Cubic equation revisited: part 1

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Abstract

Prior to revisiting cubic equation, we treat quadratic equation. Included herein are reviews on it, a root-finding algorithm, which is compared with the Newton's method, a tidbit about the Euler–Mascheroni constant, and so on.

1 Glossary

$a \in A$: a is a member of the set A .

$A := B$: A is defined as B.

$\overline{AB}$: line segment AB .

ALGOL: Algorithmic Language .

$\mathbb{C}$: the set of complex numbers .

CE: cubic equation .

CF: cubic function .

CI: constant of integration .

CP: characteristic polynomial .

$\mathbb{CP}$: complex projective space .

CQP: complex quadratic polynomial .

D: discriminant .

DE: differential equation .

$\det$ or $|\cdot|$: determinant of a matrix .

ECL: Embeddable Common Lisp .

EMC: Euler–Mascheroni constant .

env: envelope .

FS: Fibonacci sequence .

GIMP: GNU Image Manipulation Program .

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GR or φ: the golden ratio .

Hess: Hessian .

$\imath$: imaginary unit .

IF: inverse function .

iff: if and only if .

I_n: $n \times n$ identity matrix .

IP: inflection point .

LHS: left-hand side .

LMA: local maximum .

LMI: local minimum .

MR: multiple root .

MT: Möbius transformation .

NM: Newton's method .

O: the origin $(0,0)$ or $(0,0,0)$.

O_n: $n \times n$ null matrix .

QE: quadratic equation .

QED: *quod erat demonstrandum* .

QF: quadratic formula .

Qf: quadratic function .

QuE: quartic equation .

$\mathbb{R}$: the set of real numbers .

RHS: right-hand side .

$\mathbb{R}^n$: Euclidean space of dimension n .

SBCL: Steel Bank Common Lisp .

$\mathfrak{sl}(n, \mathbb{R})$: Lie algebra of special linear group of degree n over $\mathbb{R}$.

SM: symmetric matrix .

SVG: scalable vector graphics .

TL: tangent line .

TM: traceless matrix .

TP: tangent plane .

tr: trace .

TT: Tschirnhaus transformation .

UQF: univariated quadratic function .

VF: Vieta's formulas .

$Wron$: Wronskian .

wrt: with respect to .

$\mathbb{Z}$: the set of integers .

$\dagger$: conjugate transpose of a matrix .

$\Longleftrightarrow$: if and only if .

$\Longrightarrow$: implies .

$\oplus$: direct sum of matrices .

2 Introduction or QE as a propaedeutic to CE

Solving CE has been of some interest $[1-3]$. Operations such as addition, subtraction, multiplication, and so forth are employed for that purpose $[4]$. In this 'part 1', we review QE, which we regard as a propaedeutic to revisiting CE [1], and emphasise the role of recurrence related to QE. We twiddle with the following for a while.

$$ax^2 + bx + c = 0\ , \qquad a,\ b,\ c \in \mathbb{R}\ , \qquad a \neq 0. \tag{1}$$

3 By the way, what does solving QE mean?

We believe this is a 'deep' question that is worthy of tackling from a historical point of view. That said, unfortunately, what we can do for the moment seems making a 'not-so-deep' answer. To be specific, we barely remember an analogy helpful for solving second order DE's , which essentially identifies (1) with the DE $ax''(t) + bx'(t) + cx(t) = 0$, where the character $'$ stands for differentiation wrt t. Thereby, we learn that solving QE plays some role in getting solutions of such DE. This is our 'not-so-deep' answer at hand. Actually, since we have resorted to a certain kind of analogical reasoning, it is almost clear that we have failed to answer the question categorically. We thus think we have to literally detour, surveying QE. In other words, we are going to 'meander' around QE from various viewpoints in what follows.

3.1 QE in terms of CP

At the outset, CP of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$t^2 - (a+d)t + ad - bc\ .$$

Equating this with 0, one gets a QE wrt t. So we have shown that QE has something to do with CP. Next, as a special case of A, we consider the matrix

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[1] Since QE and CE *do* seem fairly common materials, we don't rule out the possibility that things described herein have already been mentioned or studied elsewhere (at least partially), while we are unaware.

$$B = \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right)^{2\ ,\ 3}.$$

Remark 3.1.1. $\mathrm{tr}(B) = 0$; we refer to such a property as ‘ traceless ness ’.

Remark 3.1.2. CP of B is given by $B(t) = t^2 - a^2 - bc$.

Incidentally, dividing the LHS of (1) by a [4], we get the monic polynomial $x^2 + \frac{b}{a}x + \frac{c}{a}$. If we subject it to TT, it becomes

$$x^2 + d^{\ 5}\ . \tag{2}$$

Since (2) is similar to $B(t)$ in that it is also the sum of the square of some variable and constant, we have shown the relevance of TT to ‘tracelessness’, for that matter.

3.2 QE in terms of D

We *a priori* consider

$$(\sqrt{a}x - t)^2 + y - ax^2 - bx - c = 0, \qquad a, b, c \in \mathbb{R}.$$

We expand its LHS to get

$$t^2 - 2\sqrt{a}xt + y - bx - c = 0. \tag{3}$$

Regarding (3) as a QE wrt t, one computes to get $D = (-2\sqrt{a}x)^2 - 4 \cdot 1 \cdot (y - bx - c) = 4(ax^2 + bx + c - y)$ [6] . By the way, it is known that D of a polynomial is 0 iff the polynomial has an MR . So taking this opportunity, we should like to prove the following.

Claim 3.2.1. QE has an MR $\Longleftrightarrow$ D of QE equals 0.

Proof.

$\Longrightarrow$: Since QE under consideration has an MR, letting A denote the MR, we can write it as $a(x - A)^2 = 0$, where $a \neq 0$. We expand the LHS of this equation to get $ax^2 - 2aAx + aA^2 = 0$. Then, D of this equation is $(-2aA)^2 - 4 \cdot a \cdot aA^2 = 0$.

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[2] By the way, the special linear Lie algebra of order n over a field $\mathbb{F}$, or $\mathfrak{sl}(n, \mathbb{F})$, consists of all $n \times n$ TM’s [5].

[3] B is involutory, if $a^2 + bc = 1$.

[4] This division is possible, because $a \neq 0$.

[5] See **Appendix 12.1** for computational details.

[6] See **Appendix 12.2** for an alternative.

$\Longleftarrow$: After some computation, the LHS of (1) becomes $a(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a}$. Since $D=b^2-4ac$, which amounts to 0, (1) eventually becomes $a(x+\frac{b}{2a})^2=0$, whose MR is $-\frac{b}{2a}$ [7]. QED [8].

As mentioned earlier, D of (3) is $4(ax^2+bx+c-y)$, which we equate with 0 to get the UQF

$$y=ax^2+bx+c. \tag{4}$$

The RHS of (4) is equal to the LHS of (1). In this way, QE can be related to D [9].

By the way, since D of (1) is b^2-4ac , we should like to raise a question about the relation

$$b^2=4ac. \tag{5}$$

Question 3.2.2. What does (5) mean at all?

This question looks as 'deep' as that raised in the beginning of this section. So we 'meander' again, mentioning a few viewpoints.

3.2.1 Deriving (5) from 2×2 matrices

We try to derive (5) from a few 2×2 matrices.

Example 3.2.1.1. We consider the following SM:

$$C=\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

Since $\det C$ equals $4ac-b^2$, one gets (5) by setting $\det C=0$ [10].

Remark 3.2.1.2. $C/2$ amounts to $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ [11], which reminds us of 'twos out' .

Example 3.2.1.3. We consider the following TM:

$$E\ ^{12}=\begin{pmatrix} b & -2a \\ 2c & -b \end{pmatrix}.$$

$\det E=-b^2+4ac$, which we set to be 0 to get (5).

Remark 3.2.1.4. If one rewrites E as $F=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, keeping the 'tracelessness' of E intact, one might be able to recall $\mathfrak{sl}(2,\mathbb{R})$ more easily.

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[7] *Cf.* here .

[8] As stated in footnote 1, we don't rule out the possibility that this proof is fairly common.

[9] Actually, there is another way to do this. *Cf.* footnote 6.

[10] In a sense, we have ' singular -ised' C.

[11] *Cf.* here .

[12] Here we use 'E' instead of 'D' to avoid possible confusion with discriminant. See **Glossary**.

3.2.2 Algorithmic derivation of (5)

Let us think somewhat algorithmically:

Step 1: *Differentiation*. We differentiate both sides of (1) wrt x to get

$$2ax + b = 0. \tag{6}$$

Step 2: *Solution*. Solving (6) for x yields

$$x = -\frac{b}{2a}. \tag{7}$$

Step 3: *Getting the relation under consideration*. We eliminate x between (1) and (7) and compute as follows.

$$\begin{aligned} & a \cdot (-\tfrac{b}{2a})^2 + b \cdot (-\tfrac{b}{2a}) + c = 0. \\ & \quad \downarrow \text{ Multiply both sides of the above by } 4a. \\ & b^2 - 2b^2 + 4ac = 0. \\ & \quad \downarrow \\ & b^2 = 4ac. \end{aligned}$$

Hence, we get (5).

3.2.3 Deriving (5) from *Hess*

We consider

$$h(x, y) = \frac{x^3}{6} + \frac{y^3}{6} + bxy, \qquad b \in \mathbb{R}. \tag{8}$$

Hess of (8) is

$$H^{13} = \begin{pmatrix} x & b \\ b & y \end{pmatrix}.$$

Since $\det H = xy - b^2$, substituting *e.g.,* $2a$ and $2c$ into x and y, respectively, yields $4ac - b^2$, which equals $\det C$.

Remark 3.2.3.1. Also thinkable are $(x, y) = (a, 4c)$, $(4a, c)$, etc.

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[13] We have skipped 'G', which will be later used to denote a group. See **10**.

3.2.4 Deriving (5) from $Wron$

Let $j_1(x) = ax^2 + bx + c$, $j_2(x) = dx + e$, where $a, b, c, d, e \in \mathbb{R}$ and $a, d \neq 0$. Then, let us consider the matrix J [14] $= \begin{pmatrix} j_1(x) & j_2(x) \\ j_1'(x) & j_2'(x) \end{pmatrix}$, where the character $'$ stands for differentiation wrt x. So $Wron$ of $j_1(x)$ and $j_2(x)$, or $Wron(j_1(x), j_2(x))$, is

$$\begin{aligned} \det J = \ & \det \begin{pmatrix} ax^2+bx+c & dx+e \\ 2ax+b & d \end{pmatrix} = (ax^2+bx+c)d - (dx+e)(2ax+b) \\ = & -adx^2 - 2aex - be + cd. \end{aligned} \tag{9}$$

Since $a, d \neq 0$, $-ad \neq 0$. So setting $\det J = 0$ gives a QE, and letting (9) amount to the LHS of (1), we get

$$\begin{cases} -ad = a, & (10) \\ -2ae = b, & (11) \\ -be + cd = c. & (12) \end{cases}$$

It follows from (10) that $a(d+1) = 0$. Since again, $a \neq 0$, we have $d = -1$, which we substitute into the LHS of (12) to get $-be + c \cdot (-1) = c$. That is,

$$be = -2c. \tag{13}$$

Since once again, $a \neq 0$, we can divide both sides of (11) by $-2a$ to get $e = -\frac{b}{2a}$. Eliminating e between this and (13) yields $b \cdot (-\frac{b}{2a}) = -2c$. One thus gets $b^2 - 4ac = 0$, *i.e.*, (5).

3.2.5 (5) as a necessary and sufficient condition

We consider the equation

$$ax^2 + bxy + cy^2 = 1, \qquad a, b, c \in \mathbb{R}, \qquad a > 0, \qquad c > 0 \tag{14}$$

and prove the following.

Claim 3.2.5.1. $b^2 = 4ac$ holds in (14) $\iff$ (14) is the equation of two parallel lines.

Proof. Regarding (14) as a QE wrt y and applying QF to it, one gets

$$y = \frac{-bx \pm \sqrt{(b^2 - 4ac)x^2 + 4c}}{2c}. \tag{15}$$

$\Longrightarrow$: Substituting $b^2 = 4ac$ into the RHS of (15), one gets $y = -\frac{bx}{2c} \pm \frac{1}{\sqrt{c}}$ [15] , which are geometrically two parallel lines.

$\Longleftarrow$: Since (14) is the equation of two parallel lines, we consider

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[14] We use 'J' instead of 'I' to avoid possible confusion with I_n. See **Glossary**.

[15] Since $c > 0$, this equation makes sense. See (14).

$$\begin{cases} \alpha x + \beta y + \gamma = 0, \\ \alpha\delta x + \beta\delta y + \epsilon = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\gamma\delta \neq \epsilon$ [16] , and their union

$$(\alpha x + \beta y + \gamma)(\alpha\delta x + \beta\delta y + \epsilon) = 0. \tag{16}$$

Expanding the LHS of (16) yields $\alpha^2\delta x^2 + 2\alpha\beta\delta xy + \beta^2\delta y^2 + \alpha(\epsilon + \gamma\delta)x + \beta(\epsilon + \gamma\delta)y + \gamma\epsilon = 0$, which we equate with (14) to get

$$\begin{cases} a = \alpha^2\delta, & (17) \\ b = 2\alpha\beta\delta, & (18) \\ c = \beta^2\delta, & (19) \\ 0 = \alpha(\gamma\delta + \epsilon), & (20) \\ 0 = \beta(\gamma\delta + \epsilon), & (21) \\ -1 = \gamma\epsilon. & (22) \end{cases}$$

Ignoring $(20) - (22)$ and looking at the RHS's of $(17) - (19)$, one gets the relation $(2\alpha\beta\delta)^2 = 4 \cdot (\alpha^2\delta) \cdot (\beta^2\delta)$. Now paying attention to their LHS's, one gets the relation $b^2 = 4ac$, or (5), as desired. QED [17] . Now that *Claim* 3.2.5.1 has been proven, we have shown that (5) can be a necessary and sufficient condition.

3.3 QE in terms of det

We observe that the LHS of (1) equals $\begin{vmatrix} \sqrt{a}x & \sqrt{-bx-c} \\ \sqrt{-bx-c} & \sqrt{a}x \end{vmatrix}$, other possibilities being $\begin{vmatrix} \sqrt{a}x & -\sqrt{-bx-c} \\ -\sqrt{-bx-c} & \sqrt{a}x \end{vmatrix}$, $\begin{vmatrix} -\sqrt{a}x & \sqrt{-bx-c} \\ \sqrt{-bx-c} & -\sqrt{a}x \end{vmatrix}$, and so forth. So we can say these $\det$'s are related to QE. Or more simply, we can consider $x^2 - g$ [18] , which amounts to *e.g.*, $\begin{vmatrix} x & \sqrt{g} \\ \sqrt{g} & x \end{vmatrix}$, $\begin{vmatrix} x & -\sqrt{g} \\ -\sqrt{g} & x \end{vmatrix}$, and so on. These remind us of real SM's. However, we can also consider the matrix

$$K = \begin{pmatrix} -x & \sqrt{g} \\ \sqrt{g} & -x \end{pmatrix}, \tag{23}$$

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[16] This condition precludes the occurrence of *e.g.*, the lines $x + 2y + 3 = 0$ and $2x + 4y + 6 = 2(x + 2y + 3)$ $= 0$, which are essentially the same. *Cf.* here .

[17] Like the proof of *Claim* 3.2.1, this proof might be fairly common. *Cf.* footnote 1.

