METHODS OF NONLINEAR BIFURCATION GEOMETRY IN THE STUDY OF NONAUTONOMOUS SCALAR EQUATIONS

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ABSTRACT

The study of nonautonomous scalar equations comprises a subset of solutions defining the regions of stability and instability inherent to a system. Under specific shifts in the variables of such equations, those referring to a scalar parameter, the quantity of stability points may vary. The exact value at which the quantity of stability points changes, refers to a bifurcation in the system. When a specific function, or set of functions, cannot be solved exactly through algebraic methods, an equivalence to geometric structures may provide intuitive connections to a more abstract topology that solves for those values exactly. Examples considered include $\dot{x} = x - x^2e^{-1}(1+t^2)^{-1}$, and the *Spruce-Budworm and Forest Model.*

I Introduction

In the study of differentiable systems, exists a continuously evolving field of research. One that seeks to better understand and simplify the methods of analysis that coincide with known observations in nature. Developments in this area of study are unique in that any definition given must be true by self-consistent logic, and that the structure of logic be accurate when tested with known predictable systems. A successful contribution to the study of differentiable systems is one that either, (1) solves previously unsolved problems; or (2) simplifies the steps required to solve known problems. This paper will look at a branch of dynamics dealing with such differentiable systems.

The pivot point for investigation begins with “Bifurcations in nonautonomous scalar equations”.[11] Discussion provided looks at functions of the type, $\dot{x} = f(x, t, \lambda)$. Analysis is predicated exclusively on the determination that for any function expressible by variables x, t, λ, that these functions also be equivalent to the variable, x, differentiated once with respect to variable, t. Understanding the fundamental principles governing this type of equation is useful for understanding the geometry of change; given, that one is concerned with how some variable, x, changes with respect to (or in conjunction with) the variable, t. The third variable, λ, applies a conditional unknown to any function of this type; being, an implicit requirement for determining units.

Fundamental theory related to nonautonomous scalar equations of this type, and the set of functions with variables x, t, λ, all having equivalence to $\dot{x}$, also includes the set of all autonomous functions, $\dot{x} = f(x, \lambda)$. For both autonomous and nonautonomous scalar equations, problems are often used to examine the stability or instability of a system under specific initial conditions. Correlation of variables and their units maintains information related to the system when examining changes in stability, changes in the number of stability points, rates of convergence to a stability point, &c. Maintaining the logical basis for the definitions and theorems involved is necessary when building a geometry that can simultaneously answer several questions pertaining to a single dynamics problem.

Geometry of a nonautonomous scalar equation relies on the continuity of the equation being considered. Continuity of the equation and correlation between units involved can then produce a geometric structure that takes into account how the function will behave when placed somewhere within the geometry. Determining how the system behaves in terms of unit changes, or a constant variable, is useful when analyzing regions of convergence that may otherwise be considered chaotic. The extrapolation of changes in the equation, in terms of specific units, also provides regions of significant shifts in the geometry or stabilities of the equation. The focus of this paper is to examine the foundational theory of nonautonomous scalar equations, and then to investigate the branches of solutions that stem from this approach.

II Definitions

An *autonomous* or *nonautonomous* equation is defined as the first derivative of some function or variable, x, in terms of some independent variable, t. The choice of variables is arbitrary, and if $x(t)$ is invertible, then t can also be considered as a function in terms of x.

$$\dot{x} = \frac{dx}{dt}$$

An *autonomous* equation is any function of this type that includes the variable being differentiated, but not the independent variable being differentiated with respect to: $\dot{x} = f(x)$. A *nonautonomous* equation is any function that includes the variable being differentiated, in this case, t, and the independent variable being differentiated, in this case, x: $\dot{x} = f(x, t)$. Any *nonautonomous scalar* equation includes a scalar, λ, that is neither the variable being differentiated nor the variable being differentiated with respect to. The general form for the *nonautonomous scalar* equation is written as any function defined in terms of x, t, λ.

$$\dot{x} = \frac{dx}{dt} = f(x, t, \lambda) \tag{1}$$

The set of all *autonomous scalar* equations is a subset of all *nonautonomous scalar* equations. This occurs because if $\dot{x}$ requires that x be a function of t, then substitution of one or more x in $f(x, \lambda)$, with the function defining x, in terms of t, allows for solutions of the type $f(t, \lambda)$ and $f(x, t, \lambda)$. The existence of $\dot{x}$ also requires that x be continuous with respect to t; since, the variable, $x = x(t)$, being *differentiable* with respect to t, implies that *integration* of $\dot{x}$ with respect to t has the solution, $\dot{x}dt = x(t) + c$, with c being some constant of integration.

The method for defining a *dynamical system* of this type is simplified to three unique elements: $\{\mathcal{T}, \mathcal{X}, \varphi^t\}$. This is the reduction of a *nonautonomous scalar* equation to a set of three *linearly independent* variables. All three of these elements, $\mathcal{T}, \mathcal{X}, \varphi^t$, are themselves sets that may or may not contain *cardinalities* greater than the *cardinality* of the reduction on the *dynamical system*; specifically, $\mathcal{T}$ and φ^t are sets comprising a larger *cardinality*, but $\mathcal{X}$ may contain more, less, or the same number of elements as the reduction on the *dynamical system*.[1]

$$|\{\mathcal{T}, \mathcal{X}, \varphi^t\}| = 3$$

For a *dynamical system*, the element, $\mathcal{T}$, is the *time set*: $\mathcal{T} = \{t\}$.[10] The variable, t, being a unit of time, requires that the total number of elements in the *time set*, $\mathcal{T}$, be equivalent or *bijective* to the set of all Real numbers, $\mathbb{R}$. The element, $\mathcal{X}$, is the *state space* of a *dynamical system*. This is defined to be some $n-$dimensional Real number space, $\mathbb{R}^n$, with $n \in \mathbb{N}$. The *linear dependence* on the Real number space requires that the *state space* be equivalent to the set, $\mathcal{X} = \{x : \ x = (x_1, x_2, ..., x_n)\} \widehat{=} \mathbb{R}^n$. When setting all variables, $x_1, x_2, ..., x_n \in \mathbb{R}$, the *state space* reduces to a single element, $|x_0| = 1$, allowing for any *autonomous* equation to reduce to a *singleton*.

$$\begin{gathered} \forall f : \mathbb{R}^n \to \mathbb{R}^n \quad \exists \mathcal{X} = \mathbb{R}^n : \\ \dot{x}_i = f_i(x_1, x_2, ..., x_n) \in \mathbb{R}^n \ \Rightarrow \ \dot{x} = f(x) \in \mathbb{R} \end{gathered}$$

Since the *time set* and *state space*, $\mathcal{T}$ and $\mathcal{X}$, are *linearly independent*, in order to define a *nonautonomous scalar* equation based on these two elements, requires a third element that connects the elements. For any unique $x \in \mathcal{X}$, the variable is represented by a *unit*, $\hat{x} \in \{0, 1\}$, and a *scalar-magnitude*, $||x|| \in \mathbb{R}$. The equivalence, $x = ||x|| \cdot \hat{x} = ||x||\hat{x}$, defines a *vector*, that correlates a dimensionless length in $\mathbb{R}$, with a determination that the *scalar-magnitude* is defined in terms of that *linearly independent unit*, $\hat{x} = 1$, or is not defined in terms of that *linearly independent unit*, $\hat{x} = 0$. A *nonautonomous* equation, $\dot{x} = f(x, t)$, that is in terms of $\hat{x}$ and $\hat{t}$, with $\hat{t}$ being the unit defining a *vector* in the *time set*,

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[1] See Appendix A for the proof on why the cardinality of a set containing an infinite set is not infinite.

provides a solution that is neither uniquely defined in terms of $\hat{x}$ or $\hat{t}$. The correlation between these two units then requires an *evolution operator*, φ^t. The exponential form in terms of the *time set*, is due to the *integration* of a given $\dot{x} = ||\dot{x}||\hat{x}\hat{t}^{-1}$, having the divisor unit, $\hat{t}$, be some $ln(\hat{t})$ proportionality with respect to some $\hat{x}$ element in the *state space*.

$$\varphi^t : \mathcal{X} \to \mathcal{X}$$

When solving for a *nonautonomous scalar* equation, $\dot{x} = f(x, t, \lambda)$, determining that a unique solution exists for a given, $x \in \mathbb{R}^n$, the *singleton* of this unique element, x_0, is defined to be the initial value at $t = 0$. Then, setting the *evolution operator* to have this position at $t = 0$, returns the initial unique element, x_0. Looking at the change in the system for $t > 0$ or $t < 0$, defines the *trajectory* of the *dynamical system* over time. This general setup for the *trajectory*, is defined by the unique element, x_0, at $t = 0$, with $t \neq 0$ being another element of the *state space*, x_t, at $t \in \mathbb{R}$.

$$x_t = \varphi^t x_0 \iff x_0 = \varphi^0 x_0$$

Obtaining an *evolution operator*, φ^t, for a *dynamical system*, is determined by the information inherent to the system. When given only the *nonautonomous scalar* equation, $\dot{x} = f(x, t, \lambda)$, the *trajectory* is dependent upon all three elements of the system, x, t, λ. If nothing is known about the system, then the *stability* for some x_0 is found by obtaining the values, $\dot{x} = 0$. These are all known values at which the *derivative* of the *state space*, with respect to the *time set*, will have no change: $\dot{x} = 0$. Any $x \in \mathcal{X}$ will be a *fixed point*; where, the value $\dot{x} = 0$, is equivalent to the maxima and minima of $x = x(t)$, with some *fixed point* being defined as, $x^* \in \mathcal{X}$.

The stability of a *fixed point*, x^*, is *unstable* if the element is a relative or absolute maxima of the function, $x = x(t)$; otherwise, a *fixed point*, x^*, is *stable* if the element is a relative or absolute minima. An *evolution operator*, φ^t, given an initial state, x_0, will always converge to some *stable fixed point* as $t \to +\infty$ and converge to some *unstable fixed point* as $t \to -\infty$. The principle being, that any element of a *state space* will have a continuously decreasing change in the *state space* as the elements of the *time set* increases, and that any *dynamical system* is equivalent to an energy system; where, an energy system is assumed to approach a minimum energy value as time increases instead of a maximum energy value. This is also referred to as a *dissipative dynamical system*.

