On induced modules over group rings of soluble groups of finite rank

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Abstract

The paper is a survey where we discuss various methods for obtaining results on induced modules over group rings of soluble groups of finite rank. These methods allow us to obtain results on the structure of solvable groups admitting primitive and semiprimitive faithful irreducible representations. In particular, it allows us to study the structure of irreducible representations of some classes of nilpotent groups.

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1 Introduction

A group G is said to have finite (Prufer) rank if there is a positive integer m such that any finitely generated subgroup of G may be generated by m elements; the smallest m with this property is the rank $r(G)$ of G. A group G is said to be of finite torsion-free rank if it has a finite series each of whose factor is either infinite cyclic or locally finite; the number $r_0(G)$ of infinite cyclic factors in such a series is the torsion-free rank of G.

If a group G has a finite series each of whose factor is either cyclic or quasi-cyclic then G is said to be minimax; the number $m(G)$ of infinite factors in such a series is the minimax length of G. If in such a series all infinite factors are cyclic then the group G is said to be polycyclic; the number $h(G)$ of infinite factors in such a series is the Hirsch number of G.

Let A be an abelian group and let p be a prime integer then the p-rank $r_p(A)$ of the group A is equal to $r(A_p)$, where A_p is the p-component of the torsion subgroup of A. Then we can define the total rank $r_t(A)$ of the group A by the following formula:

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$r_t(A) = r_0(A) + \sum_p r_p(A)$. A soluble group has finite abelian total rank, or is a soluble FATR-group, if it has a finite series in which each factor is abelian of finite total rank. Many results on the construction of soluble FATR-groups can be found in [8].

Let G be a group, let R be a ring and let I be a right ideal of the group ring RG. We say that a subgroup H of the group G controls the ideal I if

$$I = (I \cap RH)RG. \tag{1.1}$$

Let H be a subgroup of the group G and let U be a right RH-module. Since the group ring RG can be considered as a left RH-module, we can define the tensor product $U \otimes_{RH} RG$, which is a right RG-module named the RG-module induced from the RH-module U. Moreover, if M is an RG-module and $U \leq M$ then it follows from [6, Chap. 2, Corollary 1.2(i)] that

$$M = U \otimes_{RH} RG \tag{1.2}$$

if and only if

$$M = \oplus_{t \in T} Ut, \tag{1.3}$$

where T is a right transversal to subgroup H in G. If Equality 1.2 holds we say that the subgroup H controls the module M.

Suppose that $M = aRG$ is a cyclic right RG-module generated by a nonzero element $a \in M$. Put $I = Ann_{kG}(a)$ and let $U = akH$, where H is a subgroup of the group G. It is not difficult to note that, in these notations, Equality 1.1 holds if and only if Equality 1.3 holds. Thus, in the case where $M = aRG$, Equalities 1.1, 1.2, 1.3 mean the same.

Equality 1.3 shows that properties of the RG-module M and the RH-module U are closely related. For instance , if the module M has some chain condition (for example, is Artenian or Noetherian) then the module U also has this condition. So, Equality 1.2 may be very useful if properties of the RH-module U are well studied. For instance, in the case where the group H is polycyclic, we have a deeply developed theory (see [28]). Equality 1.2 also may be very useful in the case where the group G has finite torsion-free rank $r_0(G)$ if $r_0(H) < r_0(G)$ because we can use the induction on $r_0(G)$ then. However, on the place of $r_0(G)$ there may be another rank of the group G or the minimax length $m(G)$ if G is minimax.

If a kG-module M of some representation φ of a group G over a field k is induced from some kH-module U, where H is a subgroup of the group G, then we say that the representation φ is induced from a representation ϕ of subgroup H, where U is the module of the representation ϕ. Recall that the representation φ is said to be faithful if $Ker\varphi = 1$, it is equal to $C_G(M) = 1$.

In Section 2 we managed to characterize faithful irreducible representations of an abelian group G of finite total rank as representations induced from representations of finitely generated subgroups of G (Corollary 2.4). The proof of this result is based on properties of the multiplicative group of fields (Proposition 2.1) and generalizations (Proposition 2.2) of some results of Kummer theory (see [7], Chap. VIII, §8).