[18] This is a kind of depressed polynomial .

$\det K$ being $x^2 - g$. And what if $g < 0$ and one puts $x = k\imath$, $k \in \mathbb{R}$, in (23)? In this case, K can be rewritten as

$$L = \begin{pmatrix} -k\imath & \sqrt{-\ell} \\ \sqrt{-\ell} & -k\imath \end{pmatrix} = \begin{pmatrix} -k\imath & \sqrt{\ell}\imath \\ \sqrt{\ell}\imath & -k\imath \end{pmatrix},$$

where $\ell > 0$. This heralds $\mathbb{C}$ [19] , and since $L^\dagger = -L$, L is a skew-Hermitian matrix .

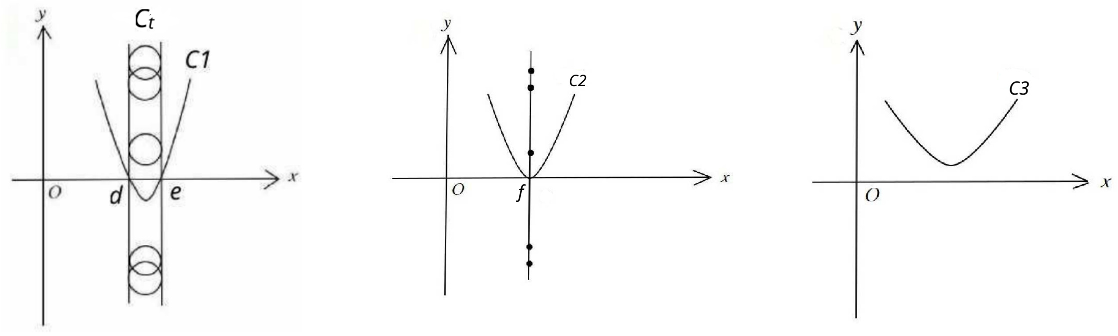
3.4 QE in terms of *env*

Putting $y = 0$ in (3), one gets

$$t^2 - 2\sqrt{a}xt - bx - c = 0. \tag{24}$$

The equation of *env* of (24) is $(-2\sqrt{a}x)^2 - 4(-bx - c) = 0$, that is, $4ax^2 + 4bx + 4c = 0$. This is essentially the same as $ax^2 + bx + c = 0$, (1). So (1) is found to virtually match such *env*, and we can say QE has something to do with *env*.

Taking this opportunity, we make *env*-based classification of the UQF $y = ax^2 + bx + c$ as shown below.



(a) (b) (c)

Fig. 1. *env*-classification of UQF. (a) Circles whose *env*'s are the lines $x = d$ and $x = e$ [20] . These lines are related to *C1*: $y = a(x - d)(x - e)$ [21] , [22]. As t changes, the family of circles $C_t : (x - \frac{d+e}{2})^2 + (y - t)^2 = (\frac{e-d}{2})^2$ moves vertically along *env*'s. (b) As d gets closer to e (or vice versa), the circles in (a) shrink to form dots. And two *env*'s fuse to become the line $x = f$. We relate this to *C2*: $y = a(x - f)^2$ [23] . (c) No *env* exists; this is related to *C3* that is devoid of real roots.

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[19] Actually, we will encounter $\mathbb{C}$. See *e.g.*, (32).

[20] Diameter of each circle is $e - d$.

[21] Here we implicitly assume that a, d, $e \in \mathbb{R}$.

[22] This is the factored form .

[23] This is the vertex form .

Remark 3.4.1. Since $a \neq 0$ [24] , we are able to rewrite $D = b^2 - 4ac$ as $a^2\{(\frac{b}{a})^2 - 4 \cdot \frac{c}{a}\}$. Let α and β be the roots of (1). Then, it follows from VF that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. So we get $D = a^2[\{-(\alpha+\beta)\}^2 - 4\alpha\beta] = a^2(\alpha - \beta)^2$. Now regarding α and β as d and e in Fig. 1(a), respectively, we realise that Fig.'s 1(a), 1(b), and 1(c) each correspond to the cases $D > 0$, $D = 0$, and $D < 0$.

Example 3.4.2. We compute *env* of the unit circle whose centre moves on the line $y = x$. The line being parametrized as $(x, y) = (u, u)$, $u \in \mathbb{R}$, we consider the family of circles

$$C_u : (x-u)^2 + (y-u)^2 = 1, \tag{25}$$

which is regarded as a QE wrt u. After some expansion and rearrangement, (25) becomes

$$2u^2 - 2(x+y)u + x^2 + y^2 - 1 = 0. \tag{26}$$

$\frac{D}{4}$ [25] of (26) amounts to

$$(x+y)^2 - 2(x^2+y^2-1) = -(x-y)^2 + 2. \tag{27}$$

Setting the RHS of (27) to be 0, we get the *env* $y = x \pm \sqrt{2}$.

3.5 QE in terms of *Hess*

We consider

$$m(x,y) = \frac{ax^4}{12} + \frac{bx^3}{6} + \frac{cx^2}{2} + dx + e + \frac{y^2}{2}, \qquad a, b, c, d, e \in \mathbb{R}. \tag{28}$$

Hess of (28) is

$$M = \begin{pmatrix} ax^2 + bx + c & 0 \\ 0 & 1 \end{pmatrix}.$$

Computing $\det M$, one gets the LHS of (1). So we can say QE has something to do with *Hess*.

3.6 QE in terms of IF

Let us divide both sides of (1) by a [26] . We then get

$$x^2 + \gamma x + \delta = 0, \qquad \text{where } \gamma = \frac{b}{a},\ \ \delta = \frac{c}{a}. \tag{29}$$

(29) is rewritten as $-\gamma x = x^2 + \delta$ [27] . Swapping x in its LHS with y yields

$$-\gamma y = x^2 + \delta. \tag{30}$$

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[24] See (1).

[25] On the other hand, we set $D = 0$ in *env*-computation of (24).

[26] *Cf.* footnote 4.

[27] The RHS of this equation is a kind of depressed polynomial. *Cf.* footnote 18.

The IF of (30) is obtained by swapping x in its RHS with y in its LHS. That is, one considers

$$-\gamma x = y^2 + \delta. \tag{31}$$

Trying to solve the system composed of (30) and (31), we subtract (31) from (30) to get $\gamma(x-y) = x^2 - y^2$, which we rearrange and factor as $(x-y)(x+y-\gamma) = 0$. We thus get $y = x$ and $y = -x + \gamma$. Here we notice that by substituting the former into (31), one can let it 'revert' to (29), a QE wrt x. Hence, we have shown some relationship between IF and QE.

Example 3.6.1. Putting $\gamma = -1$ and $\delta = 1$ in (30) and (31), we consider the following system of equations.

$$\begin{cases} y = x^2 + 1, \\ x = y^2 + 1. \end{cases}$$

We get the QE $x^2 - x + 1 = 0$ by replacing y by x in each equation.

3.7 QE in terms of MT

Consider an MT

$$f(z) = \frac{az+b}{cz+d}, \qquad z \in \mathbb{C}, \qquad ad - bc \neq 0, \tag{32}$$

and its fixed points . Then, it follows from (32) that $\frac{az+b}{cz+d} = z$. That is, we consider

$$cz^2 + (d-a)z - b = 0. \tag{33}$$

If $c = 0$, (33) becomes $(d-a)z - b = 0$, a linear equation wrt z [28] , which is not a QE. So we assume $c \neq 0$ to regard (33) as a QE wrt z, thereby showing some relationship between MT and QE.

Remark 3.7.1. Let us further assume $ad - bc = 1$ [6]. Then,

$$D \text{ of } (33) = (d-a)^2 + 4bc = (d-a)^2 + 4(ad-1) = (a+d)^2 - 4.$$

So it seems natural that the following classification should arise [29] .

$$|a+d| \begin{cases} > 2, \\ = 2, \\ < 2 \text{ [6]}. \end{cases}$$

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[28] If $a = d$, b needs to amount to 0, and the equation becomes $0 = 0$, *i.e.*, trivial.

[29] *Cf.* here .

3.8 QE in terms of *SING* [7]

Consider *e.g.*, the QE $x^2 - 1 = 0$. On solving it, one gets $x = \pm 1$. Setting $\phi = x^2 - 1 = 0$, one computes $\frac{d\phi}{dx} = \frac{d}{dx}(x^2 - 1) = 2x$ to get $\omega = d\phi = 2xdx$. Thus, the *SING* is $x = 0$. We note that it is the midpoint between 1 and -1 in $\mathbb{R}^1$, which we classify into the category **IN** and visualise as follows.

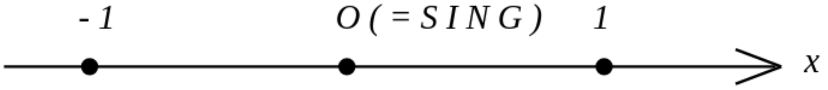

Fig. 2. *SING* as the midpoint [30]

Drawing the lines $x = -1$, 0, 1 in $\mathbb{R}^2$ yields the following:

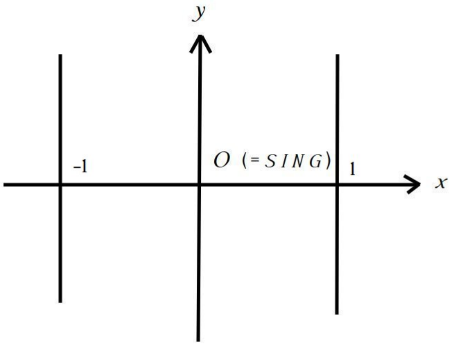

Fig. 3. Fig. 2 '$\mathbb{R}^2$-ised' [31], [32]

Remark 3.8.1. If we treat a Qf $y = x^2 - 1$ instead of the QE, its *SING* is grouped into **NO** [33] .

We conclude our 'various and sundry reviews' on QE here. For better or worse, it seems that via such reviews, we have obtained some clues to *Question* 3.2.2, on which we will elaborate.

4 Possible answers to *Question* 3.2.2

As stated earlier, we don't think we are totally clueless about *Question* 3.2.2 around which we used to merely 'meander'; we now try to answer it from a few viewpoints.

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[30] An SVG file was converted to an EPS file, which was included in a usual LaTeX manner .

[31] Terms like '$\mathbb{R}^2$-ise', '$\mathbb{R}^2$-ising', etc. will sometimes be used for describing similar procedures.

[32] '*O*' in this figure can be interpreted as the origin $(0, 0)$.

[33] *Cf.* [7, **3.3**].

4.1 Answer from a viewpoint of CP

Consider *e.g.*, CP of $\begin{pmatrix} b & -a \\ c & 0 \end{pmatrix}$, which is $t^2 - bt + ac$. Its roots are $\frac{b \pm \sqrt{b^2-4ac}}{2}$, and setting $b^2 = ac$ gives the MR $\frac{b}{2}$. Since the roots of the characteristic equation are eigenvalues , our response is

Answer 4.1.1. (5) is related to uniqueness of eigenvalue of a certain matrix.

Remark 4.1.2. CP of $\begin{pmatrix} b & a \\ -c & 0 \end{pmatrix}$, $\begin{pmatrix} b & a\imath \\ c\imath & 0 \end{pmatrix}$, etc is also $t^2 - bt + ac$.

4.2 Answer from a viewpoint of integration

Our idea is just integrate $b^2 - 4ac$ that comes from (5) wrt b. That is, we consider

$$\int b^2 - 4ac \, db, \tag{34}$$

which yields

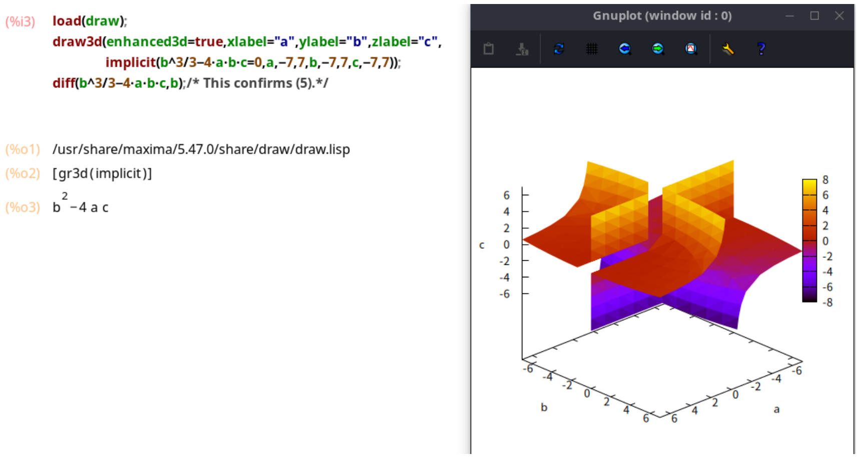
$$\frac{b^3}{3} - 4abc + \mathrm{CI}. \tag{35}$$

Hence,

Answer 4.2.1. (5) is a partial derivative of (35).

Remark 4.2.2. Partial differentiation of *e.g.*, $\frac{a^3}{3} + \frac{b^3}{3} - 4abc$ wrt b also yields the integrand of (34). So other possibilities than (35) are also thinkable.

We plot (35), regarding (a, b, c) as coordinates and ignoring its CI, for that matter:



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(%i3)   load(draw);
        draw3d(enhanced3d=true,xlabel="a",ylabel="b",zlabel="c",
                implicit(b^3/3–4·a·b·c=0,a,–7,7,b,–7,7,c,–7,7));
        diff(b^3/3–4·a·b·c,b);/* This confirms (5).*/


(%o1)   /usr/share/maxima/5.47.0/share/draw/draw.lisp
(%o2)   [gr3d(implicit)]
         2
(%o3)   b  −4 a c
```

Fig. 4. Visualisation of $\frac{b^3}{3} - 4abc = 0$ [34]

---

[34] wxMaxima ver. 24.02.1 is used for this kind of visualisation, and yielded images are sometimes edited for the sake of simplicity.

4.3 Answer from a viewpoint of ratio

Inspired by Fig. 2, we imagine the following points and the number line.

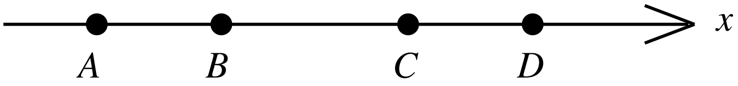


Fig. 5. Four points on the number line [35]

In the above Fig., we set $\overline{AC} = \overline{BD} = b$, $\overline{BC} = 2a$, $\overline{AD} = 2c$. Then, $\frac{\overline{AC} \cdot \overline{BD}}{\overline{BC} \cdot \overline{AD}} = \frac{b \cdot b}{2a \cdot 2c} = \frac{b^2}{4ac}$. Equating this ratio with 1, we get $\frac{b^2}{4ac} = 1$, that is, $b^2 = 4ac$. Hence,

Answer 4.3.1. (5) is related to a certain ratio.

4.4 Answer from a viewpoint of *SING* [7]

We consider

$$\phi = x^3 + y^3 + z^3 - 12xyz = 0 \tag{36}$$

to compute

$\frac{d\phi}{dx} = \frac{d}{dx}(x^3 + y^3 + z^3 - 12xyz) = 3x^2 + 3y^2 \frac{dy}{dx} + 3z^2 \frac{dz}{dx} - 12yz - 12zx \frac{dy}{dx} - 12xy \frac{dz}{dx}$.

We thus get $\omega = d\phi = 3x^2dx + 3y^2dy + 3z^2dz - 12yzdx - 12zxdy - 12xydz =$

$$3(x^2 - 4yz)dx + 3(y^2 - 4zx)dy + 3(z^2 - 4xy)dz. \tag{37}$$

Equating (37) with 0, one gets the following [36] .

$$\begin{cases} x^2 - 4yz = 0, & (38) \\ y^2 - 4zx = 0, & (39) \\ z^2 - 4xy = 0. & (40) \end{cases}$$

Replacing x, y, z in (39) by a, b, c, respectively yields $b^2 - 4ca = 0$, which is essentially the same as (5). Hence,

Answer 4.4.1. (5) is related to *SING* computation.

Remark 4.4.2. Replacing x, y, z in (38) by b, c, a, respectively also gives (5).

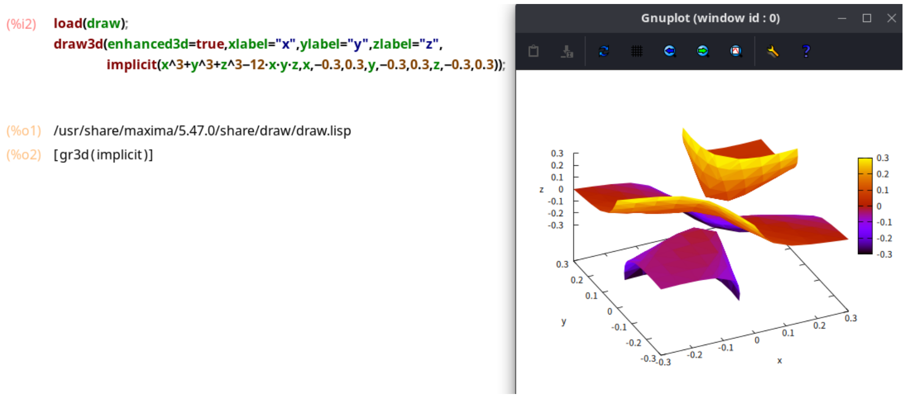
Remark 4.4.3. Replacing x, y, z in (40) by c, a, b, respectively also gives (5).

Like Fig. 4, we plot (36):

---

[35] Like Fig. 2, this comes from an SVG file.

[36] *Cf.* [7, (21) – (23)].



