Contents

Raising a complex number z_1 to the power of another complex number z_2

Author: Muhammad Talha
Institute: Tufail Shaheed Army College, Mardan Cantt.
Grade: XII-C
E-Mail: khankkhan123apsacs@gmail.com

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Abstract

This paper presents a comprehensive derivation of a calculator-friendly expression for computing the power of one complex number raised to another, specifically $z_1 = a + bi$ raised to $z_2 = c + di$. By leveraging the fundamental properties and formulae intrinsic to complex numbers, we develop an explicit, practical method that can be implemented on modern scientific and graphical calculators, such as the Casio FX-991ES or newer models. The derivation emphasizes the mathematical rigor required to handle the inherent complexities of exponentiation in the complex plane, while also providing a user-friendly format that simplifies direct calculation. This work not only bridges the gap between theoretical mathematics and practical computation but also offers a valuable tool for students, educators, and professionals who frequently engage in complex number arithmetic. With this approach, the need for advanced computational tools like Maple or Mathematica for complex exponentiation is significantly reduced. The derived formula is capable of calculating complex exponents with precision up to six decimal places, closely matching the accuracy of these sophisticated tools. Thus, it enhances the efficiency of problem-solving in various mathematical and engineering applications.

1 Deriving a calculator-friendly expression for $z_1^{z_2}$:

Suppose any two complex numbers say $z_1 = a + bi$ and $z_2 = c + di$.
Consider another complex number $z_p = z_1^{z_2}$, then taking ln on both sides

$$\ln z_p = \ln z_1^{z_2}$$
$$\because \ln x^y = y \ln x$$

$$\implies \ln z_p = z_2 \ln z_1 \tag{1}$$

Now, writing z_1 and z_2 in polar form,

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and}$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

But by Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

z_1 and z_2 are more compactly written as,

$$z_1 = r_1 e^{i\theta_1} \text{ and}$$
$$z_2 = r_2 e^{i\theta_2}$$

putting these values of z_1 and z_2 in equation (1) , we get

$$\ln z_p = r_2 \cdot e^{i\theta_2} \times \ln\left(r_1 \cdot e^{i\theta_1}\right)$$
Taking ln again on both sides,
$$\ln\left(\ln z_p\right) = \ln\left(r_2 \cdot e^{i\theta_2} \times \ln\left(r_1 \cdot e^{i\theta_1}\right)\right)$$
$$\because \ln xy = \ln x + \ln y$$
$$\implies \ln\left(\ln z_p\right) = \ln\left(r_2 \cdot e^{i\theta_2}\right) + \ln\left(\ln\left(r_1 \cdot e^{i\theta_1}\right)\right)$$
$$\ln\left(\ln z_p\right) = \ln r_2 + \ln e^{i\theta_2} + \ln\left(\ln r_1 + \ln e^{i\theta_1}\right)$$
$$\ln\left(\ln z_p\right) = \ln r_2 + i\theta_2 \ln e + \ln\left(\ln r_1 + i\theta_1 \ln e\right)$$
$$\because \ln e = 1$$
$$\ln\left(\ln z_p\right) = \ln r_2 + i\theta_2 + \ln\left(\ln r_1 + i\theta_1\right)$$

Since, natural logarithm of any complex number $z = x + yi$ is defined as,

$$\ln z = \ln r + i\theta \text{ , where r} = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = \arg z = \arctan\left(\frac{y}{x}\right)$$

Therefore, $\ln(\ln z_p) = \ln r_2 + i\theta_2 + \ln(\ln r_1 + i\theta_1)$ can be written as,

$$\ln(\ln z_p) = \ln r_2 + i\theta_2 + \ln\{\sqrt{(\ln r_1)^2 + (\theta_1)^2}\} + i \arctan\left(\frac{\theta_1}{\ln r_1}\right)$$

$$\text{Suppose , } \phi = \arctan\left(\frac{\theta_1}{\ln r_1}\right) \text{ and R} = \sqrt{(\ln r_1)^2 + (\theta_1)^2}$$

