Q}) \tag{1.3}$$

is surjective. Then a pre-image u' of u is a cohomological class of $\tilde{A}$ of degree 0. So, u' must be represented by a rational linear combinations of irreducible components of $\tilde{A}$. Therefore $u = (I \circ J)_!(u')$ is represented by a rational, linear combination of irreducible components of A. The proof is completed.

□

We'll organize the proof of the main theorem as follows. In section 2, we give a lemma on the support of currents which is the basis of our approach. In section 3, we address the orientation of vector bundles and the infinitesimal positivity. In section 4, we show that the infinitesimal positivity defined in Section 3 implies the effective decomposition of Hodge classes. In section 5, we conclude the proof.

2 Current-support

The philosophy is that Hodge's problem is about topological cycles lying on algebraic varieties. Lying on algebraic sets is much weaker than itself being algebraic. So, our intention is to see if this weaker set-theoretical containment has any structural impact on cohomology. Looking back in this view, we see that the support used by Hodge in his original article [9] is the set of singular cycles. However, this type of the support may not be sufficient. Thus we extend our interest to general currents. In this paper, we focus on a particular type of currents, namely the positive currents. The support of positive currents has been studied consistently and cumulatively in the past starting from the origin in Lelong's work ([11]). We'll only list those directly relevant in the proof: [3], [5], [6], [7], [8], [12].

We should recall some of the definitions in the references. On any complex manifold Ω, a form ϕ of bidimension (k,k) is said to be decomposable (or strongly positive) if at each point, it lies in the positive cone generated by the forms

$$\sqrt{-1}\alpha_1 \wedge \overline{\alpha}_1 \cdots \wedge \cdots \sqrt{-1}\alpha_k \wedge \overline{\alpha}_k$$

where $\alpha_1, \cdots, \alpha_k$ are differential forms of bidegree $(1,0)$ on Ω. A positive (k,k) current is a current $T \in \mathscr{D}'_{k,k}(\Omega)$ of bidimension (k,k) such that

$$T[(\bullet)\phi]$$

for any decomposable form ϕ defines a measure on Ω. If Ω is Kähler, let ω be the Kähler form and T a bidimension (k,k), closed positive current. The Lelong number of T at a point a, denoted by $\nu(T,a)$, is defined as the limit

$$\lim_{r\to 0} \frac{T[\chi_r \omega^k]}{(\pi r^2)^k}$$

where χ_r is the characteristic function of a ball centered around a with radius r. The upper level set for those positive closed T as above, denoted by $E_c(T)$ for $c \geq 0$ is the set

$$\{x \in \Omega : \nu(T,x) \geq c\}$$

which is proved to be complex analytic ([12]). A holomorphic chain is a particular type of positive currents which are modeled on complex analytic sets. The following is the precise definition.

Definition 2.1. *Let Ω be a complex manifold. A current $\mathcal{T}$ is called a $\mathbb{G}$-holomorphic chain of complex codimension p on Ω if there are irreducible subvarieties $V_i, i \in \mathbb{Z}^+$ of complex codimension p in Y and corresponding coefficients $a_i \in \mathbb{G}$ for $\mathbb{G} = \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ such that*

(1) $\cup_{i\in\mathbb{Z}^+} V_i$ is a subvariety in Ω,

(2) the current $\mathcal{T}$ is equal to the current of the integration over

$$\sum_{i\in\mathbb{Z}^+} a_i V_i$$

(the integral is well-defined since the family $\{V_i\}$ is locally finite due to (1))

If $a_i > 0$ for all i, we say $\mathcal{T}$ is positive.

Notation: for a singular chain σ, the current of integration over σ will be denoted by T_σ.

Lemma 2.2. *Let X be a compact Kähler manifold, and p an integer $\in [0, dim_{\mathbb{C}} X]$. If the current of the integration over a real, oriented singular cycle C is of bidegree (p,p) type and positive, then there is a complex analytic set S of complex codimension p such that*

$$supp(C) \subset S. \tag{2.1}$$

We call the support in (2.1) the current-support (versus the class-support in Main theorem 1.1).

Remark: A current-support is a class-support. But the converse is not true.