```
(%i2) load(draw);
      draw3d(enhanced3d=true,xlabel="x",ylabel="y",zlabel="z",
             implicit(x^3+y^3+z^3−12·x·y·z,x,−0.3,0.3,y,−0.3,0.3,z,−0.3,0.3));

(%o1) /usr/share/maxima/5.47.0/share/draw/draw.lisp
(%o2) [gr3d(implicit)]
```

Fig. 6. Visualisation of (36)

Taken together, we have answered the question of what (5) means in a few ways in this section.

5 Emphasising the role of recurrence

Inspired by **3.2.2**, we deal with algorithms and recurrences in what follows. To be specific, we put an emphasis on the role of recurrence related to QE.

5.1 Relationship between recurrence and TT

At the outset, we get interested in whether TT we touched upon in **3.1** is related to recurrence; we consider the following recurrences.

Recurrence 1

$$a_1 = 1, \qquad a_{n+1} = xa_n + 1, \qquad \text{for } n = 1, 2, 3 \ldots. \tag{41}$$

Recurrence 2

$$a_1 = 1, \qquad a_{n+1} = xa_n - 1, \qquad \text{for } n = 1, 2, 3 \ldots. \tag{42}$$

Remark 5.1.1. Explicitly, *Recurrence* 1 is $a_1 = 1$, $a_2 = x + 1$, $a_3 = x^2 + x + 1$, $a_4 = x^3 + x^2 + x + 1 \ldots$.

Remark 5.1.2. Explicitly, *Recurrence* 2 is $a_1 = 1$, $a_2 = x - 1$, $a_3 = x^2 - x - 1$, $a_4 = x^3 - x^2 - x - 1 \ldots$.

We generalise (41) and (42) slightly.

$$a_1 = 1, \qquad a_{n+1} = xa_n + b_n, \qquad \text{for } n = 1, 2, 3 \ldots, \tag{43}$$

where b_n is some constant.

Remark 5.1.3. (41) and (42) correspond to the cases where b_n in (43) is replaced by 1 and -1, respectively.

Now we are ready to consider two cases, depending on whether $b_1 = 0$.

Case 1. $b_1 = 0$. Schematically,

$$\begin{gathered} 1 \\ \downarrow \\ x + 0\ (= b_1) \\ \downarrow \\ x^2 + b_2 \\ \downarrow \\ x^3 + b_2 x + b_3 \\ \downarrow \\ x^4 + b_2 x^2 + b_3 x + b_4 \\ \downarrow \\ \ldots. \end{gathered}$$

Case 2. $b_1 \neq 0$. Schematically,

$$\begin{gathered} 1 \\ \downarrow \\ x + b_1 (\neq 0) \\ \downarrow \\ x^2 + b_1 x + b_2 \\ \downarrow \\ x^3 + b_1 x^2 + b_2 x + b_3 \\ \downarrow \\ \ldots. \end{gathered}$$

Since we see a train of depressed polynomial s in *Case* 1, we have shown that TT has something to do with recurrence.

5.2 From "ϕ" to φ

We once mentioned "ϕ"-curves [8], from which we now derive φ:

$$x^2 + xy + y^2 - 1 = 0, \text{ a part of "}\phi\text{"-curve } [8, (2)]$$

$$\downarrow \text{ Rearrangement}$$

$$x^2 + xy + y^2 = 1$$

$\downarrow$ Dilation by replacing x, y with $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, respectively

$$x^2 + xy + y^2 = 2$$

$\downarrow$ Replacement of y by -1

$$x^2 - x + 1 = 2$$

$\downarrow$ Rearrangement

$$x^2 - x - 1 = 0. \tag{44}$$

It is known that the solutions to (44) are φ and $-\frac{1}{\varphi}$. We thus derived φ from "ϕ" [37] , recalling the Fibonacci recurrence .

$$a_{n+2} = a_{n+1} + a_n. \tag{45}$$

Remark 5.2.1. The LHS of (44) coincides with a_3 in (42).

5.3 Coming across a root-finding algorithm

Speaking of φ, we have

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}^{38} \ [9]. \tag{46}$$

Inspired by this infinitely nested radical , we get the following idea.

We first twiddle with (1) to get

$$x^2 = -\frac{b}{a}x - \frac{c}{a}.^{39} \tag{47}$$

Then, we identify (47) with the recurrence

$$x_{n+2} = -\frac{b}{a}x_{n+1} - \frac{c}{a}x_n, \qquad n = 0, 1, 2, 3 \ldots. \tag{48}$$

Remark 5.3.1. If one puts *e.g.*, $a = -1$, $b = c = 1$ in (47), one gets (44).

---

[37] See **Appendix 12.3** for an alternative.

[38] See **Appendix 12.4** for a somewhat detailed explanation on why this holds.

[39] Since $a \neq 0$, we can divide both sides of $ax^2 = -bx - c$ by a and get (47). *Cf.* (1).

Remark 5.3.2. If one puts *e.g.*, $a = -1$, $b = c = 1$ in (48), one gets the recurrence $x_{n+2} = x_{n+1} + x_n$, which is essentially the same as (45).

We now compute X_1, or one root of (1), as follows:

$$X_1 = \sqrt{-\tfrac{c}{a} - \tfrac{b}{a} \cdot \sqrt{-\tfrac{c}{a} - \tfrac{b}{a} \cdot \sqrt{\cdots}}}.$$

What about X_2, another root of (1)? Since we have $X_1 + X_2 = -\frac{b}{a}$, and $X_1 X_2 = \frac{c}{a}$ due to VF , we can later get X_2 by computing $-\frac{b}{a} - X_1$, etc. And we present an example:

Example 5.3.3. $a = -1$, $b = 1$, and $c = 72$. One gets $X_1 = \sqrt{72 + \sqrt{72 + \sqrt{72 + \cdots}}}$.

The following algorithm describes the procedures to get X_1.

Step 1: *Preparation.* Set $x_0 = \sqrt{72} = 8.485\ldots$.

Step 2: *Computation.* Compute $\sqrt{72 + x_0}$ and get $x_1 = \sqrt{72 + 8.485\ldots} = 8.971\ldots$.

Step 3: *Iteration.* Likewise, repeat computing $x_{n+1} = \sqrt{72 + x_n}$ for $n = 1, 2, 3\ldots$.

We now verify *Example* 5.3.3 and each step using Clojure [40, 41] .

Verification 5.3.4.

```
% echo $0
/usr/bin/tcsh
% tcsh --version
tcsh 6.24.13 (Astron) 2024-06-12 (x86_64-unknown-linux)
options wide,nls,dl,al,kan,sm,rh,nd,color,filec
% more -V
more from util-linux 2.40.2
% more cubic_eq_part_1_v_5_3_4.clj
(defn v_5_3_4[i]
(if(< i 2)(Math/sqrt 72)
(Math/sqrt(+ 72(v_5_3_4(dec i))))))
(doseq[x(range 1 22)]
(prn(v_5_3_4 x)))
% clojure -e '(clojure-version)'
"1.11.2"
```

---

[40] We perform our computations on 8-core AMD processors of an Ubuntu 24.10 machine with 64 gigabytes of RAM .

[41] Terminal output is sometimes edited for the sake of simplicity. For instance, Erlang output in **6.1.1** is not always the same as the original one.

```
% time clojure cubic_eq_part_1_v_5_3_4.clj
8.48528137423857
8.971358948021118
8.998408689764048
8.999911593441574
8.99999508852319
8.999999727140173
8.999999984841121
8.99999999915784
8.999999999953213
8.999999999997401
8.999999999999856
8.999999999999993
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
1.229u 0.127s 0:00.53 252.8% 0+0k 976+0io 0pf+0w
```

Once the computation reaches '9.0', repetition of the computation $\sqrt{72 + 9.0} = \sqrt{72 + 9.0} = \ldots$ just continues. So we can say the computation has converged to the value 9.0; we have thus got $X_1 = 9$. Consequently, $X_2 =$ [42] $- \frac{b}{a} - X_1 = -\frac{1}{-1} - 9 = 1 - 9 = -8$. Alternatively, $X_2 =$ [43] $\frac{c}{a} \cdot \frac{1}{X_1} = \frac{72}{-1} \cdot \frac{1}{9} = -8$. In either case, we obtain the values 9 and -8, or the roots of the QE $-x^2 + x + 72 = 0$ [44] .

We will deal with CE's in a full-blown manner in 'part 2'; we let CF's herald them in what follows, approximating numerical data we obtained by CF's, and make a definition.

Definition 5.3.5. The last output in a data set is 'end point' (EP) [45] .

Example 5.3.6. EP in *Verification* 5.3.4 is 'final 9.0' that is immediately before `1.229u`.

First, we visualise the numerical data we have got as follows.

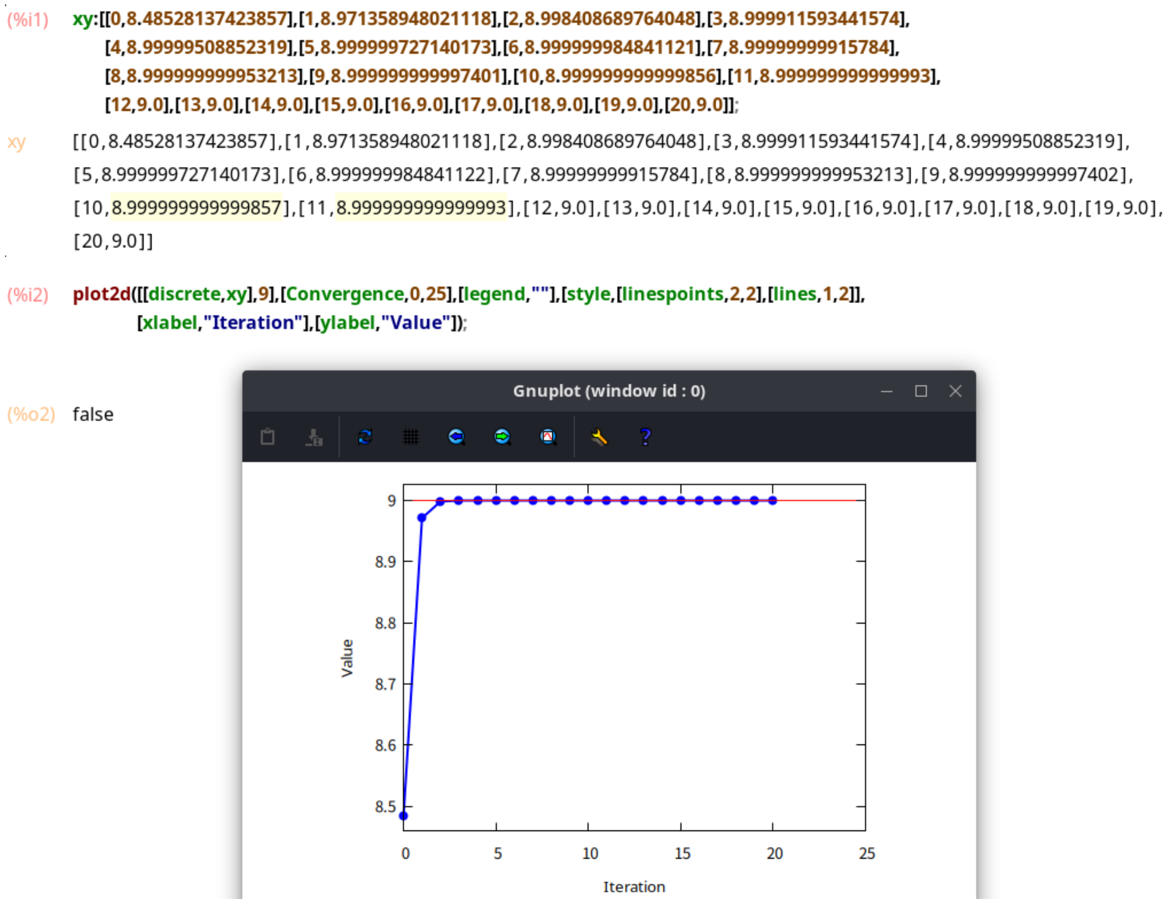
---

[42] We have used VF.

[43] Ditto.

[44] We will check these in **6.2**.

[45] Unfortunately, we aren't sure whether the term EP is widely accepted.