Given some function, $\dot{x} = f(x, t, \lambda)$, any solution can be represented by the set of all coordinate values in $\mathbb{R}^4$; such that, $\{(\dot{x}, x, t, \lambda)\} \subseteq \mathbb{R}^4$. If the function is reduced to an *autonomous scalar equation*, $\dot{x} = f(x, \lambda)$, then the set of all coordinates can be represented in $\mathbb{R}^3$; such that, $\{(\dot{x}, x, \lambda) : \ x = x(t)\} \subseteq \mathbb{R}^3$. A *vector space* provides the *trajectory path* that an initial state, x_0 will take given some unique, λ. The *trajectory path* is a continuous subset of the function, $\dot{x} = f(x, \lambda)$, with a given λ, that starts at the position x_0, and ends at the first *stable fixed point*: $\{\dot{x} : \ \lambda \in \mathbb{R}\} \times [x_0, x^*) \subseteq \mathbb{R}^2$. Then a *trajectory path* can be defined by the interval of elements in the *state space* that map to some *autonomous* equation, and converge to some *stable fixed point*. When a *trajectory path* begins at some initial state that is not a *fixed point*, then that *trajectory path* does not contain any *fixed points*.

$$\begin{gathered} \forall x_i^* \in \{x : \ x \mapsto \dot{x}\} \subseteq \mathcal{X} \quad \exists x_0, x_j^* \in \mathcal{X}, \ x_0 \neq x_j^* : \\ \lim_{t \to \pm\infty} \varphi^t x_0 = x_j^* \in \{x_i^*\} \ \Rightarrow \ \{x_i^*\} \cap [x_0, x_j^*) = \emptyset \end{gathered} \tag{106}$$

For *unstable fixed points*, the same statement applies, except that $t \to -\infty$ instead of $t \to +\infty$. This statement also does not include the set of initial states, x_0, that do not converge to a defined *fixed point*. For the case when there exists a unique *unstable fixed point* for some initial state, x_0, but no *stable fixed point*, the limit as $t \to +\infty$ will diverge to positive or negative infinity, instead of converging to a unique $x^* \in \mathcal{X}$. For the unique case that an initial state is equal to some *fixed point*, $x_0 = x^*$, the *trajectory path* is equivalent to the *fixed point* for all elements of the *time set*. The set of all time-varying solutions having equivalence to the initial state is the *family* of *invariant* elements in the space that do not vary from the *fixed point*.

$$\varphi^t x_0 = x^* = x_t(\forall t \in \mathcal{T}) \iff x_0 = x^*$$

The setup for an *evolution operator* at an initial state, $\varphi^t x_0 = x_t$, can be considered in terms of a general construction; where, the set of initial state elements converge to *global attractors*.[22] Methods for analyzing a *global attractor* allows for the system to be treated as a set of subsets on the *state space*, with any initial state from one subset remaining exclusively within that subset and converging to a *fixed point*, being the *global attractor* of that subset. Specifically, a *global attractor* will be a set of *stable fixed points*. The method for employing a definition on *global attractors* is useful when considering broader solutions given an *evolution operator* that does not strictly rely on *Cauchy-Convergence* of the original equation, or may otherwise disparage approximate *trajectory paths* for a solution. The broader claim on the definition of *global attractor* takes into account *topological* statements and properties of a *dynamical system*; otherwise, being equivalent to the statements preceding.[22] The use of new variables defining a *global attractor* is to be explained by the following definition.

Definition: For a set $\mathcal{A} \subset \mathcal{E}$, $\mathcal{A}$ is called a *global attractor* of the *semigroup* $\{\mathcal{F}(t)\}$ if it has the following properties:

1.) $\mathcal{A}$ is a *compact set* in the *topology* of the space $\mathcal{E}$;

2.) $\mathcal{A}$ *attracts* or *translates* $\mathcal{F}(t)\mathcal{B}$ of any *bounded* subset $\mathcal{B} \subset \mathcal{E}$ in the *topology* of $\mathcal{E}$ as $t \to +\infty$;

3.) The set $\mathcal{A}$ is strictly *invariant* under the *semigroup* $\{\mathcal{F}(t)\}$; such that,

$$\mathcal{F}(t)\mathcal{A} = \mathcal{A}(\forall t \geq 0).$$

Setting a *dynamical system* to have these properties takes into consideration equivalent spaces that may arise, or those which are equivalent to some *nonautonomous scalar* equation, $f(x, t, \lambda)$, but are uniquely quantified. For example, after some analytical process, a *fixed point*, x^*, or the set of all *fixed points*, $\{x^*\}$, may map to the subset, $\mathcal{A}$. The subset, $\mathcal{B}$, being regions or sets of initial states, x_0, that converge towards the set, $\mathcal{A}$. The *semigroup*, $\mathcal{F}(t)$, operates on the *state space* in terms of the *compact set*, $\mathcal{E}$, and *evolution operator*, φ^t.

Solutions that look to the *trajectory path* of a *dynamical system* given by the set, $\{\mathcal{X}, \mathcal{T}, \varphi^t\}$, relies on information which is obtainable from the system. For all cases, it may not be plausible to define the exact *trajectory*. For example, suppose an accurate prediction method requires that the *evolution operator* comprise more than just the initial state at $t = 0$, and the final state, x^*. Then a *dynamical system* with a set of *evolution operators* determining the *trajectory path* between the initial and final state, is referred to as a *family* of *solution operators*: $\{S(t, s) : x(s) = x_0\}_{t \geq s}$. Given two time steps, τ, s, with $\tau > s$, for an initial time, $s \in \mathcal{T}$, the *solution operator* with these two time steps is the composition of their respective *solution operators*.

$$S(t, \tau)S(\tau, s)x_s = S(t, s)x_s \iff \varphi^t\varphi^s x_0 = \varphi^{t+s}x_0 \iff \mathcal{F}(t)\mathcal{F}(s)\mathcal{B} = \mathcal{F}(t+s)\mathcal{B}$$

If a continuous map of elements in the *state space*, $x : \mathbb{R} \to \mathbb{R}^n$, for all t, s, provides a solution equivalent to the composition of *solution operators*, the *trajectory* is referred to as a *complete trajectory*: $S(t, s)x(s) = x(t)(\forall t, s \in \mathbb{R})$. By determining the regions of convergence for a *dynamical system*, or for *ϵ-neighborhoods* as subsets of the *state space* between *fixed points* in a *nonautonomous scalar* equation, a stronger picture is obtained for *trajectory paths* and their geometry. For example, a system that has some *invarient* element, x^*, can also comprise a subset of that *invariant* that is not unique for all $t \in \mathcal{T}$. If a *fixed point*, $x^* \in \mathcal{X}$, contains a subset of *vectors*, $\{\langle x_1, x_2\rangle_t\} \subseteq x^*$, with some $\tau > t$ that returns the function back to its initial state, $\varphi^{t+\tau}\langle x_1, x_2\rangle_t = \varphi^t\langle x_1, x_2\rangle_t$, then x^* contains a *periodic orbit*. The *trajectory path* for the *periodic orbit* of a *fixed point* is defined by the function, $\mathcal{O}(x_1, x_2) \in \mathbb{R}^2$.

Any *invariant* of a *nonautonomous scalar* equation may be treated as a *bounded* subset of the *dynamical system*; such that, an initial state, x_0, that is not an element of the *bounded* subset, is not an element of the *family* of *invariants*. Further, when looking at a function, $\dot{x} = f(x, t, \lambda)$, the *scalar* element of the function, λ, when taken to be the set of all Real numbers is similar to the *time set*, except that the *ordinary differential equation*, $\dot{x}$, only requires there to exist *linear independence* between the *time set* and the *state space*. Therefore, the *scalar* acts as a *parameter* that retains the units given for the *nonautonomous scalar* equation and accurately solves for the *trajectory path*. The interwoven nature that a variable *parameter* has on a function, $f(x, t, \lambda)$, develops a more intricate geometry to a *dynamical system*. Specific interest is given for the continuous variations that λ has on the set of *invariants*, x^*, and the *bounded* regions that converge to a unique x^*. In a *nonautonomous scalar* equation, if some $\lambda_0 \in \{\lambda\}$ changes the *cardinality* of *invariants*, and that for an $\epsilon-$neighborhood about the *parameter*, λ_0, there exists a $\lambda_1 \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $\lambda_1 \neq \lambda_0$, that contains a unique *cardinality* of *invariants* from $f(x, t, \lambda_0)$, then the element λ_0 defines a *bifurcation point* for the *dynamical system*.

III Bifurcation Geometry

For a *nonautonomous scalar* equation, $\dot{x} = f(x, t, \lambda)$, there are three variables considered, x, t, λ. The set of *invariants*, $\{x^*\} \subseteq \mathcal{X}$, are all elements in the *state space* when $\dot{x} = 0$. If the *nonautonomous scalar* equation can be reduced to an *autonomous scalar* equation, $\dot{x} = f(x, t, \lambda) = f(x(t), \lambda) = f(x, \lambda)$, then the set of all coordinates $\{(x, \lambda) : \dot{x} = 0\} \in \mathbb{R}^2$, when plotted for the *stable* and *unstable fixed points*, defines a *codim 1 bifurcation diagram*. Analytical, numerical, or graphical representations for the set of *fixed points* for a variable, λ, provides insight to the *trajectory paths* and *bounded* regions that any initial state will have in the system. In a system that requires both the *time set* and *state space* to determine *trajectories* and *invariants*, if no direct solution for reducing the function to an *autonomous* form is obtainable, then unit analysis can supply a more intuitive understanding. Consider the following *nonautonomous* equation.