Proof. All singular cycles in this paper are regarded as those whose singular maps are C^∞ embeddings including the frontier points. If $C = 0$, the lemma is trivial. So we assume $C \neq 0$. For the bidegree (p,p) positive closed T_C on X, there is an unique decomposition (for instance, see Theorem 2.4, [6] or formula (2.3), [7]):

$$T_C = T_Z + R \tag{2.2}$$

where Z is either 0 or a positive holomorphic chain with real coefficients, R is a positive current of bidegree (p,p) such that the complex codimension of the upper level set

$$E_c(R)$$

for each positive number c is at least $p+1$. Since the family is locally finite, the local sum is finite. Since X is compact, Z only consists of finitely many irreducible subvarieties as components. Hence $supp(Z)$ is a subvariety which in particular is topologically closed in the usual topology. Notice that

$$supp(C) \subset supp(Z) \cup supp(R). \tag{2.3}$$

So, it suffices to prove

$$supp(R) \subset supp(Z).$$

Let

$$supp(R) = E_0[R] \cup E_+(R)$$

where

$$E_0[R] = supp(R) \cap E_0(R)$$

and

$$E_+ = \cup_{c>0} E_c(R)$$

where all $E_c(R)$ with $c > 0$ are complex analytic subvarieties of complex codimension $\geq p+1$.

Case 1: Let $a \in E_0[R]$ but $a \notin supp(Z)$. Then the left hand side of (2.2) shows that $\nu(T_C, a) = 0$. Since $C \neq 0$, there is a non-empty cell c_q in the chain C with the real coefficient $\lambda_q \neq 0$. Assume a is an interior point of c_q. We observe the formula of Lelong number $\nu(T_C, a)$ is

$$\lim_{r\to 0} \frac{\int_{B_r \cap c_q} \lambda_q \omega^k}{(\pi r^2)^k} \tag{2.4}$$

where B_r is the ball of radius r in a complex analytic chart, and $k = n - p$ for $n = dim_{\mathbb{C}}(X)$. In local coordinates, we express

$$\omega^k = \sum_I (1 + O_I(2)) d\mu_{\mathbf{z}_I}$$

where $\mathbf{z}$ is the complex chart

$$(z_1, \cdots, z_n)$$

I is the multi-index,

$$\begin{aligned} & d\mu_{\mathbf{z}_I} = (\frac{\sqrt{-1}}{2})^k dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge d\bar{z}_{i_k}, \\ & with\ i_1 \leq \cdots \leq i_k \end{aligned}$$

is the volume form of length $2k$, and $O_I(2)$ is the higher order terms around a. Since c_q is a polyhedron, there exists an index I' such that projection $c_q \to V_{I'}$ has a non degenerate Jacobian $J^q_{I'}(a)$, where $V_{I'}$ is the coordinate's plane with coordinates $\mathbf{z}_{I'}$. Since the current $\lambda_q T_{c_q}$ is positive, the Jacobian satisfies $\lambda_q J^q_{I'}(a) > 0$. To the rest of other coordinate's planes, the multiples of Jacobians $\lambda_q J^q_I(a)$ of the projection are all non-negative around a (since $T_{\lambda_q c_q}$ is positive). Then we can convert the integral (2.4) to the sum of the Lebesgue integrals

$$\nu(T_C, a) = \sum_{all\ I} \lim_{r\to 0} \frac{\int_{B^q_r} \lambda_q J^q_I(\mathbf{z}) \big(1 + \bar{O}_I)\big) d\mu_{\mathbf{z}_I}}{(\pi r^2)^k} \tag{2.5}$$

where B^q_r is projection of $B_r \cap c_q$ to the coordinate's plane, and for all I,

$$1 + \bar{O}_I \geq 0$$

is an integrable function. Hence we have the integral estimate of (2.5),

$$\nu(T_C, a) = \sum_{all\ I} \lambda_q J^q_I(a) \geq \lambda_q J^q_{I'}(a) > 0. \tag{2.6}$$