```
(%i1) xy:[[0,8.48528137423857],[1,8.971358948021118],[2,8.998408689764048],[3,8.999911593441574],
        [4,8.99999508852319],[5,8.999999727140173],[6,8.999999984841121],[7,8.99999999915784],
        [8,8.999999999953213],[9,8.999999999997401],[10,8.999999999999856],[11,8.999999999999993],
        [12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]];
xy      [[0,8.48528137423857],[1,8.971358948021118],[2,8.998408689764048],[3,8.999911593441574],[4,8.99999508852319],
        [5,8.999999727140173],[6,8.999999984841122],[7,8.99999999915784],[8,8.999999999953213],[9,8.999999999997402],
        [10,8.999999999999857],[11,8.999999999999993],[12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],
        [20,9.0]]

(%i2) plot2d([[discrete,xy],9],[Convergence,0,25],[legend,""],[style,[linespoints,2,2],[lines,1,2]],
            [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2) false
```

Fig. 7. Visualisation of numerical outputs in *Verification* 5.3.4. Convergence is emphasised by a red line.

Then, we make another definition.

Definition 5.3.7. The line $y =$ EP in the Cartesian plane is ‘convergence line’ (CL)[46].

Example 5.3.8. CL in Fig. 7 is the red line.

We try to let EP and CL match IP and TL, respectively. See below for a schematic illustration.

---

[46] We aren’t sure whether the term CL is widely accepted, either.

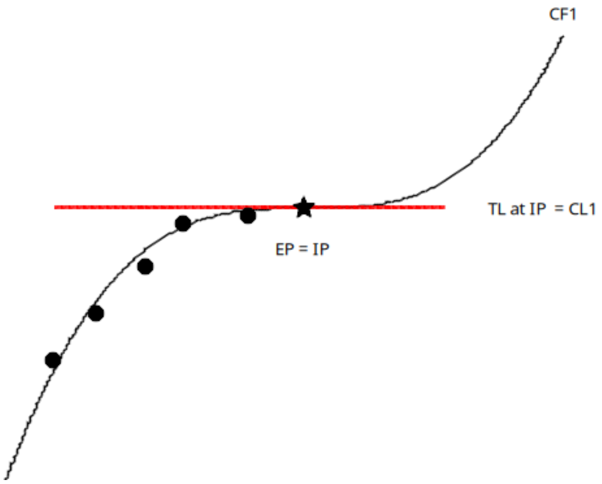
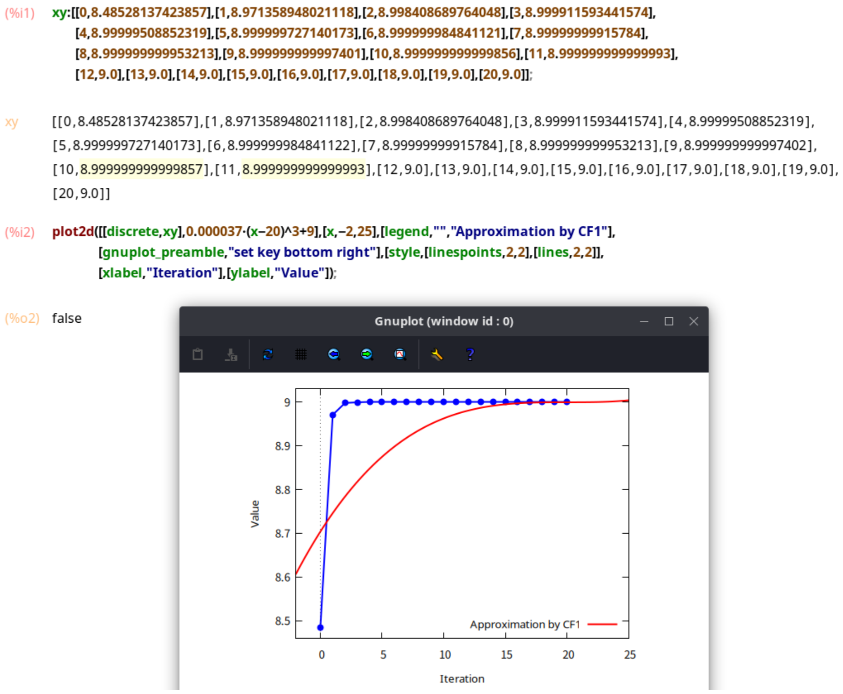


Fig. 8. Schematic approximation of program outputs using a CF [47], [48]. Outputs and EP are indicated by black dots and a black star, respectively. Original wxMaxima image was retouched by using GIMP ver. 2.10.38.

A concrete example is shown below.



```
(%i1) xy:[[0,8.48528137423857],[1,8.971358948021118],[2,8.998408689764048],[3,8.999911593441574],
        [4,8.99999508852319],[5,8.999999727140173],[6,8.999999984841121],[7,8.99999999915784],
        [8,8.999999999953213],[9,8.999999999997401],[10,8.999999999999856],[11,8.999999999999993],
        [12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]];

xy    [[0,8.48528137423857],[1,8.971358948021118],[2,8.998408689764048],[3,8.999911593441574],[4,8.99999508852319],
      [5,8.999999727140173],[6,8.999999984841122],[7,8.99999999915784],[8,8.999999999953213],[9,8.999999999997402],
      [10,8.999999999999857],[11,8.999999999999993],[12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],
      [20,9.0]]

(%i2) plot2d([[discrete,xy],0.000037·(x−20)^3+9],[x,−2,25],[legend,"","Approximation by CF1"],
            [gnuplot_preamble,"set key bottom right"],[style,[linespoints,2,2],[lines,2,2]],
            [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2) false
```

Fig. 9. Approximation of numerical data plotted in Fig. 7 by CF1 $y = 0.000037(x-20)^3 + 9$

But what if we appeal to the (celebrated) NM? Tailoring the recurrence

---

[47] *Cf.* [10, Fig. 4.6(b)] and [11, Fig. 4.16].

[48] For other CF-related figures, see Fig. 11 and **Appendix 12.5**.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{49}$$

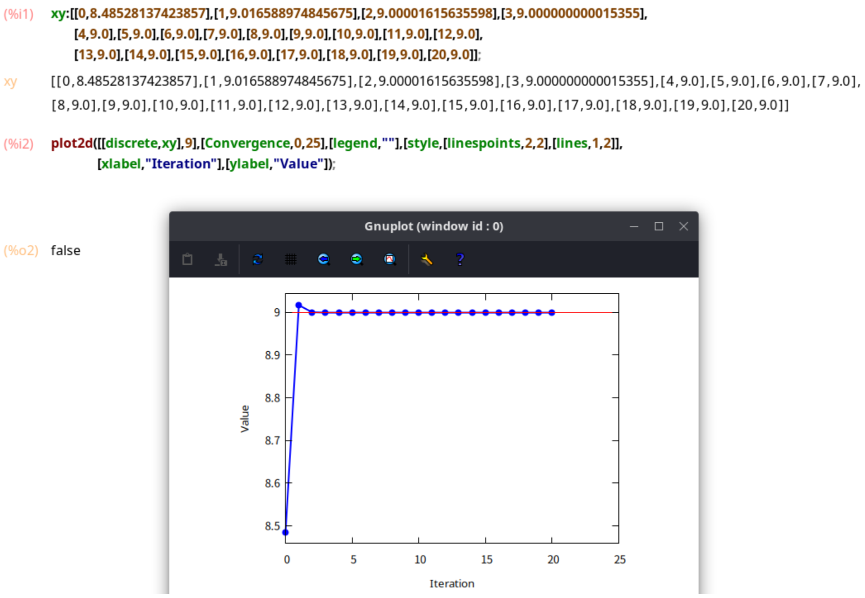
used in NM to fit *Example* 5.3.3, we get

$$x_{n+1} = x_n - \frac{x_n^2 - x_n - 72}{2x_n - 1} = \frac{x_n^2 + 72}{2x_n - 1}. \tag{50}$$

Verification 5.3.9. We write and run another clojure code which reflects (50):

```
% more cubic_eq_part_1_v_5_3_9.clj
(defn v_5_3_9[i]
(if(< i 2)(Math/sqrt 72)
(/(+(*(v_5_3_9(dec i))
(v_5_3_9(dec i)))72)
(-(* 2(v_5_3_9(dec i)))1))))
(doseq[x(range 1 22)]
(prn(v_5_3_9 x)))
% time clojure cubic_eq_part_1_v_5_3_9.clj
8.48528137423857
9.016588974845675
9.00001615635598
9.000000000015355
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
9.0
40.447u 0.532s 0:39.71 103.1% 0+0k 83336+0io 184pf+0w
```

Thus, we get $X_1 = 9$ also by NM and likewise visualise the data:



```
(%i1)	xy:[[0,8.48528137423857],[1,9.016588974845675],[2,9.00001615635598],[3,9.000000000015355],
	  [4,9.0],[5,9.0],[6,9.0],[7,9.0],[8,9.0],[9,9.0],[10,9.0],[11,9.0],[12,9.0],
	  [13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]];
xy	[[0,8.48528137423857],[1,9.016588974845675],[2,9.00001615635598],[3,9.000000000015355],[4,9.0],[5,9.0],[6,9.0],[7,9.0],
	[8,9.0],[9,9.0],[10,9.0],[11,9.0],[12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]]

(%i2)	plot2d([[discrete,xy],9],[Convergence,0,25],[legend,""],[style,[linespoints,2,2],[lines,1,2]],
	    [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2)	false
```

Fig. 10. Visualisation of numerical outputs in *Verification* 5.3.9. Convergence is emphasised by a red line.

We approximate the data by another CF, CF2. See below for a schematic illustration.

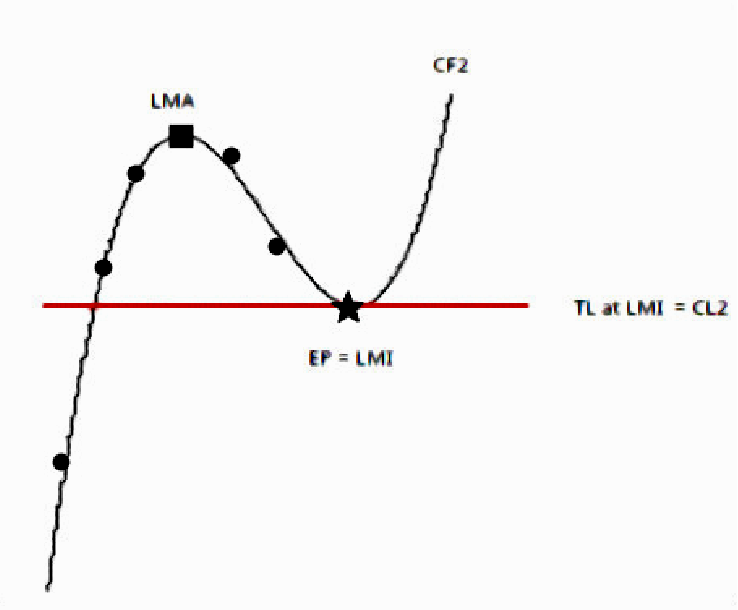
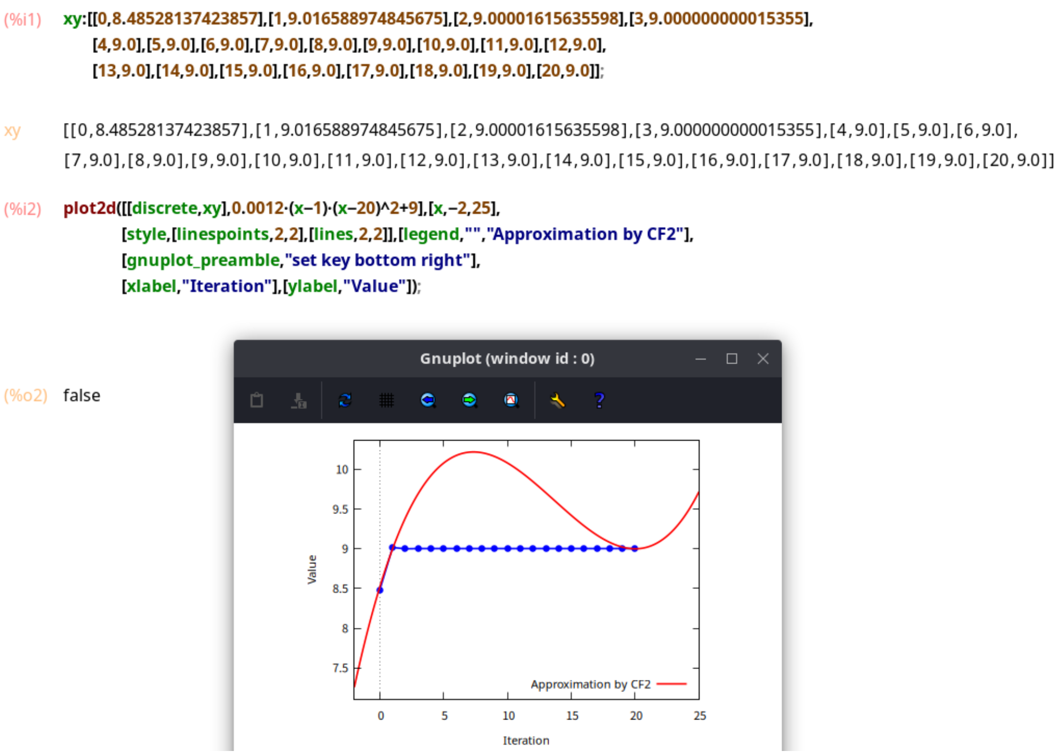


Fig. 11. Schematic approximation of program outputs using another CF. Each output is indicated like Fig. 8, LMA being indicated by a black square [49] . Original wxMaxima image was retouched like Fig. 8.

---

[49] Some might recall [10, Fig. 4.6(a)].

A concrete example is shown below.



```
(%i1)  xy:[[0,8.48528137423857],[1,9.016588974845675],[2,9.00001615635598],[3,9.000000000015355],
          [4,9.0],[5,9.0],[6,9.0],[7,9.0],[8,9.0],[9,9.0],[10,9.0],[11,9.0],[12,9.0],
          [13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]];

xy     [[0,8.48528137423857],[1,9.016588974845675],[2,9.00001615635598],[3,9.000000000015355],[4,9.0],[5,9.0],[6,9.0],
       [7,9.0],[8,9.0],[9,9.0],[10,9.0],[11,9.0],[12,9.0],[13,9.0],[14,9.0],[15,9.0],[16,9.0],[17,9.0],[18,9.0],[19,9.0],[20,9.0]]

(%i2)  plot2d([[discrete,xy],0.0012·(x−1)·(x−20)^2+9],[x,−2,25],
              [style,[linespoints,2,2],[lines,2,2]],[legend,"","Approximation by CF2"],
              [gnuplot_preamble,"set key bottom right"],
              [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2)  false
```

Fig. 12. Approximation of numerical data plotted in Fig. 10 by CF2 $y = 0.0012(x-1)(x-20)^2$ $+9$

We go on to another example.

Example 5.3.10. $a = 4$, $b = -5$, and $c = -9$. One gets $X_1 = \sqrt{\frac{9}{4} + \frac{5}{4}\cdot\sqrt{\frac{9}{4} + \frac{5}{4}\cdot\sqrt{\frac{9}{4} + \cdots}}}$.

Algorithm is as follows.

Step 1: *Preparation*. Set $x_0 = \sqrt{\frac{9}{4}} = \frac{3}{2}$.

Step 2: *Computation*. Compute $\sqrt{\frac{9}{4} + \frac{5}{4}\cdot x_0}$ and get $x_1 = \sqrt{\frac{33}{8}}\ = 2.031\ldots$.

Step 3: *Iteration*. Likewise, repeat computing $x_{n+1} = \sqrt{\frac{9}{4} + \frac{5}{4}\cdot x_n}$ for $n = 1, 2, 3\ldots$.

These steps are essentially the same as those we have already described, but this time, we write an SBCL code and run it for verification:

Verification 5.3.11.