$$\implies \ln(\ln z_p) = \ln r_2 + i\theta_2 + \ln R + i\phi$$

On re-arranging the above equation we get,

$$\ln(\ln z_p) = (\ln r_2 + \ln R) + i(\theta_2 + \phi)$$

$$\ln(\ln z_p) = (\ln(R \cdot r_2)) + i(\theta_2 + \phi)$$

Taking exp() on both sides, the above expression is simplified as,

$$\exp\{(\ln(\ln z_p))\} = \exp\{(\ln(R \cdot r_2)) + i(\theta_2 + \phi))\}$$

$$\because \exp\{\ln x\} = x$$

$$\implies \ln z_p = \exp\{(\ln(R \cdot r_2)) + i(\theta_2 + \phi))\}$$

$$\text{Since, } \exp\{a + b\} = \exp\{a\} \cdot \exp\{b\}$$

$$\implies \ln z_p = \exp\{\ln(R \cdot r_2)\} \times \exp\{i(\phi + \theta_2)\}$$

$$\ln z_p = Rr_2 + e^{i(\phi+\theta_2)} \quad \because [\exp(x) = e^x]$$

Using equation (2), it is then simplified to

$$\implies \ln z_p = Rr_2(\cos(\phi + \theta_2) + i \sin(\phi + \theta_2))$$

Taking base 'e' on both sides,we get

$$\because e^{\ln x} = x$$

Therefore we get,

$$z_p = e^{(Rr_2)\{\cos(\phi+\theta_2)+i\sin(\phi+\theta_2)\}}$$

Which is further simplified as,

$$z_p = e^{Rr_2\cos(\phi+\theta_2)+iRr_2\sin(\phi+\theta_2)}$$

$$\because [x^{y+z} = x^y \cdot x^z]$$

$$\implies z_p = e^{Rr_2\cos(\phi+\theta_2)} \cdot e^{i(Rr_2\sin(\phi+\theta_2)}$$

Suppose $R_p = e^{Rr_2\cos(\phi+\theta_2)}$ and $\theta_p = Rr_2\sin(\phi+\theta_2)$, then the above expression implies

$$z_p = R_p e^{i\theta_p}$$

Using Euler's Formula i-e equation (2), we get

$$z_p = R_p(\cos\theta_p + i\sin\theta_p) \tag{3}$$

Which is our desired expression in polar form, however the following variables must be taken into account
Later in the document, we call it formula-specific variables.

$$R_p = e^{Rr_2\cos(\phi+\theta_2)} \qquad \textit{Magnitude of } z_p$$

$$\theta_p = Rr_2\sin(\phi+\theta_2) \qquad \textit{Co-terminal argument of } z_p$$

$$R = \sqrt{(\ln r_1)^2 + (\theta_1)^2}$$

$$\phi = \arctan\left(\frac{\theta_1}{\ln r_1}\right)$$

$$r_1 = |z_1|$$

$$r_2 = |z_2|$$

$$\theta_1 = \arg z_1 \ (Radians)$$

$$\theta_2 = \arg z_2 \ (Radians)$$

To find the principle argument of z_p, you have to bring θ_p to the interval $[-\pi, \pi]$ by adding $2k\pi$ accordingly, where $k \in \mathbb{Z}$.
Thus we have established a formula to compute $z_1^{z_2}$ easily by using our calculators.

2 Testing our newly established formula:

2.1 Approximation of i^i:

i^i is in the form of $z_1^{z_2}$ therefore $z_1 = i$ and $z_2 = i$.
Upon the calculation of values of formula-specific variables, we get

$$\theta_1 = \theta_2 = \frac{\pi}{2}$$
$$r_1 = r_2 = 1$$
$$R = \frac{\pi}{2}$$
$$\phi = \frac{\pi}{2}$$
$$R_p = e^{\frac{-\pi}{2}}$$
$$\theta_p = 0$$

Putting all these values into our derived expression, we get

$$i^i = e^{\frac{-\pi}{2}}(\cos 0 + i \sin 0) = e^{\frac{-\pi}{2}}(1 + 0i) \quad \because \cos 0 = 1\ ,\ \sin 0 = 0$$
$$i^i = e^{\frac{-\pi}{2}} \approx 0.207879576$$