<u>Example 1</u>:

$$\dot{x} = x - \frac{e^{-t}}{1+t^2}x^2 \tag{2}$$

For the existence of this function as a *nonautonomous* equation to hold true, then the units of the system must be equivalent. That is, the two summed elements on the right-hand-side of the equation require a unit equivalence to the vector on the left-hand-side of the equation.

$$\dot{x} = ||\dot{x}||\frac{\hat{x}}{\hat{t}} \iff x = ||x||\frac{\hat{x}}{\hat{t}}, \text{ and } \frac{e^{-t}}{1+t^2}x^2 = \left|\left|\frac{e^{-t}}{1+t^2}x^2\right|\right|\frac{\hat{x}}{\hat{t}}$$

From this setup alone, the equation requires that x be a function in terms of itself and t: $x = x(x, t)$. This produces a solution where the *nonautonomous* and *autonomous* equations are invertible in terms of the *state space* and *time set*. Specifically, any function of this type that correlates the *state space* and *time set* as invertible, will be referred to as a *spacetime* equation.[2] Noting only that $x = ||x||\hat{x}\hat{t}^{-1}$ requires that x be equivalent to some *infinite*, *convergent*, *self-iterating* function that converges to itself when differentiating with respect to t. That is, x is defined as a function in terms of t, and t is defined as a function in terms of x. A simple way to show this invertibility between two variables of a *self-iterating* equation is by noting the conditions necessary for a general solution to the Lambert Function.[14]

$$\begin{gathered} t(1+x)\frac{dx}{dt} = x \;\Rightarrow\; \frac{dx}{dt} = \frac{x}{t(1+x)} = \frac{x}{t(1+t(1+x)\frac{dx}{dt})} \\ \iff \dot{x}^2t^2(1+x) + \dot{x}t - x = 0 \iff \frac{dt}{dx} = \frac{2t(1+x)}{-1 \pm \sqrt{1+4x(1+x)}} \end{gathered} \tag{3}$$

The *reflexive relation* of the function with respect to itself, the *linearly independent* variable being differentiated with respect to, and an *evolution operator*, then requires that any *spacetime* function of this type be an explicit *nonautonomous scalar* equation. For the case when the vector, $x^2e^{-t}(1+t^2)^{-1} = 0$, equation (2) can be written where, $\phi = x(t + t^2 + tx)^{-1}$; such that, $\dot{x} = f(x, t, \dot{x}) = \phi\dot{x}^{-1}$. For all other solutions when $x^2e^{-t}(1-t^2)^{-1} \neq 0$, the method for solving will then require the introduction of variables, α, β, being a subset of the *spacetime* function. This requirement on the *self-iterating* function is one that will follow with respect to equation (3), rewritten by Euler as $t^\alpha - t^\beta = (\alpha - \beta)xt^{\alpha+\beta}$.[8]

$$\begin{gathered} t^\alpha - t^\beta = (\alpha-\beta)xt^{\alpha+\beta} \;\Rightarrow\; x = \frac{t^{-\alpha} - t^{-\beta}}{\alpha - \beta} \\ \Rightarrow\; \frac{dx}{dt} = \frac{\alpha t^{-\alpha-1}}{\alpha-\beta} - \frac{\beta t^{-\beta-1}}{\alpha-\beta} \end{gathered}$$

This produces a general solution in terms of variables, α, β, that are similarly found in equation (2). Which, for a *self-iterating* function of the form $\dot{x} = p(x) + q(x)$ to exist, also requires that the functions, $t^{-\alpha}, t^{-\beta}$, be functions in terms of α, β, x.

$$\begin{gathered} t^{-\alpha} = 1 - \alpha x - \frac{1}{2}\alpha 2\beta(\alpha+\beta)x^2 - \frac{1}{24}\alpha 3\beta(\alpha+2\beta)(2\alpha+\beta)x^3 - \frac{1}{120}\alpha 4\beta(\alpha+3\beta)2\alpha+2\beta)(3\alpha+\beta)x^4 - ... \\ t^{-\beta} = 1 - \beta x - \frac{1}{2}\beta 2\alpha(\beta+\alpha)x^2 - \frac{1}{24}\beta 3\alpha(\beta+2\alpha)(2\beta+\alpha)x^3 - \frac{1}{120}\beta 4\alpha(\beta+3\alpha)2\beta+2\alpha)(3\beta+\alpha)x^4 - ... \end{gathered}$$

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[2] Though, not discussed in the **Definitions** section; here, the term *spacetime* refers to the class of functions where the *state space* and *time set* are inseparable by unit reductions. This is used in analogous form to the inseparability of the speed of light, $c = ||c||\hat{x}\hat{t}^{-1}$, from any unique variable in a *Lorenz transformation*.[13]

The reduction of this system therefore necessitating the introduction of a new set of variables for the *evolution operator* to accurately define some *trajectory path* given an initial state. Integrating equation (2) as a *first-order nonlinear ordinary differential equation* verifies this claim by the function having new variables, $r, \arctan(t)$.

$$dx = \left(x - \frac{e^{-t}}{1+t^2}x^2\right)dt$$
$$\Rightarrow \; x(t) = \frac{e^t}{r + \arctan t}$$

Differentiating the system again, and setting $r = 0$, gives an exact equivalence to equation (2). Analysis for when $r \neq 0$, provides information about the *trajectory paths* within the *bounded* region of convergence defining the *invariant* of the system.

$$\frac{d}{dt}\left[\frac{e^t}{r + \arctan t}\right] = \dot{x} = x(1-r)\left(\frac{\arctan(t)}{\arctan(t) + r}\right) - \frac{e^{-t}}{1+t^2}x^2$$

Setting the right-hand-side of the equation to have, $\arctan(t) = \theta$ and $\phi = \theta(\theta + r)^{-1}$, gives a *reflexive relation* similar to equation (3), due to the introduction of variables, θ, r.

$$\dot{x} = x(1-r)\left(\frac{\theta}{\theta - r}\right) - \frac{e^{-t}}{1+t^2}x^2 = x(1-r)\phi(\theta, r) - \frac{e^{-t}}{1+t^2}x^2 \tag{4}$$

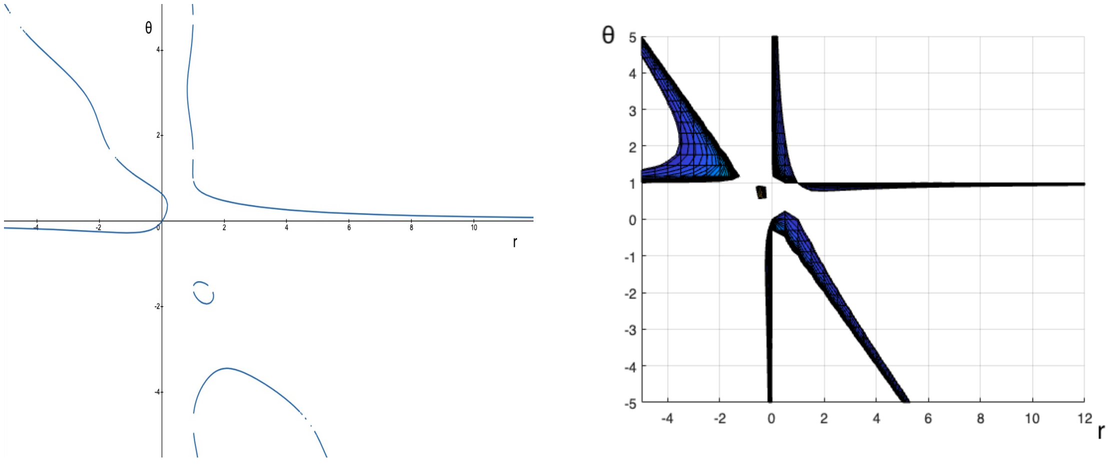
Finding a *bifurcation diagram* for this function given by the set of all *fixed points*, $x^* \in \mathcal{X}$, when $\dot{x} = 0$, being in terms of variables, t, r, θ. Since equation (4) will, for any unique, x^*, be in terms of some $(t, r, \theta) \in \mathbb{R}^3$, a first method to analyze the system is to determine known *fixed points* in the *bifurcation diagram*. The method employed here will be to set $\dot{x} = 0$ and balance the equation, $\dot{x} = p(x) - q(x)$, to $q(x) = p(x)$.

$$\frac{e^-t}{1+t^2}x^2 = x(1-r)\left(\frac{\theta}{\theta - r}\right) \tag{5}$$
$$\frac{e^-t}{1+t^2}x = (1-r)\left(\frac{\theta}{\theta - r}\right) \tag{6}$$

Plugging in the function, $x(t) = e^t(r + \theta)^{-1}$, to equation (6), reduces the total cardinality of variables in the equation by one. This defines a unique space of coordinates, $(x, t, r, \theta) \in \mathbb{R}^4$, that solves for the *bifurcation diagram* in terms of $(t, r, \theta) \in \mathbb{R}^3$; which, can be further reduced to $(t, r) \in \mathbb{R}^2$, due to θ being a function of t: $\theta(t) = \arctan(t)$.

$$\frac{1}{(1+t^2)(r+\theta)} = (1-r)\left(\frac{\theta}{\theta - r}\right)$$
$$\Rightarrow \; (1+t^2)(r+\theta)(1-r)\theta - \theta + r = 0 \tag{7}$$
$$\Rightarrow \; t^2 = \frac{\theta - r}{(r+\theta)(1-r)\theta} - 1$$

By the variable, $\theta(t)$ being equivalent to $\arctan(t)$, then for all $t \in \mathbb{R}$ the function is *bounded*: $\theta(t) \in [\frac{-\pi}{2}, \frac{\pi}{2}](\forall t \in \mathbb{R})$. Mapping these functions to a diagram with coordinates, $(t, r, \theta) \in \mathbb{R}^3$, and $(r, \theta) \in \mathbb{R}^2$, gives the geometry of the *fixed points*. This is evidence of a *global attractor* at $r = 1$, and that there exists an interior and exterior convergence and divergence space about this set of *fixed points*; which, are referred to as the *nullclines* of the *dynamical system*.

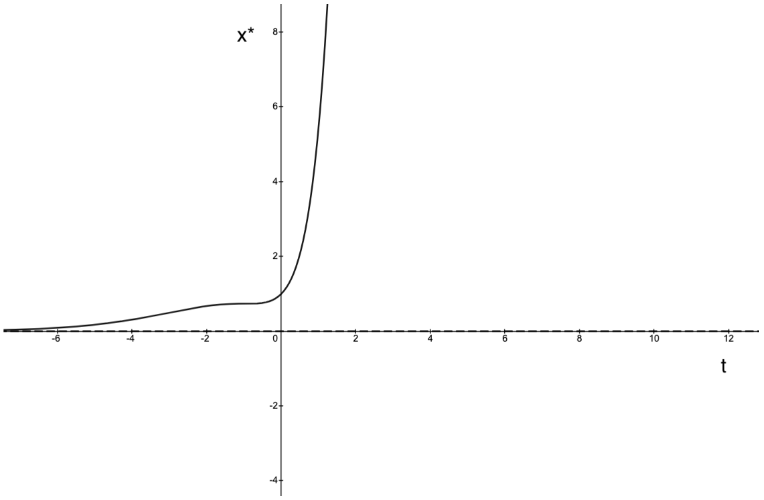


(a) Nullclines solution in $\mathbb{R}^2$ compiled using *Desmos* (b) Nullclines solution in $\mathbb{R}^3$ compiled using *MATLAB*

Figure 1: (left) $\tan(\theta)^2 = f(\theta, r)$; (right) $t^2 = f(\theta, r)$

From equation (2), the *dynamical system* is shown to have a *bifurcation diagram* that can be converted into a solution of variables, $\theta, r \in \mathbb{R}$, when beginning with variables of the *state space* and *time set*, $x, t \in \mathbb{R}$. The *nonautonomous scalar* equation, $\dot{x} = x(1 - (xe^{-t})(1+t^2)^{-1})$, having an *unstable fixed point* at $x = 0$, and a *stable fixed point* at $x = (1+t^2)e^t$, is observable on the *bifurcation diagram* when treating $t \in \mathbb{R}$, as a variable scalar, and $x^* \in \{x\}$, to be the *nullclines* of the system.