Let R be a ring, G be a group and I be a right (left) ideal of the group ring RG. It is not difficult to show that $I^{\dagger} = (1 + I) \cap G$ is a subgroup of G and if the ideal I is two-sided then $I^{\dagger}$ is a normal subgroup of G. The ideal I is said to be faithful if $I^{\dagger} = 1$. The approach based on Propositions 2.1 and 2.2 also allows us to study the properties of faithful prime ideals of group rings of abelian groups. In [12] Segal proved that under some additional conditions any faithful prime ideal of a group ring RG of an abelian minimax group G over a finitely generated commutative ring R is finitely generated. In particular, we obtained a criterion when all faithful prime ideals of a group algebra kG of an abelian group G over a finitely generated field k are finitely generated (Theorem 2.6).

The subgroup $H \leq G$ which provides 1.1 is said to be a control subgroup for the RG-module M. Results about induced modules are also called control theorems (see [19]). In Section 3 we consider various types of control subgroups for modules over group rings of soluble groups of finite rank. The obtained control subgroups appear on the basis of annihilators of elements of M in a subring RA of RG defined by a normal subgroup A of G (see Proposition 3.3). So, in Section 3 we assume that the module M is not RA-torsion-free for some normal subgroup A of G. If the module M is $R*A$-torsion-free for some crossed product $R * A$ then we use another approach considered in Section 4. The approach is based on properties of partial rings of quotients of crossed products and their modules (see Proposition 4.1). In this way we also can obtain control theorems (Theorem 4.2).

Let G be a group, let k be a field and let M be a kG -module. The module M is said to be primitive if it is not induced from any kH-submodule for any subgroup $H < G$. The module M is said to be semiprimitive if it is not induced from any kH-submodule for any subgroup $H < G$ such that $|G : H| < \infty$. A representation φ of G over k is said to be primitive (semiprimitive) if the module of the representation φ.

In Section 5 we consider the structure of nilpotent FATR-group of nilpotency class 2 which admits faithful semiprimitive irreducible representations over a finitely generated field k such that $char\, k \notin Sp(G)$ (Theorem 5.1). The obtained result allows us to describe faithful irreducible representations of nilpotent minimax groups of nilpotency class 2 (see Theorem 5.2 and Corollary 5.3). These results raise the following question.

Question 1.1. *Let G be a torsion-free minimax nilpotent group of nilpotency class 2 and let k be a finitely generated field such that $char\, k \notin Sp(G)$. Suppose that the group G admits a faithful irreducible representation φ over k. Is it true that then there exist a subgroup N of G and an irreducible primitive representation ψ of the subgroup N over k such that the representation φ is induced from ψ and the quotient group $N/Ker\psi$ is finitely generated?*

2 Prime and maximal ideals in group rings of abelian groups

The following proposition is based on results of May [9, 10] on multiplicative groups of certain fields.

Proposition 2.1. *(Cf. [27, Proposition 3.1.1]) Let F be an algebraic closed field and let k be a finitely generated subfield of F. Let K be a subfield of F generated by k and all roots of unity. Then the multiplicative group K^* of the field K may be presented in the form $K^* = T \times A$, where T is divisible torsion abelian group and A is a free abelian group.*

Let F be a field, K be a subfield of F and X be a subset of F. Then $F(X)$ denotes a subfield of F generated by F and X and $F[X]$ denotes a subring of F generated by F and X. The next proposition can be considered as a generalization of [7, Chap. VIII, Theorem 13] to the case of infinitely dimensional extensions.

Proposition 2.2. *Let K be a subfield of a field F, and suppose that the field K contains all roots of unity. Let G be a subgroup of the multiplicative group F^* of the field F and let $H = K^* \cap G$. Suppose that the quotient group G/H is periodic and $\operatorname{char} K \notin \pi(G/H)$. Then $K(G) = K \otimes_{KH} KG = \oplus_{t \in T} Kt$, where T is a transversal to H in G.*

Proof. See [23, Proposition 2.1.1] and [16, Lemma 2]. □

If G is a group then $Sp(G)$ denotes the set of prime integers p such that the group G has an infinite p-section. The combination of results of Propositions 2.1 and 2.2 gives us the following theorem.