Verification using Wolfram—Alpha:

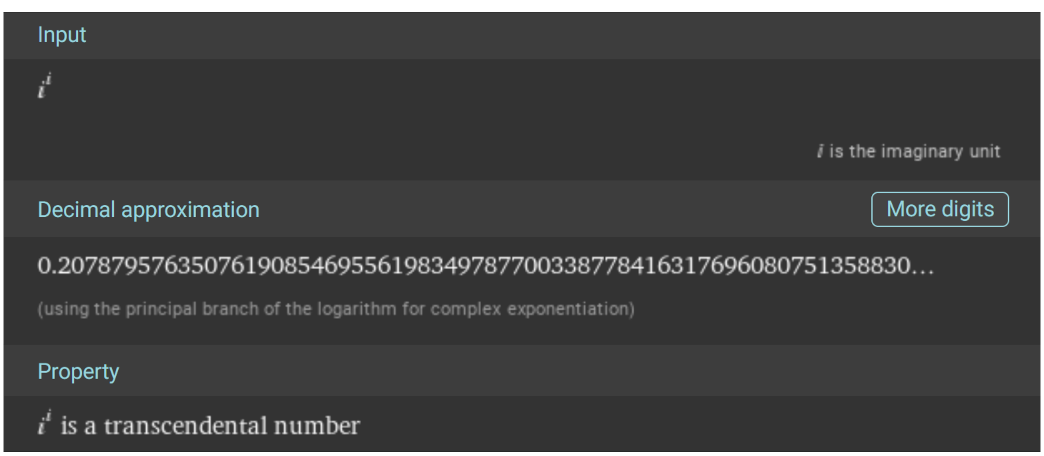

Figure 1: Figure shows the value of i^i

2.2 Approximation of $-5^{(-8+4i)}$:

Let $z_p = -5^{(-8+4i)}$, $z_1 = -5$ and $z_2 = -8 + 4i$.
The values of formula-specific variables related to our derived expression are given below:

$$r_1 = 5$$
$$r_2 - 8.944271908$$
$$\theta_1 = 3.141592654$$
$$\theta_2 = 2.677945045$$
$$R = 3.529857618$$
$$\phi = 1.097357208$$
$$R_p = 8.927596555 \times 10^{-12}$$
$$\theta_p = -18.69498960$$

Putting all these in our derived expression, we get

$$z_p = 8.821165223 \times 10^{-12} + 1.374417823 \times 10^{-12} i$$

Verification using Wolfram—Alpha:

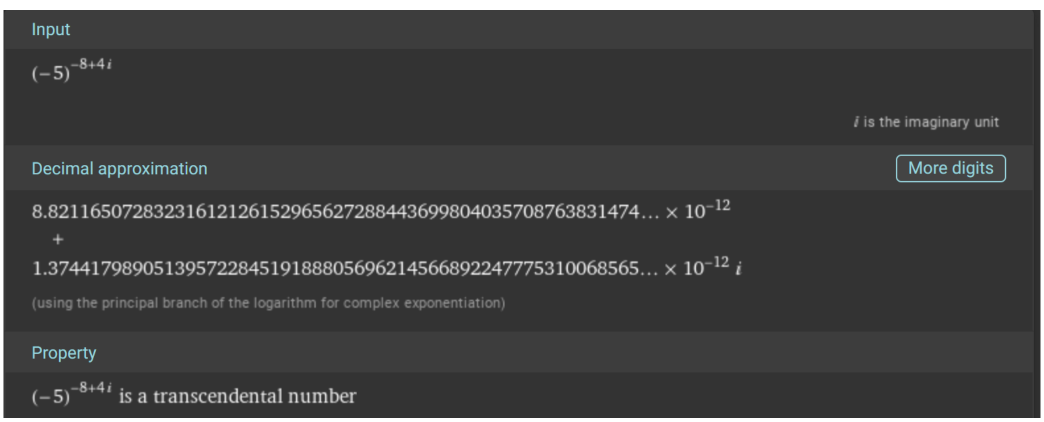

Figure 2: Figure shows the value of $(-5)^{-8+4i}$

2.3 Approximation of $(5-2i)^{2+3i}$:

Given two complex numbers $z_1 = 5-2i$ and $z_2 = 2+3i$. So $z_p = (5-2i)^{2+3i}$. The values of formula-specific variables related to our derived expression are given below:

$$r_1 = \sqrt{29} = 5.38516480713450$$
$$r_2 = \sqrt{13} = 3.60555127546399$$
$$\theta_1 = -\arctan\left(\frac{2}{5}\right) = -0.380506377112365$$
$$\theta_2 = \arctan\left(\frac{3}{2}\right) = 0.982793723247329$$
$$\phi = -0.222267153631908$$
$$R = 1.72610990515791$$
$$R_p = 90.81413643$$
$$\theta_p = 4.289930990$$

Putting these in our derived expression, we get:

$$z_p = -37.23412131 - 82.83011279i$$

Verification using Wolfram—Alpha:

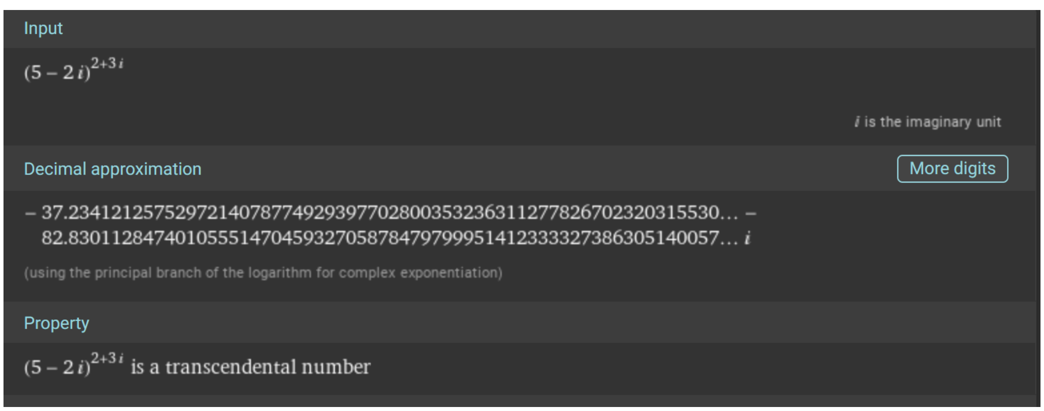

Figure 3: Figure shows the value of $(5-2i)^{2+3i}$

2.4 Approximation of 2^i:

Given that $z = 2$, the values of variables related to formula are:

$$r_1 = 2.0$$
$$\theta_1 = 0.0$$
$$r_2 = 1.0$$
$$\theta_2 = \frac{\pi}{2}$$
$$R = 0.6931471806$$
$$\phi = 0.0$$
$$R_P = 0.9999999999$$
$$\theta_p = 0.6931471806$$

Putting above values in our formula, we get

$$2^i = 0.7692389012 + 0.6389612762i$$

Verification using Wolfram—Alpha:

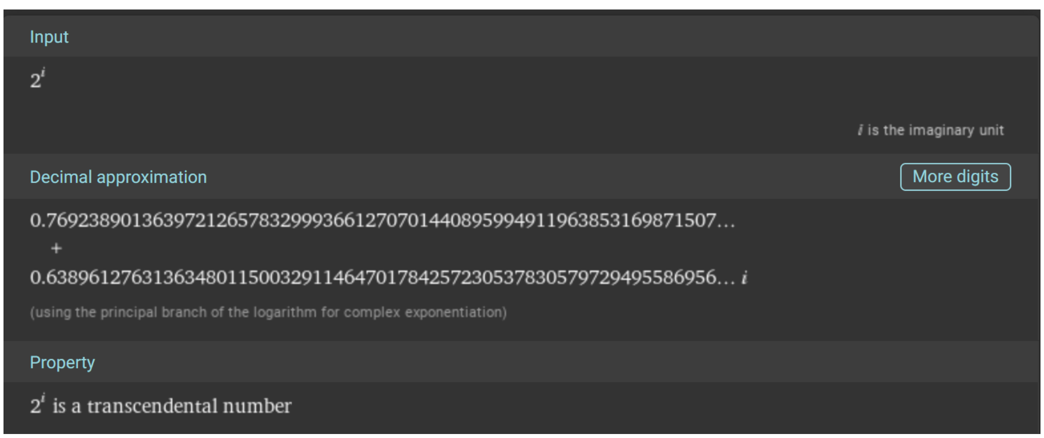

Figure 4: Figure shows approximation of 2^i using Wolfram—Alpha.

2.5 Approximation of $(3-5i)^i$:

Given that $z = 3 - 5i$, the values of variables related to our formula are:

$$r_1 = \sqrt{34} \approx 5.830951895$$
$$r_2 = 1$$
$$\theta_1 = -\arctan\left(\frac{5}{3}\right) \approx -1.030376827$$
$$\theta_2 = \frac{\pi}{2} = 1.570796327$$
$$R = 2.042175566$$
$$\phi = -0.5288590608$$
$$R_P = 2.802121551$$
$$\theta_p = 1.763180262$$

Putting all these values into our derived formula, we get:

$$(3-5i)^i = -0.5357639329 + 2.7504258210i$$

Verification using Wolfram—Alpha:

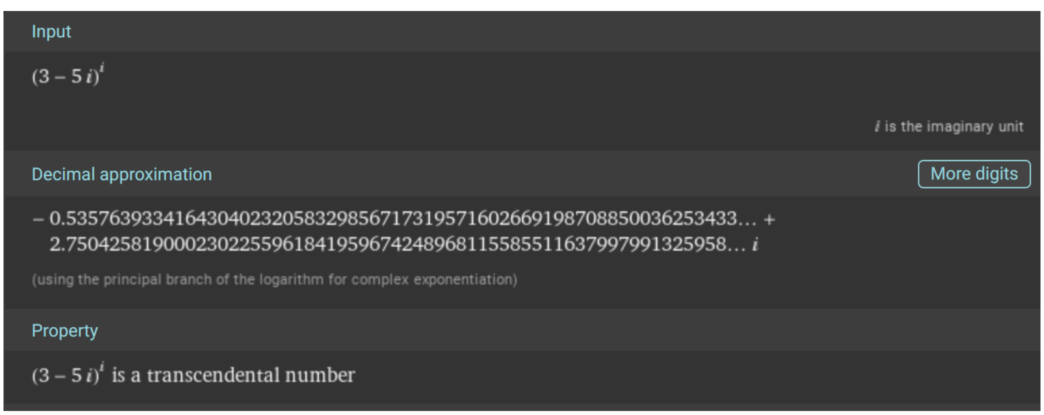

Figure 5: Figure shows approximation of $(3-5i)^i$ using Wolfram—Alpha.

3 Conclusion:

In this paper, we have successfully derived a practical, calculator-friendly expression for raising one complex number $z_1 = a + bi$ to the power of another complex number $z_2 = c + di$. By leveraging well-established mathematical principles specific to complex numbers, we have formulated a method that can be implemented directly on modern scientific and graphical calculators, such as the Casio FX-991ES and its successors. This approach not only simplifies the complex exponentiation process but also significantly reduces reliance on advanced computational software like Maple or Mathematica. Notably, the derived formula can compute complex exponents with a precision correct to six decimal places, closely matching the accuracy of results obtained from these sophisticated tools. This level of precision ensures that our method is both accurate and practical for use in educational, engineering, and scientific contexts. Ultimately, this work bridges the gap between theoretical complexity and practical computation, empowering users to perform sophisticated calculations with ease and precision directly on calculators. The accuracy and convenience of our method provide a valuable alternative to traditional computational software, making advanced complex number arithmetic more accessible to a broader audience.