Figure 2: *Unstable Nullcline* (dotted) and *Stable Nullcline* (solid) for $\dot{x} = x\left(1 - \frac{xe^{-t}}{1+t^2}\right)$



The *bifurcation* in figure 2 is observable to occur at $t \to -\infty$. This provides a method for obtaining information about the *bifurcation point* of a limit in a *dynamical system*. By use of a *transformation* of variables, the limit for equation (2), as $t \to -\infty$, is visualized in figure 1. This solution for the limit of the function with dependence on $\arctan(t)$, is *bounded* to the interval, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and is a *closed periodic orbit*. Therefore, equation (2) defines a system that is *pullback Lyapunov stable* as $t \to -\infty$, and *pullback linearly unstable* when $x = 0$.[11] Analytical theory on the topics of *Lyapunov stability*, *stability* in general, and other *topological dynamics* are evidentially supported by examples of this type.

Obtaining information about a *nonautonomous scalar* equation in terms of unit equivalence is a method which can be used to verify *stability* analysis for problems that arise in *dynamical systems*. Whether a system is necessarily *self-iterating* or dependent upon a larger set of *linearly independent* variables, the geometry at limit points for the *time set* at some initial state is applicable to understanding the *invariant set* at these limits as well. Supposing that a function is an explicit *nonautonomous scalar* equation, $\dot{x} = f(x, t, \lambda)$, or some equivalence to a more general *topology* of *linearly independent* variables, $\{\mathcal{X}, \mathcal{T}, \varphi^t\} \simeq \{\mathcal{E}, \mathcal{T}, \mathcal{F}(t)\}$, a *gradient* on the system, $\nabla \dot{x}$, can be used to consider the geometry of the system in terms of some *evolution operator*.

$$\nabla \dot{x} = \nabla f(x, t, \lambda) = \frac{\partial \dot{x}}{\partial x}\hat{x} + \frac{\partial \dot{x}}{\partial t}\hat{t} + \frac{\partial \dot{x}}{\partial \lambda}\hat{\lambda} = \left\langle \frac{\partial \dot{x}}{\partial x}, \frac{\partial \dot{x}}{\partial t}, \frac{\partial \dot{x}}{\partial \lambda} \right\rangle$$

The multiplication of each vector in the partial differential, with the unit being differentiated with respect to, retains information about the original function after the *gradient* operation. The function, $\dot{x} = f(x, t, \lambda)$, is then considered in terms of each variable in the function as some coordinate $(x, t, \lambda) \in \mathbb{R}^3$, also being a vector in terms of the *gradient*. Preserving units, correlates the *trajectory path* in terms of an initial state, $(x, t, \lambda) \in \mathbb{R}^3$, and an interpretation on *conserved*, *dissipative*, or *accumulative* changes on the vector-values with respect to the *evolution operator*. A general solution for analyzing a *trajectory path* in terms of the *gradient* is to suppose that any incremental change from the initial state will preserve the total magnitude of the *gradient*. For a system that requires the total magnitude of the *gradient* to be preserved, given some initial, $\dot{x}_0 = f(x_0, t_0, \lambda_0) \in \{(x, t, \lambda)\}$, the preserved magnitude can be written as a constant scalar value: $||\nabla \dot{x}|| = \omega$.

$$\omega(x_0, t_0, \lambda_0) = \pm\sqrt{\left|\left|\frac{\partial \dot{x}}{\partial x}\right|\right|^2 + \left|\left|\frac{\partial \dot{x}}{\partial t}\right|\right|^2 + \left|\left|\frac{\partial \dot{x}}{\partial \lambda}\right|\right|^2}$$

For a system that explicitly correlates the *trajectory path* for a given initial state, x_0, in terms of changes in the *time set*, $\lim_{t\to\infty} \dot{x}$, then solutions to the system with a conserved variable, ω, is dependent upon changes in t. From example 1, this would be given by solving the *gradient* of the function, and then taking $t = 0$, to acquire the variable, ω.

$$\begin{aligned} \nabla \dot{x}(x, t) &= \nabla \left(x - \frac{x^2 e^{-t}}{1 + t^2} \right) = \left\langle \frac{2xe^{-t}}{t^2 - 1}, \frac{x^2 e^{-t}(t^2 + 2t - 1)}{(1 - t^2)^2} \right\rangle \\ \Rightarrow\ \omega(x, t) &= \pm \frac{xe^{-t}\sqrt{4(t^2 - 1)^2 + x^2(t^2 + 2t - 1)^2}}{(t^2 - 1)^2} \end{aligned}$$

Already noting that the function has an *invariant* at $x = 0$, the system also has no given solution for $t = 1$. Therefore, when looking at the *trajectory path* as time increases, $t \geq 0$, for some $x_0 \in \mathcal{X}$, taking the initial time to be $t = 0$, the *trajectory path* is to be considered for the interval $t \in [0, 1)$. If the *trajectory path* is also to be considered for $t < 0$ and $t > 1$, then intervals, $t \in [0, -\infty)$ and $t \in (1, \infty)$ are solved for, respectively. Beginning with an initial state, $x_0 \in \mathbb{R}$ and $x_0 \neq 0$, the conserved value at $t = 0$ can be found.

$$\omega(x_0, 0) = \pm x_0\sqrt{4 + x_0^2} \tag{8}$$

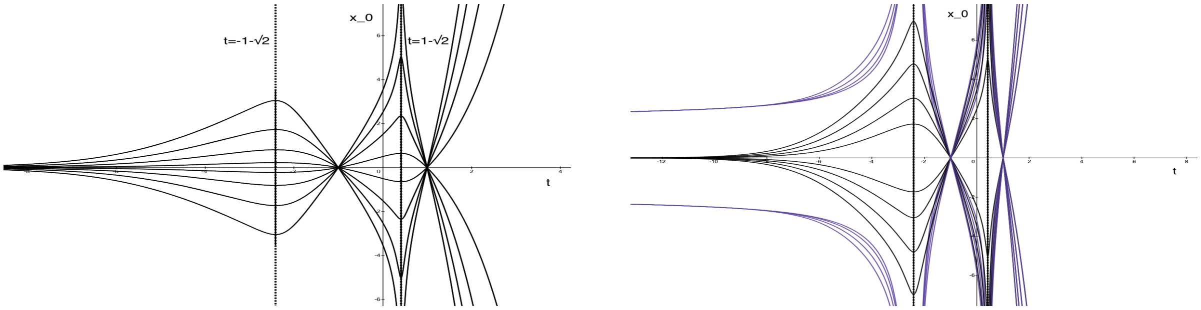
Taking the variable, $x_0 \in \mathbb{R}$, the trajectory path is then given by finding the set of $x \in \mathbb{R}$ for all $t \in [0, 1)$ when $\omega(x_0, 0) = \omega(x, t) = \omega$. Recalling that equation (2) has a *reflexive relation* between the *state space* and *time set*, reassures that solving for $x(t)$ from the *gradient* solution with constant, ω, can then be applied to the original function, $\dot{x} = f(x, t)$; where, $\dot{x} = f(x, t) = f(t(x)) = f(t)$, being a function dependent only on the *time set* when an initial state and constant, ω, are known. Otherwise, solving for $x(\omega(x_0, 0), t)$ produces the *evolution operator* defining the *trajectory path* of the system. This system then requiring that there exists at most a set of four possible *trajectory paths* given for any given scalar, $\omega(x_0, 0) \in \mathbb{R}$, when $\varphi(t)x_0 \in \mathbb{R}$. By the restriction that $\varphi(t)x_0 \in \mathbb{R}$, the set of *trajectories* will have two solutions. If the restriction is made that $\varphi(t)x_0 \in \mathbb{C}$, the other two solutions are given.

$$\varphi(t)x_0 = x_t = \pm\frac{t^2 - 1}{t^2 + 2t - 1}\sqrt{-2 \pm \sqrt{4 + x_0^2 e^{2t}(4 + x_0^2)(t^2 + 2t - 1)^2}} \tag{9}$$

$$\Rightarrow\ x_t \in \mathbb{R} \iff \varphi(t)x_0 = \pm\frac{t^2 - 1}{t^2 + 2t - 1}\sqrt{-2 + \sqrt{4 + x_0^2 e^{2t}(4 + x_0^2)(t^2 + 2t - 1)^2}}, \text{ and}$$

$$x_t \in \mathbb{C} \iff \varphi(t)x_0 = \pm\frac{(t^2-1)i}{t^2+2t-1}\sqrt{2+\sqrt{4+x_0^2e^{2t}(4+x_0^2)(t^2+2t-1)^2}}$$

For the case that the *trajectory path* is restricted to a scalar, ω, the *stable* and *unstable fixed points* from equation (9) require that equation (2) be solvable; such that, $\dot{x}=0 \Rightarrow t = \pm\sqrt{\pm 2}-1$. Then, given any element of the *time set*, $t \in \mathcal{T}$, when the *evolution operator* is solved for, the system will always have *fixed points* at $t+1 = \pm\sqrt{\pm 2}$. Which, having determined that for some fixed scalar, ω, the element, t, has two *stable fixed points* at $t+1=\pm\sqrt{2}$ for real-valued solutions, and two *unstable fixed point* at $t+1=\pm\sqrt{-2}$ for complex-valued solutions, the *nonautonomous scalar* equation, $f(x,t,\lambda)$ with scalar parameter, $x_0=\lambda$, can be referred to as being both *self-propagating* and *cyclic* by the *orthogonality* of these $t+1$ incremental time steps.