If a is a frontier point, there will be multiple cells

$$c_q$$

containing a as a general point on a $2k-1$ face. Then the same formula

$$\nu(T_C, a) = \sum_{I,q} \lim_{r\to 0} \frac{\int_{B_r^q} \lambda_q J_I^q(\mathbf{z})\big(1+\bar{O}_I\big) d\mu_{\mathbf{z}_I}}{(\pi r^2)^k} \tag{2.7}$$

still holds. Similarly, $\nu(T_C, a)$ is equal to

$$\nu(T_C, a) = \sum_{I,q} \lambda_q J_I^q(a) r_q \geq \sum_q \lambda_q J_{I'}^q(a) r_q > 0 \tag{2.8}$$

where r_q (which did exist before) is the quotient

$$\lim_{r\to 0} \frac{vol(B_q)}{(\pi r^2)^k}.$$

Hence it is also positive. The contradiction for both cases (interior points and frontier points) implies that $E_0[R] \subset supp(Z)$.

Case 2: If $a \in E_+(R)$ but $a \notin supp(Z)$, the left hand side of (2.2) shows that the Lelong number of T_C at a is not zero. Then $a \in supp(C)$. Since $supp(Z)$ is topologically closed, there is small closed neighborhood $\overline{U}$ of a such that $\bar{U} \cap supp(Z) = \varnothing$. Hence

$$\overline{U} \cap supp(C)$$

lies in $E_+(R)$. Since $\overline{U}$ is closed, $\overline{U}$ lies in $E_c(R)$ for some positive c. It implies that the set $\overline{U} \cap supp(C)$ is contained in a complex analytic set of complex codimension $\geq p+1$. But as a singular cycle C, $\overline{U} \cap supp(C)$ contains a disk of real codimension $2p$. This implies that a real disk of real codimension $2p$ is contained in a complex analytic set of complex codimension $\geq p+1$. This situation is impossible. The contradiction shows $E_+(R) \subset supp(Z)$. Hence if we let $S = supp(Z)$, (2.3) says

$$supp(C) \subset S.$$

The proof is completed.

□

Remark: Notice that the complex analytic subvariety Z always exists if C exists. Hence Zucker's counter-example in [14] provides a situation where positive singular cycles do not exist.

3 Complex-orientation and infinitesimal positivity

Let Y be a connected, orientable differential manifold of dimension m, $W \subset Y$ a closed, connected, orientable submanifold of dimension ℓ. Let $N_W \simeq N_W Y$ be the orientable normal bundle. Denote the bundle $TY|_W \to W$ by F. There is the bundle isomorphism

$$F \simeq TW \oplus N_W. \tag{3.1}$$

Then there is an expansion of the wedge product in line bundles

$$\wedge^m F^* \simeq \left(\wedge^\ell T^* W\right) \otimes \left(\wedge^{m-\ell} N_W^*\right) \tag{3.2}$$

where $*$ denotes the dual of the bundle. Thus we have line bundles

$$\left(\wedge^m F^*\right) \otimes \left(\wedge^{m-\ell} N_W\right) \simeq \wedge^\ell T^* W. \tag{3.3}$$

Hence if s_1, s_2 are nowhere vanishing continuous sections of the line bundles $\wedge^m F^*$ and $\wedge^{m-\ell} N_W$, then

$$s_1 \otimes s_2 \tag{3.4}$$

is a nowhere vanishing continuous section of T^*W. Recall that a class of nowhere vanishing continuous sections is an orientation. Thus we have the definition:

Definition 3.1. *An orientation of the bundle T^*W is said to be the quotient orientation of the representation of s_1, s_2 if it is represented by the section (3.4).*

We continue the setting above to assume Y is a complex manifold, but W is still real submanifold satisfying the conditions as above. Assume the normal bundle is a complex bundle, i.e the fibres are $\mathbb{C}^{\frac{m-l}{2}}$.

Definition 3.2. *Let s_1 represent the canonical orientation of the complex structure of Y, and s_2 represent the canonical orientation of the complex normal bundle. We call the quotient orientation of s_1, s_2 the complex orientation of W.*

Remark: If W is a complex submanifold, the complex orientation of W agrees with the canonical orientation of the complex structure.

Let $E = \mathbb{C}^N$ be a complex vector space of complex dimension N with the canonical orientation. Denote its underlined real vector space by $E_\mathbb{R}$, its complexification $E_\mathbb{R} \otimes \mathbb{C}$ by $E_\mathbb{C}$. The operator $\bar{\cdot}$ is the usual complex conjugate on $E_\mathbb{C}$. Let's work in the complex exterior algebra of $E_\mathbb{C}$.