Theorem 2.3. *(Cf. [27, Theorem 3.1.4.]) Let F be a field and let k be a finitely generated subfield of F. Let G be a subgroup of finite total rank of the multiplicative group F^* of the field F such that $\operatorname{char} k \notin Sp(G)$. Then there exist a finitely generated subgroup H of G such that $k(G) = k(H) \otimes_{kH} kG = \oplus_{t \in T} k(H)t$ and $k[G] = k[H] \otimes_{kH} kG = \oplus_{t \in T} k[H]t$, where T is a transversal to the subgroup H in the group G.*

The above theorem allows us to characterize faithful irreducible representations of abelian groups of finite total rank over a finitely generated field.

Corollary 2.4. *(Cf. [27, Corollary 3.1.5]) Let k be a finitely generated field and let G be an abelian group of finite total rank such that $\operatorname{char} k \notin Sp(G)$. Then any faithful irreducible representation φ of the group G over the field k is induced from a representation of some finitely generated subgroup of G.*

The described above techniques allows us also to study faithful prime ideal in group rings of abelian groups.

Theorem 2.5. *(Cf. [27, Theorem 3.2.1]) Let k be a finitely generated field and let G be an abelian group of finite total rank such that $\operatorname{char} k \notin Sp(G)$. Let P be a faithful prime ideal of the group ring kG. Then:*

(i) *there exists a finitely generated subgroup H of G such that $P = (P \cap kH)kG$;*

(ii) *the ideal P is finitely generated;*

(iii) *the group ring kG meets the condition of maximality for faithful prime ideals.*

The next theorem shows that the finiteness of total rank of an abelian group G is a criterion when faithful prime ideals of kG are finitely generated, where k is a finitely generated field such that $char\, k \notin Sp(G)$.

Theorem 2.6. *(Cf. [27, Theorem 3.2.2]) Let k be a finitely generated field and let G be an abelian countable group such that $char\, k \notin Sp(G)$. Suppose that the group ring kG has a faithful prime ideal then:*

(i) *each faithful prime ideal of kG is finitely generated if and only if the group G has finite total rank;*

(ii) *if the group ring kG has a faithful maximal finitely generated ideal then the group G has finite total rank.*

3 Control subgroups

Let R be a ring and let M, X and Y be R-modules. The modules X and Y are said to be separated in M if X and Y have no nonzero isomorphic sections which are isomorphic to a submodule of M. We say that the module X is solid in M if X is uniform and M have no submodules which are isomorphic to a proper section of X. The modules X and Y are said to be similar if their injective hulls $[X]$ and $[Y]$ are isomorphic. The modules X and Y are similar if and only if they have isomorphic essential submodules.

Let G be a group and A be a normal subgroup of G. Let M and W be RA-modules. A subgroup $Sep_{(G,A)}(M, W) \leq G$ generated by all elements $g \in G$ such that kA-modules W and Wg are not separated in M is said to be the separator of W in G. Evidently, for any element $h \in G$ such that $h \notin Sep_{(G,A)}(M, W)$ modules W and Wh are separated in M.

By Lemma 3.2 of [3], the stabilizer $Stab_G[W] = \{g \in G \,|\, Wg\, and\, W\, are\, similar\}$ of W in G is a subgroup of G. The properties of $Stab_G[W]$ were studied in [2, 3]. It is easy to note that $Stab_G[W] \leq Sep_{(G,A)}(M, W)$, moreover, if the submodule W is solid in M then $Stab_G[W] = Sep_{(G,A)}(M, W)$.

The following proposition shows that the notion of the separator can be very useful for obtaining results about induced modules.