```
% sbcl --version
SBCL 2.2.9.debian
```

```
% more cubic_eq_part_1_v_5_3_11.lsp
(defun v_5_3_11(n)(if(= n 0)(/ 3 2)
(sqrt(+(/ 9 4)(*(/ 5 4)(v_5_3_11(- n 1)))))))
(loop for i from 0 to 20 do
(format t"~D~%"(v_5_3_11 i)))
;Cf. http://progopedia.com/example/factorial/22/
% time sbcl --script cubic_eq_part_1_v_5_3_11.lsp
3/2
2.0310097
2.188324
2.2328022
2.2452178
2.2486713
2.249631
2.2498975
2.2499714
2.2499921
2.2499979
2.2499993
2.2499998
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
0.011u 0.031s 0:00.05 80.0% 0+0k 70976+0io 21pf+0w
```

Like *Verifications* 5.3.4 and 5.3.9, once the computation reaches '2.25', repetition of the computation $\sqrt{\frac{9}{4}+\frac{5}{4}\cdot 2.25} = \sqrt{\frac{9}{4}+\frac{5}{4}\cdot 2.25} = \ldots$ just continues. So we can say the computation has converged to the value 2.25; we have thus got $X_1 = 2.25$. Therefore, $X_2 =$ [50] $-\frac{b}{a} - X_1 = -\frac{-5}{4} - 2.25 = -1$. Alternatively, $X_2 =$ [51] $\frac{c}{aX_1} = \frac{-9}{4\cdot 2.25} = -1$. In either case, we obtain the values 2.25 and -1, or the roots of the QE $4x^2 - 5x - 9 = 0$ [52] .

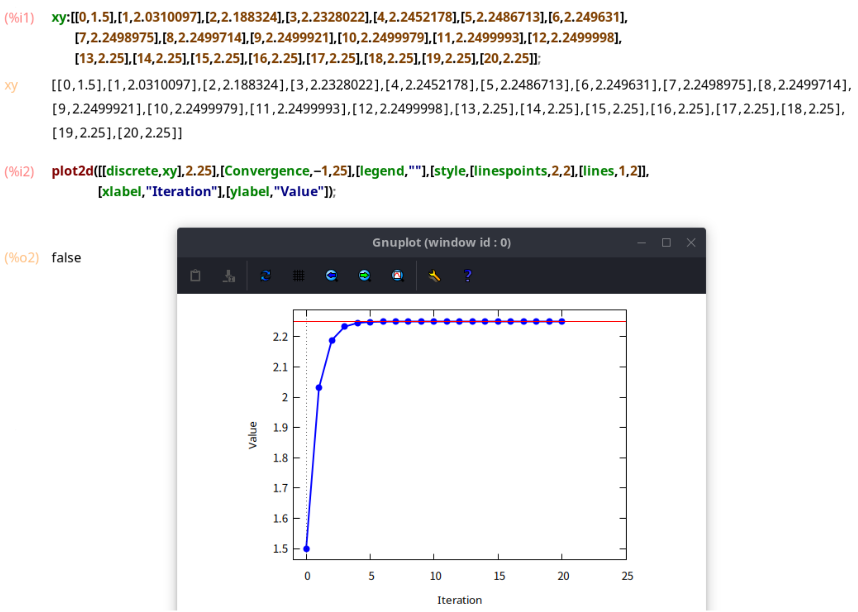
We visualise the numerical data we have obtained as follows.

---

[50] We have used VF.

[51] Ditto.

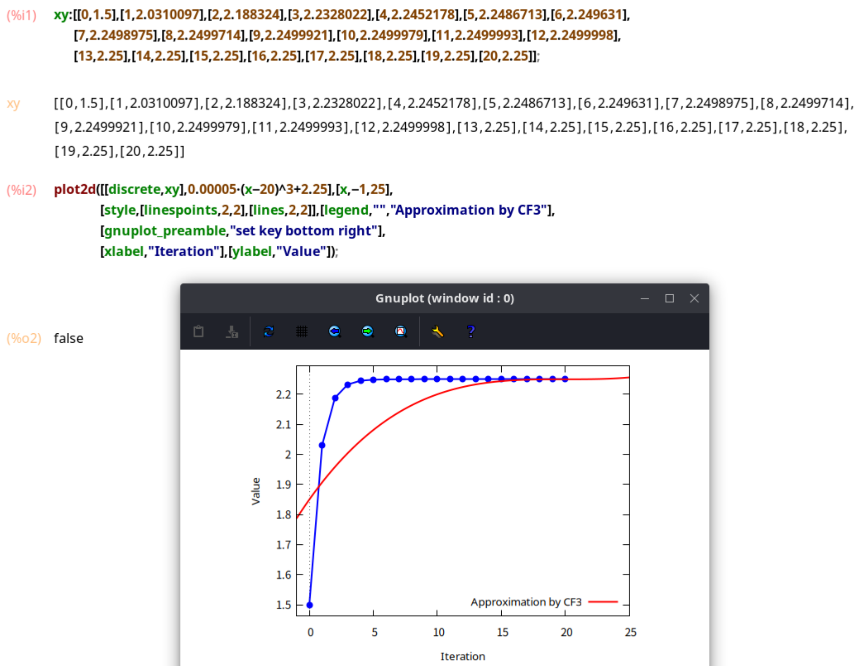
[52] We will check these in **6.2**.



```
(%i1) xy:[[0,1.5],[1,2.0310097],[2,2.188324],[3,2.2328022],[4,2.2452178],[5,2.2486713],[6,2.249631],
        [7,2.2498975],[8,2.2499714],[9,2.2499921],[10,2.2499979],[11,2.2499993],[12,2.2499998],
        [13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]];
xy    [[0,1.5],[1,2.0310097],[2,2.188324],[3,2.2328022],[4,2.2452178],[5,2.2486713],[6,2.249631],[7,2.2498975],[8,2.2499714],
      [9,2.2499921],[10,2.2499979],[11,2.2499993],[12,2.2499998],[13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],
      [19,2.25],[20,2.25]]
(%i2) plot2d([[discrete,xy],2.25],[Convergence,−1,25],[legend,""],[style,[linespoints,2,2],[lines,1,2]],
            [xlabel,"Iteration"],[ylabel,"Value"]);
(%o2) false
```

Fig. 13. Visualisation of numerical outputs in *Verification* 5.3.11. Convergence is emphasised by a red line.

Next, we try approximating them by a CF:



```
(%i1) xy:[[0,1.5],[1,2.0310097],[2,2.188324],[3,2.2328022],[4,2.2452178],[5,2.2486713],[6,2.249631],
        [7,2.2498975],[8,2.2499714],[9,2.2499921],[10,2.2499979],[11,2.2499993],[12,2.2499998],
        [13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]];

xy    [[0,1.5],[1,2.0310097],[2,2.188324],[3,2.2328022],[4,2.2452178],[5,2.2486713],[6,2.249631],[7,2.2498975],[8,2.2499714],
      [9,2.2499921],[10,2.2499979],[11,2.2499993],[12,2.2499998],[13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],
      [19,2.25],[20,2.25]]
(%i2) plot2d([[discrete,xy],0.00005·(x−20)^3+2.25],[x,−1,25],
            [style,[linespoints,2,2],[lines,2,2]],[legend,"","Approximation by CF3"],
            [gnuplot_preamble,"set key bottom right"],
            [xlabel,"Iteration"],[ylabel,"Value"]);
(%o2) false
```

Fig. 14. Approximation of numerical data plotted in Fig. 13 by CF3 $y = 0.00005(x - 20)^3$ $+2.25$

Likewise, tailoring (49) to fit *Example* 5.3.10, one gets

$$x_{n+1} = x_n - \frac{4x_n^2 - 5x_n - 9}{8x_n - 5} = \frac{4x_n^2 + 9}{8x_n - 5}. \tag{51}$$

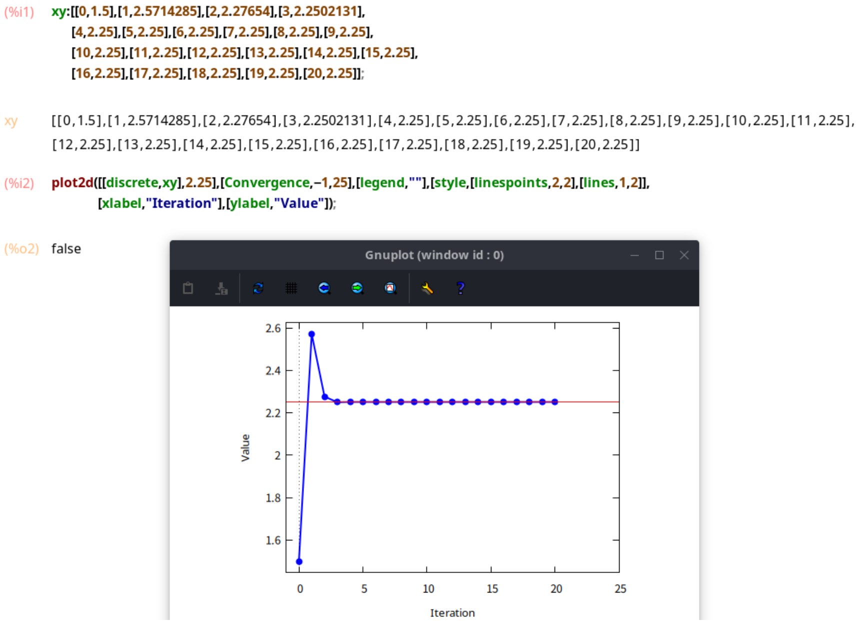
Verification 5.3.12. Likewise, we run the NM-version code.

```
% more cubic_eq_part_1_v_5_3_12.lsp
(defun v_5_3_12(n)(if(= n 0)(/ 3 2)
(/(+ (* 4(expt (v_5_3_12(- n 1)) 2))9)
(- (* 8 (v_5_3_12(- n 1)))5.0))))
(loop for i from 0 to 20 do(format t"~D~%"(v_5_3_12 i)))
% time sbcl --script cubic_eq_part_1_v_5_3_12.lsp
3/2
2.5714285
2.27654
2.2502131
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
0.151u 0.038s 0:00.19 94.7% 0+0k 0+0io 0pf+0w
```

We thus get $X_1 = 2.25$ also by NM [53] to visualise the data likewise.

---

[53] By the way, what if we subject this code to a slight change that doesn't affect its mathematical content? See **Appendix 12.6**.



```
(%i1) xy:[[0,1.5],[1,2.5714285],[2,2.27654],[3,2.2502131],
        [4,2.25],[5,2.25],[6,2.25],[7,2.25],[8,2.25],[9,2.25],
        [10,2.25],[11,2.25],[12,2.25],[13,2.25],[14,2.25],[15,2.25],
        [16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]];

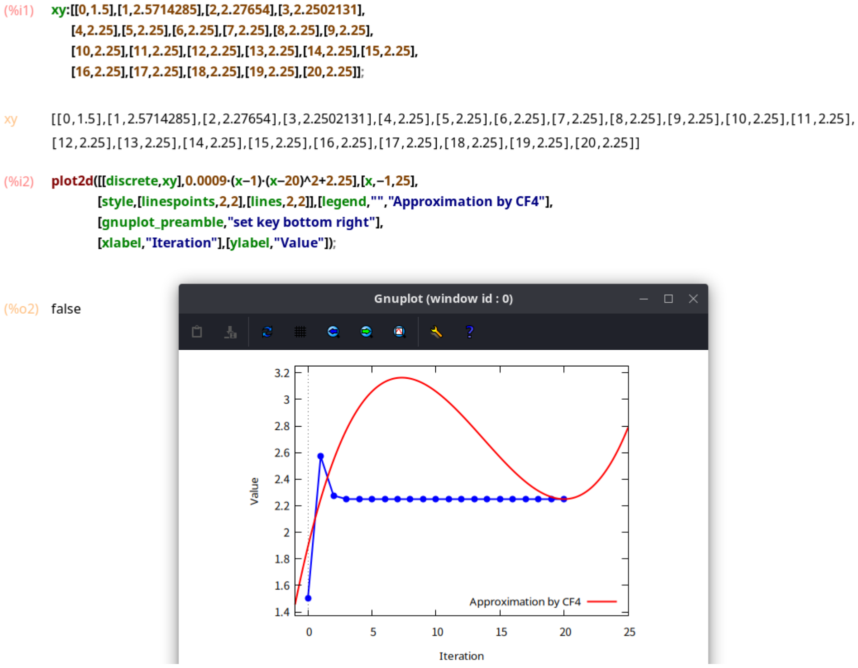
xy    [[0,1.5],[1,2.5714285],[2,2.27654],[3,2.2502131],[4,2.25],[5,2.25],[6,2.25],[7,2.25],[8,2.25],[9,2.25],[10,2.25],[11,2.25],
      [12,2.25],[13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]]

(%i2) plot2d([[discrete,xy],2.25],[Convergence,−1,25],[legend,""],[style,[linespoints,2,2],[lines,1,2]],
             [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2) false
```

Fig. 15. Visualisation of numerical outputs in *Verification* 5.3.12. Convergence is emphasised by a red line.

Likewise, we approximate the above data by a CF:



```
(%i1) xy:[[0,1.5],[1,2.5714285],[2,2.27654],[3,2.2502131],
        [4,2.25],[5,2.25],[6,2.25],[7,2.25],[8,2.25],[9,2.25],
        [10,2.25],[11,2.25],[12,2.25],[13,2.25],[14,2.25],[15,2.25],
        [16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]];

xy    [[0,1.5],[1,2.5714285],[2,2.27654],[3,2.2502131],[4,2.25],[5,2.25],[6,2.25],[7,2.25],[8,2.25],[9,2.25],[10,2.25],[11,2.25],
      [12,2.25],[13,2.25],[14,2.25],[15,2.25],[16,2.25],[17,2.25],[18,2.25],[19,2.25],[20,2.25]]

(%i2) plot2d([[discrete,xy],0.0009·(x−1)·(x−20)^2+2.25],[x,−1,25],
             [style,[linespoints,2,2],[lines,2,2]],[legend,"","Approximation by CF4"],
             [gnuplot_preamble,"set key bottom right"],
             [xlabel,"Iteration"],[ylabel,"Value"]);

(%o2) false
```

Fig. 16. Approximation of numerical data plotted in Fig. 15 by CF4 $y = 0.0009(x-1)(x-20)^2 + 2.25$

6 Example programs and a question

Here are outputs of a few programs, which we hope are not so off-topic.

6.1 Recurrence-related programs

6.1.1 Fermat numbers

We compute the recurrence

$$F_0 = 3\,, \qquad F_n = (F_{n-1} - 1)^2 + 1, \qquad n = 1, 2, 3 \ldots$$

using ECL and Erlang :

```
% ecl --version
ECL 21.2.1
% more ferm.lsp
(do((n 1(incf n)))((>= n 9))
(defun fer(n)(if(< n 2)3
(+(*(-(fer(- n 1))1)(-(fer(- n 1))1))1)))
(format t "~2d~%"(fer n)))
(format t "~%")
(quit)
% ecl --load ferm.lsp
 3
 5
17
257
65537
4294967297
18446744073709551617
340282366920938463463374607431768211457

% more ferm.erl
-module(ferm).
-export([fermat/1]).
fermat(0)->3;
fermat(N) when N>0->(fermat(N-1)-1)*(fermat(N-1)-1)+1.
% erl
Erlang/OTP 25 [erts-13.2.2.9] [source] [64-bit] [smp:16:16]
[ds:16:16:10] [async-threads:1] [jit:ns]
Eshell V13.2.2.9  (abort with ^G)
1> c(ferm).
{ok,ferm}
2> io:write([ferm:fermat(X)||X<-lists:seq(0,7)]).
[3,5,17,257,65537,4294967297,18446744073709551617,
 340282366920938463463374607431768211457]
```

6.1.2 FS

We compute the recurrence

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n = 2, 3, 4 \ldots$$

using FriCAS and Julia :

```
% fricas -nox
openServer result 0
                       FriCAS Computer Algebra System
                Version: FriCAS 1.3.11 built with gcl 2.6.14
                   Timestamp: Sun Jul 28 23:21:21 UTC 2024

(1) -> a(0)==0;a(1)==1;a(n)==a(n-1)+a(n-2)
                                                      Type: Void
(2) -> for n in 0..15 repeat output(a(n))
   Compiling function a with type Integer -> NonNegativeInteger
   Compiling function a as a recurrence relation.
   0
   1
   1
   2
   3
   5
   8
   13
   21
   34
   55
   89
   144
   233
   377
   610
                                                      Type: Void

% more fib.jl
fib(n)= n<2 ? n : fib(n-1)+fib(n-2);
for i = 0:15
println(fib(i))
end
#=
Cf. https://rosettacode.org/wiki/Comments#Julia
https://rosettacode.org/wiki/Fibonacci\_sequence\#Recursive\_43
=#
```

```
% julia -v
julia version 1.11.1
% julia fib.jl
0
1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
```

6.2 QE-related programs

We separately write the roots obtained by QF as

$$\begin{cases} r_+ = \frac{-b+\sqrt{b^2-4ac}}{2a}, & (52) \\ r_- = \frac{-b-\sqrt{b^2-4ac}}{2a} & (53) \end{cases}$$

to notice that $(52) + (53)$ gives $r_+ + r_- = -\frac{b}{a}$. That is,

$$r_- = -r_+ - \frac{b}{a}. \tag{54}$$

We check the roots of the QE's we have dealt with using $(52) - (54)$. First, we use AXIOM :

```
% axiom -nox
                        AXIOM Computer Algebra System
                          Version: Axiom (May 2016)
                Timestamp: Sunday July 28, 2024 at 23:45:19
(1) -> -1

   (1)  - 1
                                                            Type: Integer
(2) -> 1

   (2)  1
```

```
                                              Type: PositiveInteger
(3) -> 72

   (3)  72
                                              Type: PositiveInteger
(4) -> )read ce_part_1_pos_root.input
(-%%(2)+sqrt(%%(2)^2-4*%%(1)*%%(3)))/(2*%%(1))--Cf. (52).