The following is the definition of the positivity (see [5] for a different version of the definition in terms of covectors).

Definition 3.3. *A $2k$-simple vector $\zeta \in \wedge^{2k} E_{\mathbb{C}}$ is said to be weakly positive if for any set of linearly independent $(1,0)$ vectors $\alpha_1, \cdots, \alpha_{N-k}$,*

$$(\sqrt{-1})^{N-k} \zeta \wedge \bar{\alpha}_1 \wedge \alpha_1 \wedge \cdots \wedge \bar{\alpha}_{N-k} \wedge \alpha_{N-k}$$

is a positive volume form of $E_{\mathbb{R}}$, where a positive volume form of $E_{\mathbb{R}}$ is a vector $\delta(\sqrt{-1})^N \overline{\mathbf{b}}_1 \wedge \mathbf{b}_1 \wedge \cdots \wedge \overline{\mathbf{b}}_N \wedge \mathbf{b}_N$ for a non-negative real number δ and a basis $\{\mathbf{b}_1, \overline{\mathbf{b}}_1, \cdots, \mathbf{b}_N, \overline{\mathbf{b}}_N\}$ of $E_{\mathbb{C}}$.

Lemma 3.4. *Let k be a natural number $\leq N$, Let $F_1 \in \wedge^{2k} E_{\mathbb{C}}$ be simple and weakly positive. Let $e_1, \cdots, e_k$ be any linearly independent $(1,0)$ vectors in the dual space $E^*_{\mathbb{C}}$. Let*

$$G_1 = (\sqrt{-1})^k e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_k \wedge \bar{e}_k.$$

Then the pairing $\langle F_1, G_1 \rangle$ between vectors and covectors is a real non-negative number.

Proof. We denote the dual automorphism over $\mathbb{C}$ via the standard basis by $*$. If $\langle F_1, G_1 \rangle = 0$, the lemma is proved. So, we may assume $\langle F_1, G_1 \rangle \neq 0$. Then let $e_{k+1}, \bar{e}_{k+1}, \cdots, e_N, \bar{e}_N$ extend the vectors $e_1, \bar{e}_1, \cdots, e_k, \bar{e}_k$ to a basis of $E^*_{\mathbb{C}}$. Recall F_1 being simple and weakly positive means that there are $\tau_1, \cdots, \tau_{2k}$ in $E_{\mathbb{C}}$ such that

$$F_1 = \tau_1 \wedge \cdots \wedge \tau_{2k} \tag{3.5}$$

and

$$F_1 \wedge \left((\sqrt{-1})^{N-k} \overline{e^*}_{k+1} \wedge e^*_{k+1} \wedge \cdots \wedge \overline{e^*}_N \wedge e^*_N \right) \tag{3.6}$$

is a positive volume form of $\wedge^{2N} E_{\mathbb{C}}$. Since $\langle F_1, G_1 \rangle \neq 0$, (3.6) is a strictly positive volume form. Let's work with the ordered orthogonal basis

$$\sqrt{-1}\bar{e}^*_1, e^*_1, \cdots, \sqrt{-1}\bar{e}^*_k, e^*_k, \sqrt{-1}\bar{e}^*_{k+1}, e^*_{k+1}, \cdots, \sqrt{-1}\bar{e}^*_N, e^*_N \tag{3.7}$$

of $E_{\mathbb{C}}$ whose positive volume form is

$$\sqrt{-1}\bar{e}^*_1 \wedge e^*_1 \wedge \cdots \wedge \sqrt{-1}\bar{e}^*_N \wedge e^*_N.$$