Proposition 3.1. *(Cf. [21, Lemma 3.1.4]) Let G be a group and let A be a normal subgroup of G. Let k be a field, M be a kG-module and a be a nonzero element of M. Let $S = Sep_{(G,A)}(akG, akA)$ then $akG = akS \otimes_{kS} kG$, and $U \cap akS \neq 0$ for any non-zero kG-submodule U of akG.*

Since $Stab_G[W] = Sep_{(G,A)}(M,W)$ if the submodule W is solid in M, we have the following assertion.

Corollary 3.2. *Let G be a group and let A be a normal subgroup of G. Let k be a field, M be a kG-module and W be a solid kA-submodule of M. Let $S = Stab_G[W]$ then $WkG = WkS \otimes_{kS} kG$, and $U \cap WkS \neq 0$ for any non-zero kG-submodule U of akG.*

Thus, $Sep_{(G,A)}(M,W)$ and $Stab_G[W]$ are control subgroups.

Let A be an abelian torsion-free group of finite rank acted by a group G , let k be a field and let I be an ideal of kA. A subgroup $S_G(I) \leq G$ which consists of all elements $g \in G$ such that $I \cap kB = I^g \cap kB$ for some finitely generated dense subgroup $B \leq A$ is said to be the standartizer of I in G (see [1]).

A subgroup $Sep_G(I) \leq S_G(I)$ generated by all elements $g \in G$ such that $Sp(I) \cap Sp(I^g) \neq \emptyset$, where $Sp(I)$ is the prime spectrum of the ideal I, is said to be the separator of I in G (see [18, 19]).

We say that a non-zero element a of a kA-module M has locally maximal annihilator $Ann_{kA}(a)$ in kA if for some dense finitely generated subgroup B of A the ideal $Ann_{kA}(a) \cap kB$ of kB is prime and $dim(Ann_{kB}(a)) \leq dim(Ann_{kB}(b))$ for any nonzero element $0 \neq b \in M$.

Proposition 3.3. *(Cf. [21, Lemma 3.1.3]) Let G be a group, A be an abelian normal torsion-free subgroup of finite rank of G. Let k be a field, M be a kG-module and let a be a non-zero element of M having locally maximal annihilator $Ann_{kA}(a)$ in k. Then $S_G(Ann_{kA}(a)) \geq Sep_G(Ann_{kA}(a)) \geq Sep_{(G,A)}(M, akA)$.*

It immediately follows from Propositions 3.1 and 3.3 that $Sep_G(Ann_{kA}(a))$ and $S_G(Ann_{kA}(a))$ are control subgroups. The described above controlling subgroups represent the main tool for studying inducibility of modules over group rings of soluble groups.

Corollary 3.2 shows that the structure of the module M strongly depends on properties of the module kS-module WkS which meets the condition $Stab_S[W] = S$. The following proposition show that the condition $Stab_S[W] = S$ has significant influence on the construction of the subgroup A.

Let A be an abelian torsion-free group acted by a group G, we consider A as an $\mathbb{Z}G$-module. We denote by $Soc_G\bar{A}$ the socle of the $\mathbb{Q}G$-module $\bar{A} = A \otimes_{\mathbb{Z}G} \mathbb{Q}G$ and put $Soc_G A = Soc_G\bar{A} \cap A$. It is not difficult to show that $Soc_G A$ is an isolated G-invariant subgroup of A.

Proposition 3.4. *(Cf. [26, Proposition 5.1]) Let A be a nilpotent normal non-abelian minimax torsion-free subgroup of a solvable-by-finite group G such that $r_0(G) < \infty$. Let K be a G-invariant subgroup of A such that the quotient group A/K is torsion-free abelian. Let k be a field such that $char\, k = 0$. Suppose that there is an uniform kA-torsion kA-module W such that $Stab_G[W] = G$ and the module W is kX-torsion-free for any proper G-invariant subgroup X of N such that $K \leq X$ and $r_0(X) < r_0(N)$. Then $Soc_G(A/K) = A/K$.*

4 Crossed products, partial rings of quotients and induced modules

A ring $R*G$ is called a crossed product of a ring R and a group G if $R \leq R*G$ and there is an injective mapping $\varphi^* : g \mapsto \bar{g}$ of the group G to the group of units $U(R*G)$ of the ring $R*G$ such that each element $a \in R*G$ can be uniquely presented as a finite sum $a = \sum_{g\in G} a_g \bar{g}$, where $a_g \in R$. The addition of two such sums is defined component-wise. The multiplication is defined by the formulas $\bar{g}\bar{h} = t(g,h)\overline{gh}$ and $r\bar{g} = \bar{g}(\bar{g}^{-1}r\bar{g})$, where $g, h \in G$,$r \in R$, $\bar{g}^{-1}r\bar{g} \in R$ and $t(g,h)$ is a unit of the ring R (see [11]).