   (4)  - 8
                                             Type: AlgebraicNumber
(5) -> )read ce_part_1_neg_root.input
-%%(4)-(%%(2)/%%(1))--Cf. (54).


   (5)  9
                                             Type: AlgebraicNumber
```

Thus, we have confirmed that the roots of the QE $-x^2 + x + 72 = 0$ are -8 and 9 [54] . Likewise,

```
(1) -> 4

   (1)  4
                                             Type: PositiveInteger
(2) -> -5

   (2)  - 5
                                                     Type: Integer
(3) -> -9

   (3)  - 9
                                                     Type: Integer
(4) -> )read ce_part_1_pos_root.input
(-%%(2)+sqrt(%%(2)^2-4*%%(1)*%%(3)))/(2*%%(1))--Cf. (52).


        9
   (4)  -
        4
                                              Type: AlgebraicNumber
(5) -> )read ce_part_1_neg_root.input
-%%(4)-(%%(2)/%%(1))--Cf. (54).
```

---

[54] See *Example* 5.3.3, *Verification* 5.3.4, and *Verification* 5.3.9.

```
   (5)  - 1
                                                    Type: AlgebraicNumber
```

This time, we have confirmed that the roots of the QE $4x^2 - 5x - 9 = 0$ are $\frac{9}{4}$ and -1 [55] .

Next, we use SWI-Prolog .

```
% swipl --version
SWI-Prolog version 9.0.4 for x86_64-linux
% more ce_part_1_qe_1.pl
:- write('We compute the ''positive root'' (PR) and ''negative
root'' (NR) of Ax^2+Bx+C=0.'),nl.
:- write('Let A=-1, B=1, C=72.'),nl.
r+(A,B,C,PR):-PR is (-B+sqrt(B**2-4*A*C))/(2*A).
/* Cf. (52). */
r-(A,B,C,NR):-NR is (-B-sqrt(B**2-4*A*C))/(2*A).
/* Cf. (53). */
:- write('Type ''r+(-1,1,72,PR).'''),write('. '),
write('Next, ''r-(-1,1,72,NR).'''),write('. '),nl.
/*
Cf.
https://en.wikibooks.org/wiki/Computer_Programming/Hello_world#Prolog
https://stackoverflow.com/questions/40821641/writing-functions-in-prolog
*/
% swipl -f ce_part_1_qe_1.pl
We compute the 'positive root' (PR) and 'negative root' (NR) of
Ax^2+Bx+C=0.
Let A=-1, B=1, C=72.
Type 'r+(-1,1,72,PR).'. Next, 'r-(-1,1,72,NR).'.
?- r+(-1,1,72,PR).
PR = -8.0.
?- r-(-1,1,72,PR).
PR = 9.0.
```

Again, we have confirmed that the roots of the QE $-x^2+x+72 = 0$ are -8 and 9 [56] . Likewise,

```
% more ce_part_1_qe_2.pl
:- write('We compute the ''positive root'' (PR) and ''negative
root'' (NR) of Ax^2+Bx+C=0.'),nl.
:- write('Let A=4, B=-5, C=-9.'),nl.
r+(A,B,C,PR):-PR is (-B+sqrt(B**2-4*A*C))/(2*A).
/* Cf. (52). */
r-(A,B,C,NR):-NR is (-B-sqrt(B**2-4*A*C))/(2*A).
```

---

[55] See *Example* 5.3.10, *Verification* 5.3.11, and *Verification* 5.3.12.

[56] *Cf.* footnote 54.