(a) Real-valued trajectories for a set of $x_0 \in \mathcal{X}$, with fixed points at $t \to -\infty, t = \pm\sqrt{2}-1$

(b) Complex- and Real-valued trajectories for a set of $x_0 \in \mathcal{X}$, with fixed points at $t \to -\infty, t \to \pm\sqrt{2}-1$

Figure 3

Example 2:

A function with many parameters can often be analyzed as a *nonautonomous scalar* equation. Consider the *Spruce-Budworm and Forest Model*.[16]

$$\dot{B} = r_B B\left(1-\frac{B}{K_B}\right) - \frac{\beta B^2}{\alpha^2+B^2} \tag{10}$$

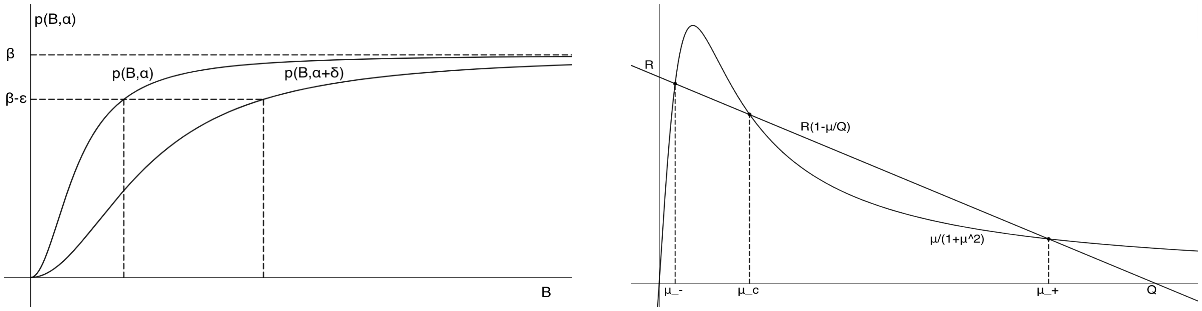
This equation comprises a set of *fixed points* that vary from one to four. dependent upon inputs of $B, r_B, K_B, \alpha, \beta \in \mathbb{R}$. The *stabilities* are observable in a *vector space*, with positive-only value. This is due to the dependent variable, B, or Budworm population density, being assumed quantifiable only through physical data collection of positive integer values. The *dynamical system* defined by equation (10) has, that for an increasing Budworm population density, an *unstable fixed point* at $B=0$. By the polynomial nature of the function, $\dot{B}$, the system will always comprise between two and four *fixed points* for all $B \geq 0$. For the case when there are three total *fixed points*, one of the three *fixed points* is a *bifurcation* of the system. The *bifurcation points* are *semistable fixed points*; where, for any small *perturbation* about one of the variables, the total number of *fixed points* will either increase or decrease by one. If the *perturbation* increases the total number of *fixed points* by one, then one of the *fixed points* will be *stable*, and the other, *unstable*.

The *nullclines* of this function are the set of all $\dot{B}B^{-1}=0$. Balancing this equation, $r_B - r_B B K_B^{-1} = \beta B(\alpha^2+B^2)^{-1}$, allows for analyzable shifts of one or more variables on one side of the equation in terms of variables on the other side. The division of B from the equation also reduces the complexity of solutions. This is because, for all solutions to $\dot{B}=0$, when $B=0$, the *fixed point* does not change. When attempting to reduce the function to as few variables as possible, the function can be rewritten in terms of three new variable *parameters* which are expressible as a combination of the original variables: $\mu = B\alpha^{-1}$, $R = \alpha r_B \beta^{-1}$, and $Q = K_B\alpha^{-1}$.[16]

$$r_B\left(1-\frac{B}{K_B}\right) = \frac{\beta B}{\alpha^2+B^2} \tag{11}$$

$$\mu = \frac{B}{\alpha},\ R = \frac{\alpha r_B}{B},\ Q = \frac{K_B}{\alpha} \iff R\left(1-\frac{\mu}{Q}\right) = \frac{\mu}{1+\mu^2}$$

Utilizing a method of unit analysis for equation (10), the variables are defined, such that, B represents the Budworm population density, r_B is the linear birth rate, and K_B is the carrying capacity of the Budworm population density proportional to tree foliage. The subtracted term, $p(B) = (\beta B^2)(\alpha^2 + B^2)^{-1}$, is the rate of predation on the Budworm population density by mostly avian predators. The variable, β, is the upper limit of the predation as $\beta \to +\infty$, and α determines the rate at which predation reaches the upper limit, β. The variable, α, can also be considered when determining the minimum population density of Budworms at which the predation rate, $p(B)$, is within some $\epsilon > 0$ distance from the upper limit, β.



(a) Influence of variable shift, $\alpha \to \alpha + \delta$, in predation function, $p(B)$.

(b) Reduced parameter graph of $\dot{B}B^{-1} = 0$, with three fixed points, μ_-, μ_c, μ_+.

Figure 4

From the initial model given by equation (10), the function is analyzable as a *nonautonomous scalar* equation. This is done by letting $B \in \mathcal{B}$ act as the *state space*; such that, $\mathcal{B} \cong \mathcal{X}$, and letting at least one or more of the other variables be elements of the *time set*, $\mathcal{T}$. From a method of direct-unit analysis, only the linear birth rate, r_B, is a function of the *time set*. By this method, the variables, r_B, K_B, α, must be elements of the *state space* as well. The only element that is not uniquely defined by linear dependence to only the *state space* or *time set* is the upper limit of predation, β; which, has linear dependence to $\dot{B}$. The inclusion of β in the function $\dot{B}$, requires that the system be invertible in terms of the *state space* and *time set*. Equation (10) is therefore a *nonautonomous scalar spacetime* equation similar to example 1. Some analysis was done regarding this conclusion, and is worth noting that surprisingly, the composition of a time-derivative of β with respect to the *time set* will normalize without having influence on any time-derivative of $\dot{B}$.[3] This is relevant when considering the expansion of the variable, R, into a function of t.

Although the determination that $\dot{B}$ can, for any variable in the equation, comprise invertibility with the *state space* and *time set*, the implicit function defining each variable is unknown. Therefore, when analyzing solutions regarding the reduced expression in terms of R, μ, Q, understanding the initial unit-dependence is applicable to developing exact solutions and expansions of the function.

$$R\left(1 - \frac{\mu}{Q}\right) = \frac{\mu}{1 + \mu^2} \tag{12}$$

Equation (12) has a maximum $R > 0$, being $R_c \in \{R\}$; where, for any shift in variable, $Q > 0$, for all $R \geq R_c$, there exists exactly one *fixed point* and no *bifurcations*. Then for all $0 < R < R_c$, there exists at least one interval of elements, $Q > 0$, with exactly one *fixed point*, and at least one interval of elements, $Q > 0$ that comprises three total *fixed points*. The value R_c is obtained by determining the maximum slope of the right-hand-side of equation (12), the coordinate position of the slope, and then finding the value of R at which the left-hand-side of equation (12) crosses this value; such that, the value for R will be R_c. The maximum slope is found from taking the derivative of $\mu(1 + \mu^2)^{-1}$ in terms of μ. The coordinate position is found by solving the second derivative in terms of μ and setting equal to zero, since this will be the maximum of the first derivative when $\mu > 0$. Then, solving for a general linear function that is equivalent to the left-hand-side of equation (12) produces the element, R_c, for coordinate, $(0, R_c)$, with intersection at μ^*.

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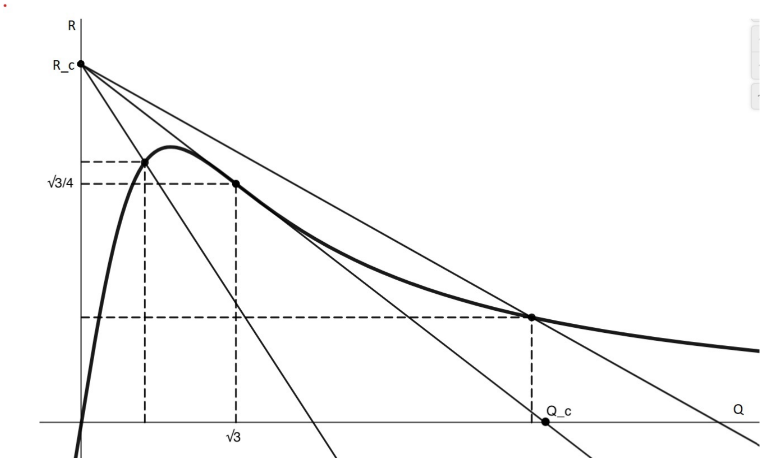
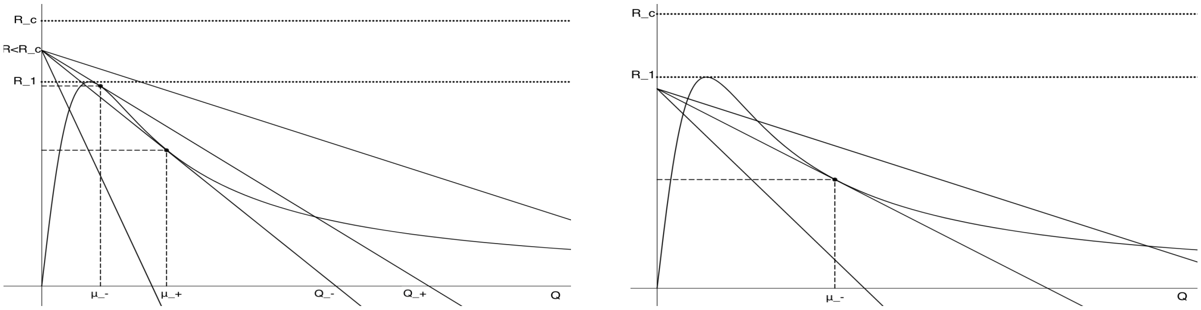
[3] See Appendix B for a proof on setting $\beta = 1$.



$$\begin{aligned} &\frac{d^2}{d\mu^2}\left[\frac{\mu}{1+\mu^2}\right] = 0 \;\Rightarrow\; \mu^* = \sqrt{3} \\ \Rightarrow\quad &\frac{d}{d\mu}\left[\frac{\mu}{1+\mu^2}\right] = -\frac{1}{8} \\ \Rightarrow\quad &\frac{\mu}{1+\mu^2} = \frac{\sqrt{3}}{4} \\ \Rightarrow\quad &R_c = \frac{3\sqrt{3}}{8} \iff Q_c = 3\sqrt{3} \end{aligned}$$

Recalling unit equivalence between reduced variables, $\mu = B\alpha^{-1}, R = \alpha r_B \beta^{-1}, Q = K_B\alpha^{-1}$, then for all $R \geq R_c$, there will be no *bifurcations* and the foliage, K_B, being dependent only on Q, will have no affect on the occurrence of a *bifurcation*. For the case that $\sqrt{3} \cdot 4^{-1} \leq R_0 \leq R_c = 3\sqrt{3} \cdot 8^{-1}$, where $R_0 \in \{R\}$, then variations on $Q > 0$ will have two regions with exactly one *fixed point* and one region with three *fixed points*. This region of three *fixed points* is dependent upon the maximum of the function, $\mu(1+\mu^2)^{-1}$; whereby, taking the derivative with respect to μ, and setting $\mu = 0$, the positive value solution is given by a minimum value of $R_1 = (2\mu)^{-1}$. The fixed variable, R_1, then defines the region at which, for any $R_0 \in [R_1, R_c)$, there are exactly two *bifurcation points*, and that for all $R_0 \in (0, R_1)$, there is exactly one *bifurcation point*.