If $a = \sum_{g\in G} a_g \bar{g} \in R*G$, then the set $Supp(a)$ of elements $g \in G$ such that $a_g \neq 0$ is called the support of the element a. Let H be a subgroup of the group G then the set of elements $a \in R*G$ such that $Supp(a) \subseteq H$ forms a crossed product $R*H$ contained in $R*G$. If the subgroup H is normal and P is a $\bar{G}$-invariant ideal of the ring $R*H$ then it is not difficult to verify that the quotient ring $R*G/PR*G$ is a crossed product $(R*H/P)*(G/H)$ of the quotient ring $R*H/P$ and the quotient group G/H. In particular, if RG is a group ring, H is a normal subgroup of the group G and P is a G-invariant ideal of the group ring RH then the quotient ring RG/PRG is a crossed product $(RH/P)*(G/H)$ of the quotient ring RH/P and the quotient group G/H. The last example shows the main way in which crossed products arise in studying of group rings and modules over them.

If V is a $R*H$-module then we can define the tensor product $V \otimes_{R*H} R*G$ which is a right $R*G$-module. The arguments of [6, Chap. 2, Corollary 1.2(i)] show that an $R*G$-module $M = V \otimes_{R*H} R*G$ if and only if $M = \oplus_{t\in T} V\bar{t}$, where T is a right transversal for H in G.

The following proposition shows how by passing to partial rings of quotients we can obtain Noetherian rings and Noetherian modules.

Proposition 4.1. *(Cf. [27, Proposition 4.1.1]) Let G be a group, D be a normal subgroup of G and R be a ring. Suppose that there exists a crossed product $R*G$ such that $R*D$ is an Ore domain. Then there exists a partial right ring of quotients $R*G(R*D)^{-1} = \{r \cdot s^{-1} | r \in R*G, s \in R*D\}$ and:*

(i) *$R*G(R*D)^{-1}$ is a crossed product $(R*D(R*D)^{-1})*(G/D) = \oplus_{t\in(G/D)}(R*D(R*D)^{-1})\bar{t} = (R*D(R*D)^{-1}) \otimes_{R*D} R*G$, where $R*D(R*D)^{-1}$ is a division ring;*

(ii) *if the quotient group G/D is polycyclic-by-finite then the ring $R*G(R*D)^{-1}$ is Noetherian;*

(iii) *if $M = aR*G$ is a cyclic $R*G$-module which is $R*D$-torsion-free then there exists a cyclic $R*G(R*D)^{-1}$-module $W = aR*G(R*D)^{-1}$ such that $M \leq W$ and $W = \{ms^{-1} | m \in M, s \in R*D\}$.*

We apply the above proposition to obtain the next control theorem in a situation where the group G has normal subgroups A and D such that $A \leq D$, a kG-module M

is annihilated by a maximal G-invariant faithful ideal P of kA and at the same time the module M is kD/PkD-torsion-free. So, we consider the situation where the module M is $\tilde{k} * \tilde{D}$-torsion-free, where $\tilde{k} = kA/P$ and $\tilde{D} = D/A$.