```
/* Cf. (53). */
:- write('Type ''r+(4,-5,-9,PR).'''),write('. '),
write('Next, ''r-(4,-5,-9,NR).'''),nl.
% swipl -f ce_part_1_qe_2.pl
We compute the 'positive root' (PR) and 'negative root' (NR) of
Ax^2+Bx+C=0.
Let A=4, B=-5, C=-9.
Type 'r+(4,-5,-9,PR).'. Next, 'r-(4,-5,-9,NR).'
?- r+(4,-5,-9,PR).
PR = 2.25.
?- r-(4,-5,-9,PR).
PR = -1.0.
```

Again, we have confirmed that the roots of the QE $4x^2 - 5x - 9 = 0$ are $\frac{9}{4}$ and -1 [57] .

6.3 Question coming from 6.1.2: does (44) have something to do with quasi-symmetric QuE (QSQuE) ?

Here **6.1.2** reminds us of (45). And since it is closely related to (44), the above question has arisen. We try to answer the question: Since $x = 0$ is not a root of (44), we assume $x \neq 0$ to divide both sides of (44) by x to get

$$x - \frac{1}{x} = 1. \tag{55}$$

Then, take

$$x^4 + 4x^3 + x^2 - 4x + 1 = 0, \tag{56}$$

for example. Likewise, we can assume $x \neq 0$, and thus $x^2 \neq 0$. This enables us to divide both sides of (56) by x^2 to get $x^2 + 4x + 1 - \frac{4}{x} + \frac{1}{x^2} = 0$. After some manipulation, this becomes

$$(x - \frac{1}{x})^2 + 4(x - \frac{1}{x}) + 3 = 0. \tag{57}$$

Then, using (55), we rewrite (57) as $t^2 + 4t + 3 = 0$, whose roots are -1 and -3. We thus deal with $x - \frac{1}{x} = -1, -3$, *i.e.,* $x^2 + x - 1 = 0$ and $x^2 + 3x - 1 = 0$. Applying QF to these QE's, one gets $x = \frac{-1 \pm \sqrt{5}}{2}, \frac{-3 \pm \sqrt{13}}{2}$, solutions of (56) [58] . Hence,

Answer 6.3.1. (44) can play a role in solving (56), a QSQuE.

---

[57] *Cf.* footnote 55.

[58] By the way, $\frac{-1-\sqrt{5}}{2}$, one of the solutions, amounts to $-\ \varphi$.

7 On $n \times n$ 'trace matrix' (T_n)

Inspired by (45), we consider the following.

$$\left(\begin{array}{c} a_{n+1} \\ a_{n-1} \end{array}\right) = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) \left(\begin{array}{c} a_n \\ a_{n-1} \end{array}\right).$$

We observe that the above 2×2 matrix is a kind of 'special trace matrix' ST_2 $[12, \mathbf{5}]$, since its tr and $\det$ are 2 and 1, respectively. Taking this opportunity, we should like to extend ST_n to T_n, *i.e.*, $n \times n$ 'trace matrix' by dropping the requirement that $\det$ of the matrix under consideration be 1 [12, *Def.* 5.1] (on purpose). In other words, we are constrained only by

$$\mathrm{tr}(\mathrm{T}_n) = n. \tag{58}$$

Example 7.1. $N = \left(\begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array}\right)$ is a kind of T_2, since it is a 2×2 matrix, $\mathrm{tr}(N) = 2$, which satisfies (58), and $\det(N) = -1$ $(\neq 1)$.

Notation 7.2. Like $\mathrm{ST}_{n,\mathbb{R}}$ [12, *Notation* 5.6], we can write $\mathrm{T}_{n,\mathbb{R}}$ instead of T_n, when we wish to emphasise the fact that each entry of T_n is a real number .

Example 7.3. $P^{59} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ is a kind of $\mathrm{T}_{3,\mathbb{R}}$, since it is a 3×3 matrix, and $\mathrm{tr}(P)$ and $\det(P)$ are 3 and 0 ($\neq$ 1), respectively.

Notation 7.4. Like $\mathrm{ST}_{n,\mathbb{C}}$ [12, *Notation* 5.8], we can write $\mathrm{T}_{n,\mathbb{C}}$ instead of T_n, when we wish to emphasise the fact that each entry of T_n is a complex number .

Example 7.5. $Q = \left(\begin{array}{ccc} 1-\imath & 3+\imath & -\imath \\ \imath & 1+2\imath & -2\imath \\ 2-\imath & \imath & 1-\imath \end{array}\right)$ is a kind of $\mathrm{T}_{3,\mathbb{C}}$, since it is a 3×3 matrix, and $\mathrm{tr}(Q)$ and $\det(Q)$ are 3 and $-5 - 13\imath$ ($\neq$ 1), respectively.

We now prove a theorem.

THEOREM 7.6. *Direct sum of 'special trace matrices' is a 'special trace matrix', again.*

Proof. Let us consider the $m \times m$ matrix R which is the direct sum of n 'special trace submatrices', *i.e.*, $S_1, \cdots, S_i, \cdots, S_n$. Explicitly,

---

[59] We use 'P' instead of 'O' to avoid possible confusion with O_n. See **Glossary**.

$$R=\bigoplus_{i=1}^{n}S_i=\begin{pmatrix}\overbrace{S_1}^{s_1\text{ columns}}\}s_1\text{ rows} & & & & \\ & \ddots & & \mathbf{0} & \\ & & \overbrace{S_i}^{s_i\text{ columns}}\}s_i\text{ rows} & & \\ & \mathbf{0} & & \ddots & \\ & & & & s_n\text{ rows}\{\overbrace{S_n}^{s_n\text{ columns}}\end{pmatrix},$$

where $\mathbf{0}$ stands for entries that are 0's. We immediately see that $\mathrm{tr}(R)=\mathrm{tr}(S_1)+\cdots+\mathrm{tr}(S_i)+\cdots+\mathrm{tr}(S_n)$. S_i's being 'trace submatrices', we have $\mathrm{tr}(S_1)=s_1$, $\cdots$, $\mathrm{tr}(S_i)=s_i$,$\cdots$, and $\mathrm{tr}(S_n)=s_n$. So $\mathrm{tr}(R)=s_1+\cdots+s_i+\cdots+s_n$, which amounts to m. And since $\det(R)=\det(S_1)\times\cdots\times\det(S_i)\times\cdots\times\det(S_n)$, we have $\det(R)=1\times\cdots\times 1\times\cdots\times 1=1$. QED.

Example 7.7. Consider ST$_2$'s $\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$ and $\begin{pmatrix}-1 & -1\\ 4 & 3\end{pmatrix}$. Then, their direct sum is $\begin{pmatrix}3 & -4 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & -1 & -1\\ 0 & 0 & 4 & 3\end{pmatrix}$. This is a kind of ST$_4$, since it is a 4×4 matrix, and its tr and $\det$ are 4 and 1, respectively.

We further prove the following.

COROLLARY 7.8. Direct sum of 'trace matrices' is a 'trace matrix', unless tr of such a direct sum amounts to 1.

Proof. 'Forget' arguments about $\det(R)=1$ in *Proof* of THEOREM 7.6. QED.

Example 7.9. Consider T$_2$'s $\begin{pmatrix}3 & -1\\ 1 & -1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}$. Then, their direct sum is $\begin{pmatrix}3 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}$. This is a kind of T$_4$, since it is a 4×4 matrix, and its tr and $\det$ are 4 and 2 $(\neq 1)$, respectively.

Remark 7.10. In other words, such a direct sum can be a 'special trace matrix'.

Example 7.11. Consider T_2's $\left(\begin{array}{cc} 3 & -5 \\ 1 & -1 \end{array}\right)$ and $\left(\begin{array}{cc} 0 & -1 \\ \frac{1}{2} & 2 \end{array}\right)$. Then, their direct sum is $\left(\begin{array}{cccc} 3 & -5 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 2 \end{array}\right)$. This is a kind of ST_4, since it is a 4×4 matrix, and its tr and $\det$ are 4 and 1, respectively.

We also prove the following.

LEMMA 7.12. *Let* U[60] $= \left(\begin{array}{cc} 1 & 0 \\ 0 & x \end{array}\right)$ *and* $V = U \oplus xU \oplus x^2U \oplus \cdots \oplus x^{n-1}U \oplus x^nU. \; Next, \; let$ $V \; be \; a \; kind \; of \; \mathrm{T}_{2n+2}. \; Then, \; one \; root \; of \; the \; equation \; \mathrm{tr}(V) = 2n+2 \; is \; 1.$

Proof. A straightforward calculation. Writing the equation under consideration explicitly yields $1+x+x(1+x)+\cdots+x^{n-1}(1+x)+x^n(1+x) = 2n+2$. So one gets the equation $W(x) = x^{n+1} + \underbrace{2x^n + \cdots + 2x}_{n \text{ terms}} + 1 - 2n - 2 = 0$. Then, we observe $W(1) = 1^{n+1} + \underbrace{2 \cdot 1^n + \cdots + 2 \cdot 1}_{n \text{ terms}} + 1$ $-2n - 2 = 1 + 2n + 1 - 2n - 2 = 0$. Hence, one root of the equation $\mathrm{tr}(V) = 2n+2$ is 1. QED.

Example 7.13. The case $n = 1$. Explicitly, $X = \left(\begin{array}{cc} 1 & 0 \\ 0 & x \end{array}\right) \oplus x \left(\begin{array}{cc} 1 & 0 \\ 0 & x \end{array}\right) = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2 \end{array}\right)$.

Since $\mathrm{tr}(X) = 2 \cdot 1 + 2 = 4$, we have $1 + x + x + x^2 = 4$. Solving the resultant equation $x^2 + 2x - 3 = 0$, one gets the roots 1 and -3.

8 Going into $\mathbb{C}$

We gradually take $\mathbb{C}$ into consideration, as suggested around the end of **3.3** [61, 62]. In the beginning, since we have mentioned a TM in **3.1**, we take the following TM as example.

$$Y = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right).$$

---

[60] Here we use 'U' instead of 'T' to avoid possible confusion with T_2, T_4, etc.

[61] Strictly speaking, we have *implicitly* mentioned $\mathbb{C}$ in Fig. 1(c), where UQF has no real roots, since such UQF can be written as $y = a(x-\alpha)(x-\beta)$, where $\alpha, \beta \in \mathbb{C}$.

[62] Actually, we have already caught a glimpse of $\mathbb{C}$ in *Notation* 7.4 and *Example* 7.5, in which we touched upon complex number s.

CP of Y is λ^2+1. Setting $\lambda^2+1=0$, we get the roots $\pm\imath$. In this way, starting from a real matrix, or Y, one is smoothly led to the realm of $\mathbb{C}$ (hopefully).

Now let us proceed to a bit more complicated example, which is another TM. We consider

$$Z = \begin{pmatrix} 1 & \imath & -1 & -\imath \\ \imath & -1 & -\imath & 1 \\ -1 & -\imath & 1 & \imath \\ -\imath & -1 & \imath & -1 \end{pmatrix}.$$

Then,

$$|Z-\lambda I_4| = \begin{vmatrix} 1-\lambda & \imath & -1 & -\imath \\ \imath & -1-\lambda & -\imath & 1 \\ -1 & -\imath & 1-\lambda & \imath \\ -\imath & -1 & \imath & -1-\lambda \end{vmatrix} =^{63} \begin{vmatrix} 0 & \lambda\imath & \lambda^2-2\lambda & -\lambda\imath \\ 0 & -\lambda & -\lambda\imath & 0 \\ -1 & -\imath & 1-\lambda & \imath \\ 0 & -2 & \lambda\imath & -\lambda \end{vmatrix}$$

$$= (-1)^{1+1}\times 0 \times \begin{vmatrix} -\lambda & -\lambda\imath & 0 \\ -\imath & 1-\lambda & \imath \\ -2 & \lambda\imath & -\lambda \end{vmatrix} + (-1)^{2+1}\times 0 \times \begin{vmatrix} \lambda\imath & \lambda^2-2\lambda & -\lambda\imath \\ -\imath & 1-\lambda & \imath \\ -2 & \lambda\imath & -\lambda \end{vmatrix}$$

$$+(-1)^{3+1}\times(-1)\times \begin{vmatrix} \lambda\imath & \lambda^2-2\lambda & -\lambda\imath \\ -\lambda & -\lambda\imath & 0 \\ -2 & \lambda\imath & -\lambda \end{vmatrix} + (-1)^{4+1}\times 0\times \begin{vmatrix} \lambda\imath & \lambda^2-2\lambda & -\lambda\imath \\ -\lambda & -\lambda\imath & 0 \\ -\imath & 1-\lambda & \imath \end{vmatrix}$$

$$= -\begin{vmatrix} \lambda\imath & \lambda^2-2\lambda & -\lambda\imath \\ -\lambda & -\lambda\imath & 0 \\ -2 & \lambda\imath & -\lambda \end{vmatrix}$$

$$\begin{aligned} =^{64} & -\{\lambda\imath\cdot(-\lambda\imath)\cdot(-\lambda) + (\lambda^2-2\lambda)\cdot 0\cdot(-2) + (-\lambda\imath)\cdot(-\lambda)\cdot\lambda\imath \\ & \quad -(-\lambda\imath)\cdot(-\lambda\imath)\cdot(-2) - (\lambda^2-2\lambda)\cdot(-\lambda)\cdot(-\lambda) - \lambda\imath\cdot 0\cdot\lambda\imath\} \end{aligned}$$

$$= -(-\lambda^3-\lambda^3-2\lambda^2-\lambda^4+2\lambda^3) = \lambda^4+2\lambda^2 = \lambda^2(\lambda^2+2).$$

---

[63] We have performed some elementary row operations to increase ‘0 entries’, which will make our computation (slightly) easier.

[64] Here we use the rule of Sarrus .

One thus gets $\lambda^2(\lambda^2+2)$, CP of Z. Setting $\lambda^2(\lambda^2+2)=0$, we get the MR 0 (multiplicity 2) and $\pm\sqrt{2}\imath$. This time, making use of a 4×4 TM, we treated a QuE, which can be regarded as the product of two QE's, *i.e.*, $\lambda^2=0$ and $\lambda^2+2=0$. And its roots were shown to include purely imaginary number s.

Now let $\alpha\pm\beta\imath$, where α, $\beta\in\mathbb{R}$, be the roots of a certain QE. Then, $\alpha+\beta\imath$ is symmetric to $\alpha-\beta\imath$ wrt real axis in the complex plane . What about the symmetry wrt imaginary ($\Im$) axis ? Consider *e.g.*, the CQP $x^2-2\delta\imath x-\gamma^2-\delta^2$, where γ, $\delta\in\mathbb{R}$. Since QF is applicable to the QE whose coefficients are complex numbers , we get the roots of the CQP equation, *i.e.*, $\frac{-(-2\delta\imath)\pm\sqrt{(-2\delta\imath)^2-4\cdot 1\cdot(-\gamma^2-\delta^2)}}{2}=\delta\imath\pm\gamma$ to note solving a certain CQP equation yields solutions symmetric wrt the $\Im$-axis. As a result of this, we get aware of the possibility that $\alpha+\beta\imath$ is to $\alpha-\beta\imath$ what $\gamma+\delta\imath$ is to $-\gamma+\delta\imath$, which makes both axes look (more and more) impressive; we thus get 'acquainted' with the complex plane (and $\mathbb{C}$).

8.1 On '$\mathbb{C}$-isation'

Now that we have got some familiarity with $\mathbb{C}$, we should like to try to '$\mathbb{C}$-ise' (4). Specifically, by replacing x and/or y in (4) with $x+\imath y$, one considers the following.

Case 1. Replacing x in (4) by $x+\imath y$, one gets $y=a(x+\imath y)^2+b(x+\imath y)+c$.

Case 2. Replacing y in (4) by $x+\imath y$, one gets $x+\imath y=ax^2+bx+c$.

Case 3. Replacing both x and y in (4) by $x+\imath y$, one gets $x+\imath y=a(x+\imath y)^2+b(x+\imath y)+c$.

In what follows, we make case-by-case treatments.

8.1.1 $x\longrightarrow x+\imath y$

After some computation, one gets

$$ax^2+bx+c-ay^2-y+\imath y(2ax+b)=0. \tag{59}$$

It follows from (59) that

$$\begin{cases} ax^2+bx+c-ay^2-y=0, & (60)\\ y(2ax+b)=0. & (61)\end{cases}$$

Remark 8.1.1.1. Putting $y=0$ in (60) results in (1).

Remark 8.1.1.2. Doing so in (61) gives $0=0$, which is trivial.

8.1.2 $y \longrightarrow x + \imath y$

Likewise, one gets

$$ax^2 + (b-1)x + c - \imath y = 0. \tag{62}$$

It follows from (62) that

$$\begin{cases} ax^2 + (b-1)x + c = 0, & (63) \\ y = 0. & (64) \end{cases}$$

Remark 8.1.2.1. We notice in (63) [65] that $(b-1)^2 - 4ac$ can be a 'new' kind of D we obtained via '$\mathbb{C}$-isation'.

Remark 8.1.2.2. Setting $2ax + b = 1$ in (61) yields (64).

8.1.3 $x, y \longrightarrow x + \imath y$

Likewise, one gets

$$ax^2 + (b-1)x + c - ay^2 + \imath y(2ax + b - 1) = 0. \tag{65}$$

It follows from (65) that

$$\begin{cases} ax^2 + (b-1)x + c - ay^2 = 0, & (66) \\ y(2ax + b - 1) = 0. & (67) \end{cases}$$

Remark 8.1.3.1. Putting $y = 0$ in (66) yields (63).

Remark 8.1.3.2. Replacing $2ax + b - 1$ by 1 in (67) yields (64).

8.2 On '$\mathbb{P}$-isation'

We try to 'higher-dimensionalise' (4) via a procedure which we tentatively call '$\mathbb{P}$-isation' [66]. Specifically, one replaces the variables x and y in (4) by $\frac{x}{z}$ and $\frac{y}{z}$, respectively, to get

$$y/z = a(x/z)^2 + b(x/z) + c.$$

---

[65] Putting $y = 0$ in (66) yields this.

[66] Some might recall projective space .

That is, we consider

$$yz = ax^2 + bzx + cz^2 \tag{68}$$

to take up three-dimensional space (3D space).

Remark 8.2.1. Setting $z = 0$ in (68) yields $ax^2 = 0$. Assuming $\neq 0$, we solve it to get $x = 0$, the y-axis.

Remark 8.2.2. Replacing z in (68) by 1, one gets $y = ax^2 + bx + c$. One thus 'retrieves' (4).

Example 8.2.3. '$\mathbb{P}$-ising' $y = x^2 + 3x + 2$ yields $y/z = (x/z)^2 + 3(x/z) + 2$. That is, one gets the equation $yz = x^2 + 3zx + 2z^2$ via '$\mathbb{P}$-isation'.

8.3 On '$\mathbb{CP}$-isation'

Inspired by $\mathbb{CP}^n$, here we first '$\mathbb{C}$-ise' and then '$\mathbb{P}$-ise' something, which we tentatively call '$\mathbb{CP}$-isation'.

Example 8.3.1. '$\mathbb{P}$-ising' (60) and (61), which have already been '$\mathbb{C}$-ised', yields

$$\begin{cases} a(\frac{x}{z})^2 + b(\frac{x}{z}) + c - a(\frac{y}{z})^2 - a(\frac{y}{z}) = 0, \\ \frac{y}{z} \cdot (2a \cdot \frac{x}{z} + b) = 0. \end{cases}$$

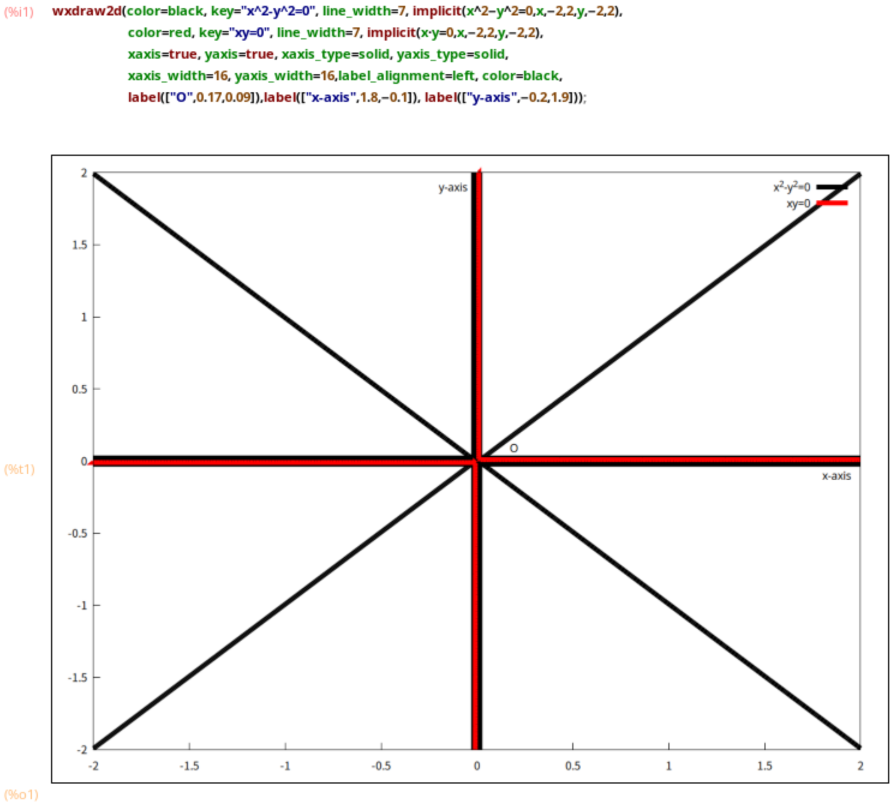
Clearing denominators of the above, one gets

$$\begin{cases} ax^2 + bzx + cz^2 - ay^2 - ayz = 0, & (69) \\ y(2ax + bz) = 0. & (70) \end{cases}$$

Remark 8.3.2. Setting $z = 0$ in (69) and (70) yields $ax^2 - ay^2 = 0$ and $2axy = 0$. Assuming $a, 2a \neq 0$, we divide both sides of them by a and $2a$, respectively to get

$$\begin{cases} x^2 - y^2 = 0, & (71) \\ xy = 0. & (72) \end{cases}$$

In either case, we have got the two lines intersecting perpendicularly as visualised below.



```
(%i1) wxdraw2d(color=black, key="x^2-y^2=0", line_width=7, implicit(x^2−y^2=0,x,−2,2,y,−2,2),
                color=red, key="xy=0", line_width=7, implicit(x·y=0,x,−2,2,y,−2,2),
                xaxis=true, yaxis=true, xaxis_type=solid, yaxis_type=solid,
                xaxis_width=16, yaxis_width=16,label_alignment=left, color=black,
                label(["O",0.17,0.09]),label(["x-axis",1.8,−0.1]), label(["y-axis",−0.2,1.9]));
```

Fig. 17. (71) and (72) visualised.

Remark 8.3.3. Some might recall pencil and/or $[8$, Fig. $9]$.

Remark 8.3.4. Putting $a = 1$ in $ax^2 - ay^2 = 0$ in *Remark* 8.3.2 gives $x^2 - y^2 = 0$. This is also the '$z = 1$ section' of Whitney's umbrella $x^2 - y^2 z = 0$, since replacing z in the 'umbrella' by 1 gives $x^2 - y^2 = 0$, too.

Remark 8.3.5. Putting $z = 1$ in (70) makes it 'revert' to (61) [67] .

8.4 On '$\mathbb{P}\mathbb{C}$-isation'

Now a natural question arises:

Question 8.4.1. Can we perform '$\mathbb{C}\mathbb{P}$-isation' the other way around?

In other words,

Question 8.4.2. Is what we tentatively call '$\mathbb{P}\mathbb{C}$-isation' feasible?

To address these questions, we first '$\mathbb{P}$-ise' (4) to get (68). Next, the following ways to '$\mathbb{C}$-ise' (68) are thinkable.

---

[67] *Cf. Remark* 8.2.2.

- $x \longrightarrow x + \imath y$, $y \longrightarrow x + \imath y$, $z \longrightarrow x + \imath y$;

- $x, y \longrightarrow x + \imath y$, $y, z \longrightarrow x + \imath y$, $z, x \longrightarrow x + \imath y$;

- $x, y, z \longrightarrow x + \imath y$.

In what follows, we make case-by-case treatments.

8.4.1 $x \longrightarrow x + \imath y$

First, we replace x in (68) by $x + \imath y$, a kind of '$\mathbb{C}$-isation' [68] . After some computation, we get

$$a(x^2 - y^2) + bzx + cz^2 - yz + \imath(2axy + bzy) = 0. \tag{73}$$

It follows from (73) that

$$\begin{cases} a(x^2 - y^2) + bzx + cz^2 - yz, & (74) \\ y(2ax + bz) = 0. & (75) \end{cases}$$

Remark 8.4.1.1. Putting $z = 0$ in (74) gives $ax^2 - ay^2 = 0$ in *Remark* 8.3.2.

Remark 8.4.1.2. (75) is the same as (70).

8.4.2 $y \longrightarrow x + \imath y$

Likewise, we get

$$ax^2 + bzx + cz^2 - zx - \imath yz = 0. \tag{76}$$

It follows from (76) that

$$\begin{cases} ax^2 + (b-1)zx + cz^2 = 0, & (77) \\ yz = 0. & (78) \end{cases}$$

Remark 8.4.2.1. Putting $z = 1$ in (77) gives (63).

---

[68] *Cf. Case* 1 in **8.1**.

Remark 8.4.2.2. Putting $z = 1$ in (78) gives (64).

8.4.3 $z \longrightarrow x + \imath y$

Likewise, we get

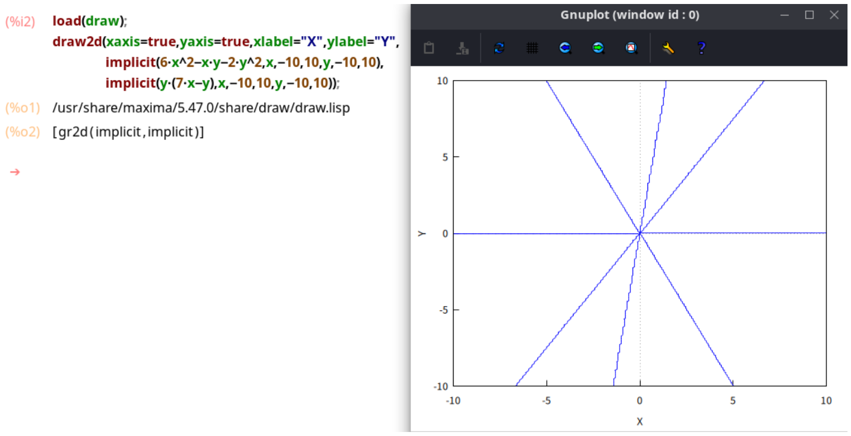
$$(a+b+c)x^2 - xy - cy^2 + \imath y\{(b+2c)x - y\} = 0. \tag{79}$$

It follows from (79) that

$$\begin{cases} (a+b+c)x^2 - xy - cy^2 = 0, & (80) \\ y\{(b+2c)x - y\} = 0. & (81) \end{cases}$$

Here we wonder if (80) and (81) are concurrent lines geometrically, and try visualising a few examples:

Example 8.4.3.1. $a = 1, b = 3, c = 2.$