(a) Fixed $R_0 \in \mathbb{R}$ in the interval, $R_1 < R < R_c$, with *bifurcation points* shown.

(b) Fixed $R \in \mathbb{R}$ in the interval, $0 < R < R_1$, with the unique *bifurcation point* shown.

Figure 5

Given the existence of some maximum, R_c, at which no variation in Q will cause a *bifurcation*, obtaining the interval $Q \in [Q_-, Q_+]$, for any unique, $0 < R < R_c$, determines the shifts of variable, $Q(K_B) \to Q(K_B \pm \epsilon)$, $\epsilon > 0$; such that, the function will comprise either three *bifurcation points*, or no *bifurcation points*. Recalling that the variable, $Q = K_B\alpha^{-1}$, is the only reduced *parameter* with dependence on K_B, then any change in Q can also be independently considered through changes to the variable, K_B. Then, by way of the left-hand-side of equation (12) being linear, the maximum, Q_+, and minimum, Q_-, can be determined by the same linear function equivalence that was used to find the value, R_c.

$$R_c = -\frac{1}{8}\mu + \frac{3\sqrt{3}}{8} \iff 0 = -\frac{1}{8}Q_c + \frac{3\sqrt{3}}{8} \tag{13}$$

Setting $R = R_c$ in terms of variable, μ, equal to the right-hand-side of equation (12), provides the exact solutions for μ_c, and Q_c; such that, $\mu_c = \sqrt{3}$, and for $Q_c = 3\sqrt{3}$, being the unique *sum-of-roots* solution for Q_c. Then for all $R \geq R_c$, any variation on μ will similarly provide exactly one solution. This can be considered through the use of *perturbation* variables, $\epsilon > 0$ and $\delta > 0$.

$$-\frac{\epsilon}{8}\mu + \frac{3\sqrt{3}}{4} + \delta = R = \frac{\mu}{1+\mu^2} \tag{14}$$

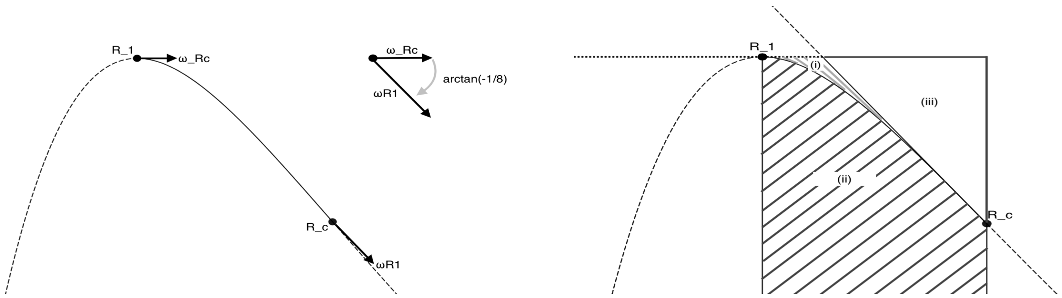
By flipping the sign, $\delta \to -\delta$, within a boundary condition that the $R-$intercept of equation (13) be greater than zero; such that, $\delta \in (0,\ 3\sqrt{3} \cdot 4^{-1})$, then for any unique δ, there will be either, one, two, or three exact solutions for μ, for all $\epsilon > 0$. Specifically, these exact solutions can be found by solving for the cubic function in terms of μ. To determine the set of all *bifurcation points* from this method, requires obtaining the set of all real-valued solutions, μ, with cardinality $|\mu| = 2$. Since the function has for any unknown set of fixed points, $1 \leq |\mu| \leq 3$, finding the set of all *bifurcation points* is given by reduction to solutions of the type $\mu = \{\mu_-, \mu_+\}$. Simplifying the left-hand-side of equation (13) to a linear function in terms of slope, m, and $R-$intercept, R_0, generalizes this function.

$$\begin{gathered}
0 < \frac{\epsilon}{8} = m \in \mathbb{R}, \text{ and} \\
\frac{3\sqrt{3}}{4} + \delta = R_0 \in \left(0, \frac{3\sqrt{3}}{4}\right) \subseteq \mathbb{R} \\
\Rightarrow\ -m\mu + R_0 = \frac{\mu}{1+\mu^2} \\
\iff\ \mu^3 - R_0\mu^2 + (1+m)\mu - R_0 = 0
\end{gathered} \tag{15}$$

Observations on the geometry of the curve, with respect to the set of all *bifurcation points*, that are defined to be *bounded* by the interval, $(\mu, R_0) \in [2^{-1}, \sqrt{3}] \times [\sqrt{3}\cdot^{-1}, 2^{-1}]$, considers the set of solutions given by the *infinum* maximum values of $Q = Q_-$, at which there exists only one *fixed point*, and the set of solutions given by the *supremum* minimum values of $Q = Q_+$ when there exists only one *fixed point*. In order to determine the geometry of the *trajectory paths* intrinsic to the shape of these curves, a determination on the *magnitude* of the *gradient* from equation (10) can be solved for in terms of the discrete solutions of the *bifurcation points*. For the upper and lower bounds of this curve given by $(R, Q, \mu) = \{(R_c, Q_c, \mu_c), (0, 1 \cdot 0^{-1}, 2^{-1})\}$, the definition, $\dot{B}B^{-1} = \dot{0}$, is used to define the reduction of equation (10) to equation (11), when setting equation (11) equal to zero. As well, it is noteworthy to recall that when solving for the *bifurcation point* given by the limits $R \to R_1$, and $\mu \to 2^{-1}$, requires that in order for Q to approach positive infinity, then $Q = 1 \cdot 0^{-1}$.

$$\begin{gathered}
\nabla\dot{0}(R, Q, \mu) = \left\langle \frac{\partial\dot{0}}{\partial R}, \frac{\partial\dot{0}}{\partial Q}, \frac{\partial\dot{0}}{\partial\mu} \right\rangle = \left\langle -1 + \frac{\mu}{Q},\ -\frac{R\mu}{Q^2},\ \frac{1-\mu^2}{(1+\mu^2)^2} + \frac{R}{Q} \right\rangle \\
\Rightarrow\ \omega(R_c, Q_c, \mu_c) = \omega_{R_c} = \frac{\sqrt{265}}{24} \\
\Rightarrow\ \omega\left(\frac{1}{2}, \frac{1}{0}, \frac{1}{2}\right) = \omega_{R_1} = \frac{2\sqrt{10}}{5}
\end{gathered}$$

From these two *fixed points*, being the upper and lower limit of the curve considered, the *magnitude* of the *gradient* given by *bifurcation point* solutions to equation (15) in the interval, $R_0 \in [R_1, R_c]$, will have a magnitude increase of $\omega_{R_c}\omega_{R_1}^{-1}$. Which, having already determined that the slope of the linear function from equation (13), when $R = R_c$ is $-m = -8^{-1}$, and that the slope at $R = R_1$ is tangent to the maxima of the right-hand-side of equation (12), then the slope at $R = R_1$ is given to be $-m = 0$. Having obtained the exact *tangent* values given at the limits of the curve-section, then the change in angle, θ, is equivalent to the *arctangent* ratio: $\theta = \arctan(-8^{-1})$. Information pertaining to the geometry of the bifurcation points with respect the right-hand-side of equation (12), in the interval, $\mu \in [\mu_{R_c}, \mu_1]$, added to the area covered by the change in the linear function from $R_c \to R_1$, produces an upper limit area within the boundary conditions. This upper limit area, being the area added onto the right-hand-side of equation (12), is found from the total area within the boundaries, subtracted by the *complement* of the area covered by the angle difference of the *gradient*.



(a) Gradient of magnitudes at R_1 and R_c with angles perpendicular to $+R-$axis.

(b) Area of regions within $\sup\{\min\{R_1 \to R_c\}\}$ with (i)+(ii)+(iii)= $R_1 \cdot (\mu_{R_c} - \mu_{R_1})$.