Theorem 4.2. *(Cf. [27, Theorem 4.1.5]) Let G a nilpotent $FART$-group and let D be a normal subgroup of G such that the quotient group G/D is polycyclic. Let k be a finitely generated field such that $\operatorname{char} k \notin Sp(G)$ and let M be a faithful kG-module. Suppose that the subgroup D contains an isolated abelian G-invariant subgroup A such that $P = Ann_{kA}(M)$ is a maximal G-invariant faithful ideal of kA. If the module M is kD/PkD-torsion-free then for any nonzero element $0 \neq a \in M$ there is a finitely generated subgroup $H \leq G$ such that $akG = akH \otimes_{kH} kG$.*

5 Primitive and semiprimitive modules

Certainly, primitive irreducible modules are a basic subject for investigations when we are dealing with induced modules and, naturally, the following question appears: what can be said on the construction of a group G if it admits a faithful primitive (or semiprimitive) irreducible representation over a field k? It should be noted that there are many results which show that the existence of a faithful irreducible representation of a group G over a field k may have essential influence on the structure of the group G (see for instance [13, 14, 15, 17, 24]).

In [4] Harper solved a problem raised by Zaleskii and proved that any not abelian-by-finite finitely generated nilpotent group has an irreducible primitive representation over a not locally finite field. In [22] we proved that if a minimax nilpotent group G of nilpotency class 2 has a faithful irreducible primitive representation over a finitely generated field of characteristic zero then the group G is finitely generated. In [5] Harper studied polycyclic groups which have faithful irreducible representations. It is well known that any polycyclic group is finitely generated soluble of finite rank and meets the maximal condition for subgroups (in particular, for normal subgroups). In [21] we showed that in the class of soluble groups of finite rank with the maximal condition for normal subgroups only polycyclic groups may have faithful irreducible primitive representations over a field of characteristic zero.

If G is a group then the FC-center $\Delta(G) = \{g \in G | \, |G : C_G(g)| < \infty\}$of G is a characteristic subgroup of G. In [5, Theorem A] Harper proved that if a polycyclic group G has a faithful primitive irreducible representation over a field k then $\Delta(G)$ is rather large in the sense that $\Delta(G) \cap H > 1$ for any subgroup $1 \neq H$ of G such that $|G : N_G(H)| < \infty$.

By Auslander theorem, any polycyclic group G is linear over the field and it is well known that the group G is finitely generated of finite rank. In [26] we study finitely generated linear (over a field of characteristic zero) groups of finite rank which have faithful irreducible primitive representations over a field of characteristic zero. We prove that if an infinite finitely generated linear group G of finite rank has a faithful irreducible primitive representation over a field of characteristic zero then $\Delta(G)$ is infinite (see [26, Theorem 6.1]).

The described in Sections 3 and 4 methods of obtaining of control theorem allows us to describe the structure of some nilpotent groups which admit faithful semiprimitive irreducible representations.

Theorem 5.1. *(Cf. [27, Theorem 5.1.5]) Let k be a finitely generated field and let G be a nilpotent FATR-group of nilpotency class 2 such that the torsion subgroup T of G is contained in the centre Z of G and $\mathrm{char}\, k \notin Sp(G)$. Suppose that the group G admits a faithful semiprimitive irreducible representation φ over the field k. Then the group G is finitely generated.*

The following theorem and corollary show that the structure of faithful irreducible representations of minimax nilpotent groups of nilpotency class 2 strongly depends on irreducible representations of finitely generated nilpotent groups.

Theorem 5.2. *(Cf. [27, Theorem 5.2.2]) Let G be a torsion free minimax nilpotent group of nilpotency class 2 and let k be a finitely generated field such that $\mathrm{char}\, k \notin Sp(G)$. Suppose that the group G admits a faithful irreducible representation φ over k. Then there exist a subgroup N of G and an irreducible semiprimitive representation ψ of the subgroup N such that the representation φ is induced from ψ and the quotient group $N/Ker\psi$ is finitely generated.*

Corollary 5.3. *(Cf. [27, Corollary 5.2.3]) Let G be a torsion-free minimax nilpotent group of nilpotency class 2 and let k be a finitely generated field such that $shar\, k = 0$. Suppose that the group G admits a faithful irreducible representation φ over k. Then there exist a subgroup N of G and an irreducible primitive representation ψ of the subgroup N over k such that the representation φ is induced from ψ and the quotient group $N/Ker\psi$ is finitely generated.*

Proof. The assertion follows from the above theorem and [25, Theorem 5.6]. □

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