We further denote vectors as follows

$$\begin{aligned}
&F_1 = \tau_1 \wedge \cdots \wedge \tau_{2k} \in \wedge^{2k} E_{\mathbb{R}} \\
&F_2 = (\sqrt{-1})^{N-k} \bar{e}^*_{k+1} \wedge e^*_{k+1} \wedge \cdots \wedge \bar{e}^*_N \wedge e^*_N \in \wedge^{2(N-k)} E_{\mathbb{R}} \\
&G_1 = (\sqrt{-1})^k e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_k \wedge \bar{e}_k \in \wedge^{2k} E^*_{\mathbb{R}} \\
&G_2 = (\sqrt{-1})^{N-k} e_{k+1} \wedge \bar{e}_{k+1} \wedge \cdots \wedge e_N \wedge \bar{e}_N \in \wedge^{2(N-k)} E^*_{\mathbb{R}}
\end{aligned}$$

Then we write down linear combinations of vectors $\tau_1, \cdots, \tau_{2k}$ in the basis (3.7), and find that $2N$-wedge product

$$\tau_1 \wedge \cdots \wedge \tau_{2k} \wedge \sqrt{-1}\bar{e}^*_{k+1} \wedge e^*_{k+1} \wedge \cdots \wedge \sqrt{-1}\bar{e}^*_N \wedge e^*_N$$

is equal to

$$\gamma\sqrt{-1}\bar{e}_1^* \wedge e_1^* \wedge \cdots \wedge \sqrt{-1}\bar{e}_N^* \wedge e_N^*$$

where γ is the determinant

$$\begin{vmatrix} \langle F_1, G_1\rangle & \langle F_1, G_2\rangle \\ \langle F_2, G_1\rangle & \langle F_2, G_2\rangle \end{vmatrix} = \begin{vmatrix} \langle F_1, G_1\rangle & \# \\ 0 & \langle F_2, G_2\rangle \end{vmatrix} = \langle F_1, G_1\rangle\langle F_2, G_2\rangle \tag{3.8}$$

where the pairings are between the vectors and covectors. Since F_1 is weakly positive, γ is positive. Therefore $\langle F_1, G_1\rangle$ is positive. We complete the proof. □

4 Effective decomposition

Our proof of the main theorem is based on a geometric description of Hodge classes. Demailly in [3] proved that on a compact Kähler manifold any Hodge class with real coefficients is represented by the difference of two positive currents. His currents are known to be smooth forms. We would like to show that if the manifold is projective, these positive currents can also be singular cycles.

We'll use the angle bracket $\langle \bullet \rangle$ to denote the cohomological class represented by $\bullet$.

Lemma 4.1. *(Effective decomposition) Let X be a complex projective manifold. Then a Hodge class $u \in H^{2p}(X;\mathbb{Q})$ has a positive representation in the following way.*

(1)

$$u = \langle \sigma_+\rangle - \langle \sigma_-\rangle \tag{4.1}$$

where σ_+, σ_- are singular cycles,

(2) both currents T_{σ_+} and T_{σ_-} are of bidegree (p, p) and positive.

Remark: Deligne once called conjectured formula (4.1) the effective decomposition. At the mean time, he also pointed out there is another positive current decomposition over $\mathbb{R}$ that holds on a compact Kähler manifold but in smooth forms (i.e. Demailly's decomposition). Indeed, Lawson in [10] proved that there is an Abelian variety with a cycle class that is represented by a positive current but not by an effective algebraic cycle.

Proof. Cohomology of X is generated by pseudomanifolds. So, we let u be represented by a rational, linear combination of connected, oriented pseudomanifolds whose currents of integration is of bidegree (p, p). Let σ be one of these psuedomanifolds with the positive coefficient λ (with an appropriate orientation of σ).

Step 1 (complex normal bundle): Let σ^o be the set of smooth points of σ, and $\mathcal{S} := \sigma - \sigma^o$ whose real codimension is ≥ 2. We consider two manifolds: X and $X^o := X - \mathcal{S}$. We denote the usual current of integration over the pseudomanifold σ by $\mathcal{T}$. Next we work in its open subset X°. Notice that σ^o is an open and closed submanifold of X°. Then there is a tubular neighborhood of σ^o in X°, denoted by

$$P : U \to \sigma^o$$

that has an oriented $\mathbb{R}^{2p}$-bundle structure. In the manifold U, the integral over a compact set of the chain $\lambda\sigma^o$ defines a closed current, denoted by $\mathcal{T}^o$ in U. In current's homology, $\mathcal{T}^o$ as a closed current is homologous to a closed form ω^o in U, which is compactly supported in the vertical direction of the bundle (but not in the horizontal direction). By the Thom isomorphism for the $\mathbb{R}^{2p}$-bundle, the fibre integral $P_*(\cdot)$ yields an isomorphism