Figure 6

From figure 7(b), the area of region (ii) is obtained by integration of the right-hand-side of equation (13) from $\mu_{R_1} = 2^{-1} \to \mu_{R_c} = \sqrt{3}$. The area of region (ii) is given by the subtraction of areas (i) and (iii), from the total area being considered; such that, the total area is a rectangle with height, $R_1 - 0$, and width, $\mu_{R_1} - \mu_{R_c}$, being equal to the sum of all three area components, (i)+(ii)+(iii)= $R_1 \cdot (\mu_{R_c} - \mu_{R_1})$. Area (iii) is then a triangle with height, $R_c - R_1 = (2 - \sqrt{3}) \cdot 4^{-1}$, and base found from the intersection of the linear function of equation (13) with $R = 2^{-1}$, subtracted from R_c. This provides the area of (iii) to have equivalence, $2^{-1} \cdot (4 - 2\sqrt{3}) \cdot ((2 - \sqrt{3}) \cdot 4^{-1}) = (7 - 4\sqrt{3}) \cdot 4^{-1}$. The area of (ii) is the resultant areas of (ii) and (iii), subtracted from the total area (i)+(ii)+(iii).

$$\begin{aligned}
\text{(i)} + \text{(ii)} + \text{(iii)} = R_1(\mu_{R_c} - \mu_{R_1}) &= \frac{\sqrt{3} - 1}{2} \\
\text{(ii)} &= \int_{\frac{1}{2}}^{\sqrt{3}} \frac{\mu}{1 + \mu^2} d\mu = \frac{\ln 2}{2} \\
\text{(iii)} &= \frac{7 - 4\sqrt{3}}{4} \\
\text{(i)} &= \frac{\sqrt{3} - 1}{4} - \text{(ii)} - \text{(iii)} = \frac{6\sqrt{3} - 9 - 2\ln(2)}{4}
\end{aligned}$$

Having found an equivalence relation between variable, Q, and variable, μ, with respect to equation (11), then integration of the left-hand-side of equation (11) in terms of Q, for $Q_c \to +\infty$, when $R_c \to R_1$, produces the same area as (i). The solution to this equivalence, being some convergent value, requires that R and Q be *invertible*; such that, $R = R(Q)$, and $Q = Q(R)$, where the set of all *bifurcation points* are governed by solutions which are defined by the set of $\sup\{\min\{R_1 \to R_c\}\}$ over the curve. This solution, which has three *linearly independent* variables, also being convergent over integration, is implicitly defined to have, R, be proportional to some exponential function with parameter value, $\lambda \in \mathbb{R}$. This implicit proportionality to an exponential function in terms of Q, is demonstrated by the unit-dependence of $R = r_B \alpha B^{-1}$, to the variable, r_B; where, $r_B = ||r_B|| \hat{t}^{-1}$, requires that R have a unit-dependent variable in terms of the *time set*. Then, by the variables, μ and Q, not comprising a unit-equivalence to r_B, the implicit requirement that R be proportional to an exponential function is exemplified by the restriction that for some $Q \to +\infty$, a proportionality of $e^{-Q} \ln(Q)$ is convergent. As well, a boundary condition is provided for $R^* = R + 2^{-1}$, due to the *bifurcation diagram* being convergent at one-half instead of zero; where, without this boundary condition, the function would not have equivalence to area (i).

$$\int_{3\sqrt{3}}^{+\infty} \left[R^* \left(1 - \frac{R^*}{Q} \right) \right] dQ = \left[R^*(Q - R^* \ln(Q)) \right]_{3\sqrt{3}}^{+\infty}$$

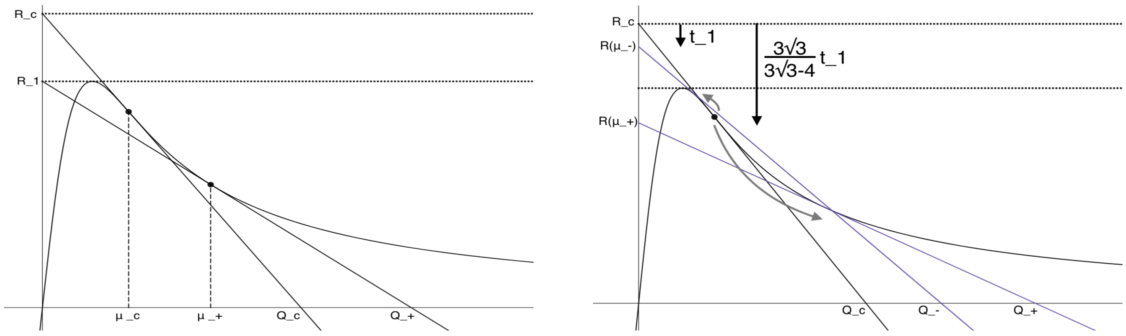
$$\Rightarrow \lim_{Q \to +\infty} \left[R^* \left(Q - R^* \ln\left(Q\right) \right) - R^*(3\sqrt{3} - R^* \ln(3\sqrt{3})) \right] = \frac{6\sqrt{3} - 9 - 2\ln(2)}{4}$$

$$\Rightarrow \lim_{Q \to +\infty} \left[2Q - \ln(Q) \right] - \lim_{Q \to +\infty} \left[2R \ln(Q) \right] = \frac{12\sqrt{3}(R + 1) - 9 - \ln(2)}{2R + 1} - (2R + 1) \ln(3\sqrt{3}) \tag{16}$$

The convergence of equation (15) requires that the limits on the left-hand-side of equation (16) be equivalent to the right-hand-side.[4] Since the *bifurcation diagram* will have dependence on variables, R and Q, then R can be expressed as a function in terms of Q; such that, $R = R(Q)$, and that the limits converge. Recalling that $R = \alpha r_B \beta^{-1}$, $Q = K_B \alpha^{-1}$, and that the unit-dependence of these elements from equation (10), will have the variable, β, be unit-dependent to $\hat{x}\hat{t}^{-1}$, then requires that β be expressible as some function in terms of $\dot{B}$. Then equation (10) is *self-iterating*; such that, $R(r_B, B, \beta)$, must have a proportionality to Lambert's problem; where, solutions to Lambert's problem requires a proportionality to the exponential function.[5] Specifically, that R is expressible by some $R = \lambda_0 Q e^{-\lambda_1 Q}$ function, with $\lambda_0, \lambda_1 \in \mathbb{R}$. The *reflexive relation* of variable, R, with respect to variable, Q, being proportional to units of $\hat{B}$ and $\hat{t}$, will, by the *self-iterating* nature of equation (10), require there to exist some time-dependent solution; where, Q may be treated as a *nonautonomous scalar* equation with some solution dependence to the *time set*, $\mathcal{T}$. Which, for unknown scalar values, λ_0, λ_1, and unknown variable dependence of Q, with respect to the *time set*, $\mathcal{T}$, the expression can be considered by the expression for the limit as $Q \to +\infty$, in terms of the finite sum of elements that converge to $\ln(3\sqrt{3})$; where, $R = \lambda_0 Q^{-\lambda_1 Q}$.[6]

$$\lim_{Q\to+\infty}\left[2Q - \ln(Q) - 2\lambda_0 Q e^{-\lambda_1 Q}\ln(Q) + \frac{9+\ln(2)}{2\lambda_0 Q e^{-\lambda_1 Q}+1} + 2\lambda_0 Q e^{-\lambda_1 Q}\ln(3\sqrt{3})\right] \\ = \\ \ln(3\sqrt{3}) + 6\sqrt{3} \tag{17}$$

A complete solution for the *bifurcation diagram* of the lower *bounded* function is the set of all *fixed points* for *supremum* minimum *bifurcation points* in the interval, $R_0 \in [2^{-1},\ 3\sqrt{3}\cdot 8^{-1}]$, and the *infinum* maximum *bifurcation points* in the interval $R_0 \in [0,\ 2^{-1}]$, for which there exists exactly two *fixed points*. From figure 7(a), this will be the set of all Q_+ when $R_1 < R_0 < R_c$, and Q_- when $0 < R_0 \leq R_1$. Then, for $R_0 = R_1$, the derivative of the right-hand-side of equation (14) will have some $\mu \in \mathbb{R}$ equivalence to the slope of the linear function $R = -mQ + 2^{-1}$; where, $(R, Q) \in \mathbb{R}^2$ can be solved for at $0 = -mQ_+ + 2^{-1}$. The solution for Q_+ as a function of μ, when substituted into equation (11) for $R_0 = R_1$, provides two complex-valued solutions: $\mu = 2^{-1} \pm i\sqrt{7}\cdot 2^{-1}$. From this information, an approximate estimation can be given at $\mu_+ = (4\sqrt{7}+3)\cdot 4^{-1}$. With the method provided, this approximation for μ_+ may be utilized with respect to figure 8(a) to better determine an approximate geometry of the *bifurcation points*; however, since integration from $\mu_c \to \mu_+$, subtracted from the area covered from $R_c \to R_1$, will not produce an exact equivalence to equation (16) unless the approximation is determind to be exact, then a stronger approach for obtaining the *bifurcation diagram*, is to consider relative slope change with respect to the *supremum* minimum, and *infinum* minimum *bifurcation points*, $\{Q_-, Q_+\} \in (Q_c, +\infty)$. These are found by comparing the ratios of $R_c \to R_1$ and $R_c \to 0$ for some equivalent iterative step equivalent to the *time set*, $t \in \mathcal{T}$. If the time step is set to occur



(a) (b)

Figure 7

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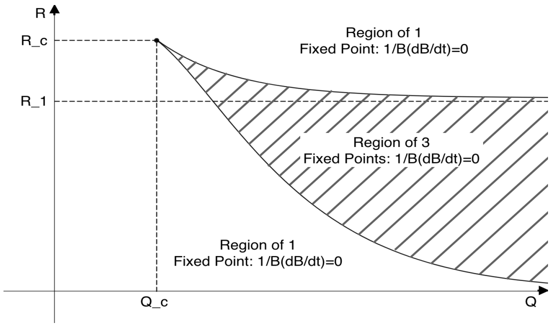
[4] **Question 1** to the reader: Can $\lim_{Q\to+\infty}[Q - ln(Q)]$ be solved for some $Q = Q(t)$, with the limit of t producing an equivalence to the *Euler-Mascheroni* constant, $\gamma = 0.57721...$?

[5] See example 1 for reasoning.

[6] **Question 2** to the reader: If Q is expressible by some function of variable, t, and $\lambda_0, \lambda_1 \in \mathbb{C}$, what is the function $Q(t)$; such that, $\lim_{t\to+\infty}$ of equation (16) is equivalent to $\ln(3\sqrt{3}) + 6\sqrt{3}$? Does equation (15) have an exact solution for R; such that, the equation is balanced?

exclusively within the same boundary conditions as $R_c \to R_1$, and $R_c \to 0$, then for any $R_c - t_1$ that maps to μ_-, the proportionality ratio is $R_c(R_c - 2^{-1})^{-1} = 3\sqrt{3}(3\sqrt{3} - 4)^{-1}$. Observable information from figure (b), and relative time-proportionality shifts determining Q_+ with respect to Q_-, provides the appearance of everywhere being the second *fixed point* in the interval, (R_c, R_1). Indication of this equivalence is noteworthy, as the maximum change in the rate of the slope, $-m$, from the linear equation, $R = -m\mu + R_0$, being the third derivative of $\mu(1 + \mu^2)^{-1} = 0$, with respect to μ, is the maximum change in rate of the function, $\mu_- = \sqrt{9 \cdot 2^{-1} + \sqrt{73} \cdot 2^{-1}}$. This reasons by evidence, that the difference, $Q_+ - Q_-$, for any $t \in (R_c, R_1)$, will have a lesser change in difference for the variables, μ_-, μ_+, up to the limit of the maximum change in rate of the function. The *reflexive relation* of variables in the original *nonautonomous scalar* equation with unit dependence, then allows for the generalization of $t = t(B)$, and $B = B(t)$; since, $\mu = B\alpha^{-1}$, and both, B, and α, are unit dependent on the *state space*, $\mathcal{B}$. Then the variance of the *bifurcation points*, and region of three *fixed points* with respect to Budworm population density will have some shift in the difference of B_+ and B_- for variable, $R \in \mathbb{R}$. The geometry provided for the *bifurcation diagram* in terms of variables, R, Q, B, can be generalized under these conditions.[16]

Figure 8



IV Discussion

Topics and methods provided in this text are of the exploratory type. Language and definitions are constructed from verifiable reference material. Any deviation from the minutiae of linguistic interpretation is unintentional. The goal of this writing is to utilize the language provided as effectively as possible to present a narrative on the numerical and geometric evidence presented. All sequences of steps and processes have been considered from an exhaustive list of trial and error attempts. They are not claimed to be unique or novel. All solutions presented, if correct, are presumed solvable by differing strategies.