$$H^\bullet_{cv}(U;\mathbb{Q}) \simeq H^{\bullet-2p}(\sigma^o;\mathbb{Q}) \tag{4.2}$$

where "*cv*" denotes the cohomology with the vertically compact support. Applying to the cohomology $\langle\omega^o\rangle$, we obtain that $P_*(\omega^o)$ is a constant $\lambda \neq 0$, the rational coefficient associated to σ. We observe that the restricted current $\mathcal{T}$ to U is exactly $\mathcal{T}^o$. Hence the representative form ω^o (which is in U) is the restriction of a differential form representing the Hodge class u in X. Since u is Hodge, ω^o can be chosen to be a real (p,p) form. Let $a \in \sigma^o$ be a point. Let U_a be a sufficiently small neighborhood of a. Let $\sigma_a = U_a \cap \sigma^o$. The real (p,p) form ω^o when restricted to U_a can be point-wisely expressed as a real linear combination of decomposable forms in $\wedge^{(p,p)}T^*_{\mathbb{C}}X$ (Proposition 1.9, [5]). Also notice that each decomposable form has a compact support along the fibre direction. Then to have a non-zero fibre integral $P_*(\omega^o)$, the complexified tangent space $T_{\mathbb{C},u}(P^{-1}(a))$ must be a complex p-codimensional subspace where $u \in P^{-1}(a)$ in the neighborhood U_a. Thus there is a choice of a tubular neighborhood U such that $P : U \to \sigma^o$ is a complex vector bundle.

Step 2 (Global extension): The normal bundle above has a canonical orientation on the complex fibres, i.e. a continuous, non-vanishing global section of the bundle. So, we assume σ^o has the complex orientation(see Definition 3.2 above). Let $X \subset \mathbb{P}^N$ be the projective embedding over $\mathbb{C}$. Let $\mathbb{C}^N$ be a generic affine open set of $\mathbb{P}^N$. We'll denote the intersection $\mathbb{C}^N \cap \sigma^o$ by $\widehat{\sigma^o}$. Let $z_1, \cdots, z_N$ be the affine coordinates of $\mathbb{C}^N$. Then $\mathbb{R}^{2N} = \mathbb{C}^N$ is equipped with the Eulidean metric. Next we'll apply geometric measure theory in the Euclidean space $\mathbb{C}^N$. Let $a \in \widehat{\sigma^o}$. Let $U_a \subset \mathbb{C}^N$ be a neighborhood of a and $\sigma_a = \widehat{\sigma^o} \cap U_a$. Since σ^o is a C^∞ manifold, T_{σ_a} is a locally rectifiable current in $\mathbb{C}^N$. Notice that the evaluation $T_{\lambda\sigma_a}[\psi]$ is the same as $\lambda T_{\lambda_a}[\psi]$ if ψ is a test form compactly supported in the neighborhood. Let

$$\phi = \sqrt{-1}dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \sqrt{-1}dz_k \wedge d\bar{z}_k \tag{4.3}$$

be the generic decomposable form (recall $k = n - p$). Let f be a positive C^∞ function compactly supported in U_a. So, we have the evaluation formula for the locally rectifiable current $T_{\lambda\sigma_a}$,

$$T_{\lambda\sigma_a}[f\phi] = \lambda \int_{\sigma_a} \langle f\phi, \xi\rangle\ \theta\ d\mathcal{H}^{2k} \tag{4.4}$$

(Definition 3.1, Chapter 6, [13] or p. 558, [8]) where ξ is the orientation, θ is the multiplicity and $\mathcal{H}^{2k}$ is the $2k$-Hausdorff measure of $\mathbb{C}^N$. The orientation ξ is a $\mathcal{H}^{2k}$-measurable function

$$\sigma_a \dashrightarrow \wedge^{2k}\mathbb{C}^N$$

where the dash arrow denotes a well-defined map on the $\mathcal{H}^{2k}$-almost all points $x \in \hat{\sigma^o}$. Let ξ send $\mathcal{H}^{2k}$-almost all points $x \in \sigma_a$ to

$$\tau_1 \wedge \cdots \wedge \tau_{2k}$$

where τ_j are ordered, orthornormal vectors spanning the approximated, tangent space which coincides with the usual oriented tangent space $T_x(\sigma_a)$ of the manifold.