Of topics lightly explored, but with much interest, are those of *Rate-* and *Noise-Induced Tipping*. The ongoing understanding is one filled with questions regarding methods used to quantify the region at which, for example, a Noise-Induced Tipping of a system would occur; further, as a means to learn how the composition of two functions defining these Noise Tipping regions might increase or decrease by an arbitrary approximation or topology.

Appendix A. Quantifying Nested Sets

A set which contains an infinite number of elements can be simplified to a single statement that refers to that infinity as a unique number. This method is useful when considering a set of *linearly independent* sets that themselves are sets containing a *finite*, *countable infinity*, or *uncountable infinity* of elements. The proof of this statement contains a direct proof and a contradiction. Take the *cardinality* of a *finite* number of elements to be less than a *countable infinity* of elements to be less than an *uncountable infinity* of elements.

$$|\mathbb{R}| = |\mathbb{R}/\mathbb{Q}| > |\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| > |\{n\}| = n \in \mathbb{R}$$

This provides a statement that allows for sets of numbers to be mapped to other sets of numbers. Analysis on this topic can be simplified to three unique equivalence relations. Since the *cardinality* of a set defines the quantity of elements contained within, then the *empty set*, $\emptyset$, contains no elements, and therefore has a quantity of 0. The quantity of any *singleton* defining a number, including the number 0, will have a *cardinality* of 1. Lastly, a set containing an infinite quantity of *singletons* will have a *cardinality* greater than 1. Consider the *time set*, $\mathcal{T}$, that contains an *uncountable infinity* of elements.

$$\begin{gathered} \forall\{\tau\} = \mathcal{T} \neq \emptyset \;\; \exists\{\tau_n : n \in \mathbb{N}, \tau_n \neq \tau_{n+1}\} \subseteq \mathcal{T} : \\ \{\tau\} \widehat{=} \mathbb{N} \iff |\mathcal{T}| \geq |\{\tau_n\}| \geq |\mathbb{N}| = \aleph_0 \end{gathered}$$

The relation symbol, $\widehat{=}$, denotes a *bijection* between two sets. The symbol, $\aleph_0$, refers to the number of elements in a set with a *countable infinity* of unique elements. Since the *time set*, $\mathcal{T}$, has a *bijection* with the set of all Real numbers, $\mathbb{R}$, then the quantity of elements in $\mathcal{T}$ is greater than $\aleph_0$.

$$|\mathcal{T}| > \aleph_0 \;\Rightarrow\; |\mathcal{T}| = |\mathbb{R}| \geq \aleph_1 > \aleph_0 = |\mathbb{N}|$$

When considering a set comprising an infinite number of elements, there are two cases: $|\aleph_0| = 1$ or $|\aleph_0| = \aleph_0$. The same principle then applies to a set containing some unique number; such as, $|1| = 1$, $|\pi| = 1$, or $|0| = 1$. The last of these, $|0| = 1$, is distinct from the number of elements in the empty set: $|\emptyset| = 0$. Since a single element, $n \in \mathbb{N}$, that is not equal to 1, $n \neq 1$, does not produce a *cardinality* that is equivalent to the value, $|n| \neq n$, then the same holds true for $|\aleph_0| \neq \aleph_0$.

Proof by contradiction:

If we assume that $0 = \emptyset$ and are considering the empty set with respect to the set, $\mathcal{T} \widehat{=} \mathbb{R}$, then the *complement* of the empty set is the *time set*, $\mathcal{T}$: $\emptyset^C = \mathcal{T}$. If a unique *singleton*, $\tau \in \mathcal{T}$, has a *cardinality* of 1, and since $1 \in \mathcal{T}$, then the *complement* of this *singleton* will be the full set not including that element: $1^C = \mathcal{T}/1$. The *cardinality* of this set is then the full set, $|\mathcal{T}/1| = \mathcal{T}$. The *complement* of this set will be the empty set, $\mathcal{T}^C = \emptyset$. From this method of quantifying the sets, determining the *cardinality* of a *singleton*, then taking the *complement* by this method twice produces the empty set; where, the *complement* of the empty set is then equivalent to $\mathcal{T}$ and $\mathcal{T}/1$. This requires that $\mathcal{T} = \mathcal{T}/1$, which is clearly not true.

$$\begin{gathered} ||\tau|^C|^C = \emptyset \;\Rightarrow\; \emptyset^C = \mathcal{T},\; \emptyset^C = \mathcal{T}/1 : \\ 1 \in \mathcal{T},\; 1 \notin \mathcal{T} \;\Rightarrow \textit{contradictio} \end{gathered}$$

■

Appendix B. Normalization of Upper-Limit variable, β

Given the following equation for the *Spruce-Budworm and Forest model*.

$$\dot{B} = r_B B\left(1 - \frac{B}{K_B}\right) - \frac{\beta B^2}{\alpha^2 + B^2}$$

A unit analysis determines unit-equivalences required for the function to be considered in a *nonautonomous scalar* equation form.

$$\begin{gathered}\dot{B} = \left|\left|\frac{dB}{dt}\right|\right| \frac{\hat{B}}{\hat{t}} \iff \hat{r}_B\hat{B} \propto \frac{\hat{B}}{\hat{t}},\quad \frac{\hat{r}_B\hat{B}^2}{\hat{K}_B} \propto \frac{\hat{B}}{\hat{t}},\quad \frac{\hat{\beta}\hat{B}^2}{\hat{\alpha}^2 + \hat{B}^2} \propto \frac{\hat{B}}{\hat{t}},\text{ and } \hat{\alpha}^2 + \hat{B}^2 \propto \hat{B}^2 \\ \Rightarrow \hat{r}_B \propto \frac{1}{\hat{t}},\quad \hat{K_B} \propto \hat{B},\quad \hat{\alpha} \propto \hat{B},\text{ and } \hat{\beta} \propto \frac{\hat{B}}{\hat{t}}\end{gathered}$$

The variable, B, is taken to be analogous to the *state space*: $\mathcal{B} \cong X$. The variable, r_B is the only variable with unique linear dependence on the *time set*, $\mathcal{T}$. Requiring that β have dependence on $\dot{x}$, provides evidence that the function is *self-iterating*. Then, noting that units cancel for BK_B^{-1} and $B^2(\alpha^2 + B^2)^{-1}$, a reduction provides the system as a *nonautonomous scalar spacetime* function.

$$\dot{B} = \dot{x},\quad B = x,\quad r_B = \phi(t),\quad \frac{B}{K_B} = \lambda_0,\quad \beta = \beta(\dot{x}),\quad \frac{B^2}{\alpha^2 + B^2} = \lambda_1$$

$$\Rightarrow \dot{x} = x\phi(t)(1 - \lambda_0) - \lambda_1\beta(\dot{x})$$

Solving this equation with dimensionless scalar values, $\lambda_0, \lambda_1 \in \mathbb{R}$, in terms of $x, t \in \mathbb{R}$, provides a solution for the *trajectory path.*

$$x\phi(t) = \frac{\dot{x} + \lambda_1\beta(\dot{x})}{1 - \lambda_0}$$

Taking the time-derivative for each of the variables with dependence on the *time set*, produces a system of equations that individually comprise depenence to the rate of change, $\ddot{x}$, in the *nonautonomous scalar equation*.

$$\begin{aligned}\ddot{x} =& (1 - \lambda_0)\phi\dot{x} + (1 - \lambda_0)\dot{\phi}x - \lambda_1\dot{\beta} \\ \dot{x} =& \left(\frac{1}{\phi(1 - \lambda_0)}\right)\ddot{x} + \left(\frac{\lambda_1}{\phi(1 - \lambda_0)}\right)\dot{\beta} - \left(\frac{\dot{x} + \lambda_1\beta}{\phi^2(1 - \lambda_0)}\right)\dot{\phi} \\ \dot{\phi} =& \left(\frac{1}{x(1 - \lambda_0)}\right)\ddot{x} + \left(\frac{\lambda_1}{x(1 - \lambda_0)}\right)\dot{\beta} - \frac{\dot{x}^2 + \lambda_1\beta\dot{x}}{x^2(1 - \lambda_0)} \\ \dot{\beta} =& \left(-\frac{1}{\lambda_1}\right)\ddot{x} + \left(\frac{1 - \lambda_0}{\lambda_1}\right)\dot{\phi} + \frac{(1 - \lambda_0)\phi\dot{x}}{\lambda_1}\end{aligned}$$

Taking the composition of $\dot{\phi} \circ \dot{\beta}$, $\dot{x} \circ \dot{\beta}$, and $\ddot{x} \circ \dot{\beta}$ removes the variable $\ddot{x}$ from each solution.

$$\begin{aligned}\dot{\phi} \circ \dot{\beta} \;\Rightarrow\; \dot{\phi} =& \frac{(1 - \lambda_0)\phi(x - 1)x - \dot{x}(1 - \lambda_1\beta)}{(1 - \lambda_0)(x - 1)x} \\ \dot{x} \circ \dot{\beta} \;\Rightarrow\; \dot{x} =& \phi(1 - \lambda_0) - \lambda_1\beta \\ \ddot{x} \circ \dot{\beta} \;\Rightarrow\; x =& 1\end{aligned}$$

This provides reasoning to conclude that the upper limit of predation, β, being dependent on $\dot{B}$, can be normalized to a scalar value, $\beta = 1$, when considering the rates of change on the system, $\ddot{x}$, without losing information about the system overall.

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