Due to the parallel transform of a Euclidean space, we can identify tangent spaces $T_x(\mathbb{C}^N)$ at all points x with the complex vector space $E = \mathbb{C}^N$ as in the lemma 3.4. Assume that x is one of $\mathcal{H}^{2k}$-almost all points in (4.4). As usual, we regard $\tau_j \in E_{\mathbb{C}}$. We should notice that

$$\widehat{\sigma^o} \subset X^o \cap \mathbb{C}^N \subset \mathbb{C}^N.$$

The step 1 implies that the normal bundle $N_{\widehat{X^o}}\mathbb{C}^N$ is also a complex bundle. Due to this structure of the complex bundle, the orientation

$$\tau_1 \wedge \cdots \wedge \tau_{2k}$$

for the integral (4.4) is weakly positive in $\wedge^{2k}E_{\mathbb{C}}$. Notice that $f\phi$ is decomposable. Hence we can directly apply Lemma 3.4 which asserts the pairing $\langle f\phi, \xi \rangle$ at $\mathcal{H}^{2k}$-almost all points x is non-negative. This implies that the measure-theoretical integral (4.4) is non-negative. So, the evaluation of T_{σ_a} at all decomposable forms is non-negative. Therefore, the current T_{σ_a} is positive. This positivity holds around $\mathcal{H}^{2k}$-almost all points a on $\widehat{\sigma}^o$. * This implies that $T_{\widehat{\sigma}^o}$ is positive. By the convergence of the current of integration at the boundary $\sigma - \widehat{\sigma^o}$ of σ, the current $T_{\lambda\sigma}$ is also positive.

If σ has the inverse complex orientation, we reverse the orientation of the same pseudomanifold to have the complex orientation. Then above positivity works the same way. Thus overall we express u in the following way. For each pseudomanifold σ in the cycle u, if its orientation from the singular cycle structure agrees with the complex orientation, we keep the original coefficient λ and the original orientation. If its singular cycle orientation is the inverse of the complex orientation, we change its original coefficient λ to $-\lambda$, and original orientation to the complex orientation. So after the re-orienting, u can be represented by singular cycles,

$$\sum_{i=1}^{l} \lambda_i \sigma_i - \sum_{i=l+1}^{m} \lambda_i \sigma_i \tag{4.5}$$

*If $\mathbb{C}^N$ did not exist, the positivity would've only held locally.

where all rational λ_i are positive and all pseudomanifolds σ_i, with the complex orientation, represent the positive currents of bidegree (p,p). So, we let

$$T_+ = \sum_{i=1}^{l} \lambda_i T_{\sigma_i}$$

$$T_- = \sum_{i=l+1}^{m} \lambda_i T_{\sigma_i}.$$

Lemma 4.1 is proved.

□

5 Proof

Proof of Main theorem 1.1: Recall X is a complex projective manifold and $dim_{\mathbb{C}}(X) = n$. Let $deg(u) = 2p$ where $0 \leq p \leq n$. By Lemma 4.1 the class u has a current representative:

$$u = \langle T_+ \rangle - \langle T_- \rangle. \tag{5.1}$$

where $T_+ = \sum_{i=1}^{l} \lambda_i T_{\sigma_i}, T_- = \sum_{i=l+1}^{m} \lambda_i T_{\sigma_i}$. Let's consider T_+ which is the current of integration over the singular cycle

$$\sum_{i=1}^{l} \lambda_i \sigma_i.$$

By Lemma 2.2, $supp(T_+)$ is contained in an algebraic set A_+ of complex codimension p. Similarly, $supp(T_-)$ is contained in an algebraic set A_- of complex codimension p. Overall, a representative current of the class u is supported on the algebraic set $A = A_+ \cup A_-$ of complex codimension p. It is clear that a current-support implies a class-support. Then the class-support in (1.2) should also hold. The proof is completed.

□

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