```
(%i2)   load(draw);
        draw2d(xaxis=true,yaxis=true,xlabel="X",ylabel="Y",
                  implicit(6·x^2−x·y−2·y^2,x,−10,10,y,−10,10),
                  implicit(y·(7·x−y),x,−10,10,y,−10,10));
(%o1)   /usr/share/maxima/5.47.0/share/draw/draw.lisp
(%o2)   [gr2d(implicit,implicit)]
```

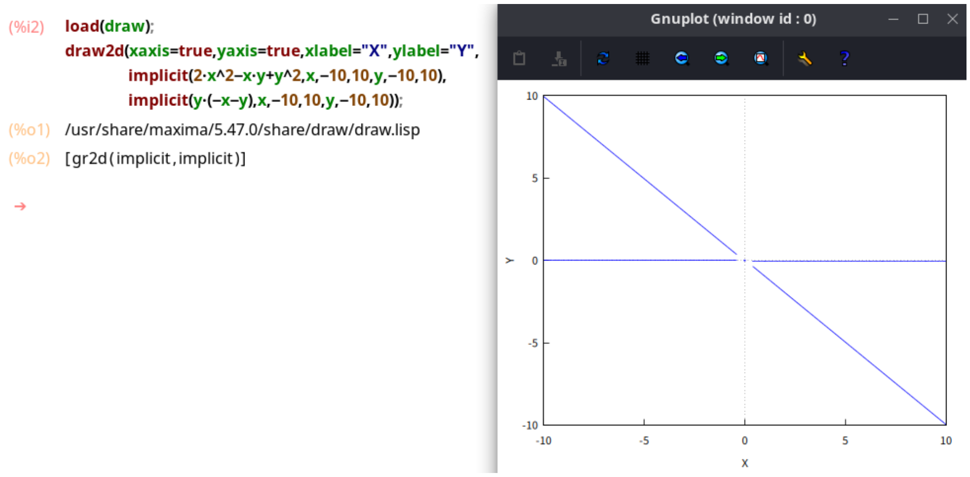
Fig. 18. Four lines that are concurrent at O

Remark 8.4.3.2. Some might recall pencil [69].

Example 8.4.3.3. $a = 2, b = 1, c = -1.$

---

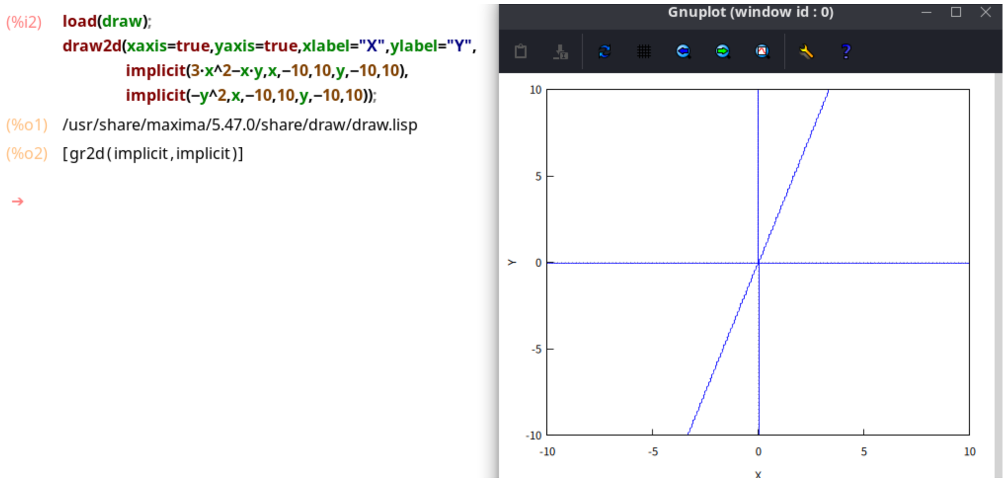
[69] *Cf. Remark* 8.3.3.



```
(%i2) load(draw);
      draw2d(xaxis=true,yaxis=true,xlabel="X",ylabel="Y",
              implicit(2·x^2−x·y+y^2,x,−10,10,y,−10,10),
              implicit(y·(−x−y),x,−10,10,y,−10,10));
(%o1) /usr/share/maxima/5.47.0/share/draw/draw.lisp
(%o2) [gr2d(implicit,implicit)]
```

Fig. 19. Two lines that are concurrent at O

Example 8.4.3.4. $a = 3$, $b = 0$, $c = 0$.



```
(%i2) load(draw);
      draw2d(xaxis=true,yaxis=true,xlabel="X",ylabel="Y",
              implicit(3·x^2−x·y,x,−10,10,y,−10,10),
              implicit(−y^2,x,−10,10,y,−10,10));
(%o1) /usr/share/maxima/5.47.0/share/draw/draw.lisp
(%o2) [gr2d(implicit,implicit)]
```

Fig. 20. Three lines that are concurrent at O

Multiplying the LHS of (80) by $4c$, we get

$$4(a+b+c)cx^2 - 4cxy - 4c^2y^2,$$

which we factor into

$$[\{\sqrt{1+4(a+b+c)c}+1\}x + 2cy][\{\sqrt{1+4(a+b+c)c}-1\}x - 2cy]. \tag{82}$$

We now consider two cases:

Case 1. $c \neq 0$. Equating (82) to 0, we get $y = -\frac{\sqrt{1+4(a+b+c)c}+1}{2c}x$, $\frac{\sqrt{1+4(a+b+c)c}-1}{2c}x$.

Case 2. $c = 0$. Putting $c = 0$ into (80), one solves $(a+b)x^2 - xy = 0$ to get $y = (a+b)x$, $x = 0$.

We then solve (81) to get $y = 0$, $(b+2c)x$, and the following.

Table Classification of (80) and (81)

	$c \neq 0$	$c = 0$
$b \neq 0$	$y = -\frac{\sqrt{1+4(a+b+c)c}+1}{2c}x$, $\frac{\sqrt{1+4(a+b+c)c}-1}{2c}x$, $(b+2c)x$, 0	$y = 0$, bx, $(a+b)x$, $x = 0$
$b = 0$	$y = -\frac{\sqrt{1+4(a+c)c}+1}{2c}x$, $\frac{\sqrt{1+4(a+c)c}-1}{2c}x$, $2cx$, 0	$y = 0$, ax, $x = 0$

Remark 8.4.3.5. It follows from the above table that we obtain at most four lines that pass O from this kind of '$\mathbb{PC}$-isation'.

8.4.4 $x, y \longrightarrow x + \imath y$

Likewise, we get

$$a(x^2 - y^2) + (b-1)zx + cz^2 + \imath y\{2ax + (b-1)z\} = 0. \tag{83}$$

It follows from (83) that

$$\begin{cases} a(x^2 - y^2) + (b-1)zx + cz^2 = 0, & (84) \\ y\{2ax + (b-1)z\} = 0. & (85) \end{cases}$$

Remark 8.4.4.1. Putting $z = 0$ in (84) gives $ax^2 - ay^2 = 0$ in *Remark* 8.3.2.

Remark 8.4.4.2. Putting $z = 0$ in (85) gives $2axy = 0$ in *Remark* 8.3.2.

8.4.5 $y, z \longrightarrow x + \imath y$

Likewise, we get

$$(a+b+c-1)x^2 + (1-c)y^2 + \imath xy\{b + 2(c-1)\} = 0. \tag{86}$$

It follows from (86) that

$$\begin{cases} (a+b+c-1)x^2 + (1-c)y^2 = 0, & (87) \\ xy\{b + 2(c-1)\} = 0. & (88) \end{cases}$$

Remark 8.4.5.1. Putting $a+b+c-1 = 1$ and $1 - c = -1$ in (87) yields (71).

Remark 8.4.5.2. Putting $b + 2(c-1) = 1$ in (88) yields (72).

8.4.6 $z, x \longrightarrow x + \imath y$

Likewise, we get

$$(a+b+c)(x^2-y^2) - xy + \imath y\{2(a+b+c)x - y\} = 0. \tag{89}$$

It follows from (89) that

$$\begin{cases} (a+b+c)x^2 - xy - (a+b+c)y^2 = 0, & (90) \\ y\{2(a+b+c)x - y\} = 0. & (91) \end{cases}$$

Remark 8.4.6.1. Replacing $(a+b+c)y^2$ in (90) by cy^2 yields (80).

Remark 8.4.6.2. Replacing $2(a+b+c)$ in (91) by $b+2c$ yields (81).

8.4.7 $x, y, z \longrightarrow x + \imath y$

Likewise, we get

$$(a+b+c-1)(x^2-y^2) + \imath y\{2(a+b+c-1)x\} = 0. \tag{92}$$

It follows from (92) that

$$\begin{cases} (a+b+c-1)(x^2-y^2) = 0, & (93) \\ (a+b+c-1)xy = 0. & (94) \end{cases}$$

Remark 8.4.7.1. Putting $a+b+c-1 = 1$in (93) gives (71).

Remark 8.4.7.2. Putting $a+b+c-1 = 1$in (94) gives (72).

Taken together, our response to *Question* 8.4.2 (or 8.4.1) is

Answer 8.4.3. Yes (at least formally).

Having answered question(s), we now wish to turn our attention to ‘trinions’ (t_r’s) [13] and raise questions about them.

9 t_r-related question 1: what if we apply t_r’s to QE?

Actually, we will try to answer this question in **Discussion**.

9.1 Preliminaries

$t_r := a+bi+cj, a, b, c \in \mathbb{R}$ $[13, \textit{Def. } 2.1.4]$. Assuming $\sqrt{b^2+c^2} \neq 0$, one makes some computation to write

$$t_r = a + \sqrt{b^2+c^2} \cdot (\tfrac{bi}{\sqrt{b^2+c^2}} + \tfrac{cj}{\sqrt{b^2+c^2}}) = a + u(\alpha i + \beta j),$$

where $u = \sqrt{b^2+c^2}$, and $\alpha^2 + \beta^2 = 1$. Rewriting

$$\begin{cases} \alpha = \cos\theta, & \\ \beta = \sin\theta, & 0 \leq \theta < 2\pi \end{cases}$$

yields

$$t_r = a + u(i\cos\theta + j\sin\theta). \tag{95}$$

We further rewrite this as $t_r = a + uv(\theta)$ and make a definition.

Definition 9.1.1. $a + uv(\theta)$ is angular form of t_r's.

Example 9.1.2. $1 + i$ in angular form is $1 + v(0)$.

Example 9.1.3. $2 + i + \sqrt{3}j$ in angular form is $2 + 2v(\frac{\pi}{3})$.

Claim 9.1.4.

$$v(\theta_1)v(\theta_2) = 0. \tag{96}$$

Proof. A straightforward calculation. Specifically, $(i\cos\theta_1 + j\sin\theta_1) \cdot (i\cos\theta_2 + j\sin\theta_2) = i^2\cos\theta_1\cos\theta_2 + ij\cos\theta_1\sin\theta_2 + ji\sin\theta_1\cos\theta_2 + j^2\sin\theta_1\sin\theta_2 = 0 \cdot \cos\theta_1\cos\theta_2 + 0 \cdot \cos\theta_1\sin\theta_2 + 0 \cdot \sin\theta_1\cos\theta_2 + 0 \cdot \sin\theta_1\sin\theta_2 = 0$ [70] . QED.

Remark 9.1.5. If one replaces θ_1, θ_2 in (96) by θ, one gets $v(\theta)^2 = 0$, that is, $v(\theta) = 0$ for any θ. In this case, t_r becomes $a + u \cdot 0 = a$, or $\mathrm{Sc}(t_r)$ [71] .

9.2 Applying t_r's to QE

We consider the following cases, depending on the properties of the roots of (1):

Case 1. Both roots are real.

Case 2. Both roots are complex.

Case 3. Roots are MR.

In what follows, we present examples corresponding to each case.

---

[70] For computation of this, see [13, **Table 1**].

[71] See [13, **2.1**].

9.2.1 The case both roots are real

Example 9.2.1.1. $x^2 - 4x + 3 = 0$. Its roots are 1 and 3. Writing x as

$$x = a + bi + cj, \tag{97}$$

we compute

$$x^2 = a^2 + 2abi + 2acj.\text{[72]} \tag{98}$$

We substitute (97) and (98) into the LHS of this example. After some manipulation, one gets

$$a^2 - 4a + 3 + 2b(a-2)i + 2c(a-2)j = 0. \tag{99}$$

We thus have

$$\begin{cases} a^2 - 4a + 3 = 0, & (100) \\ 2b(a-2) = 0, & (101) \\ 2c(a-2) = 0. & (102) \end{cases}$$

If $a = 2$, (101) and (102) hold, irrespective of b, c. But (100) doesn't, since $2^2 - 4 \cdot 2 + 3 = -1 \neq 0$. So $a \neq 2$, and we necessarily have $b = c = 0$, which means $\mathrm{Vec}(t_r)$ [73] vanishes, whereas $\mathrm{Sc}(t_r)$ remains. In other words, we are induced to 'forget' $\mathbb{T}_r$ [74] and left with $\mathbb{R}$. Now that (101) and (102) are trivial, we just solve (100) to get the roots 1 and 3, as desired.

9.2.2 The case both roots are complex

Example 9.2.2.1. $x^2 + 2x + 5 = 0$. Its roots are $-1 \pm 2\imath$. In a manner similar to the previous subsubsection, one gets $a^2 + 2a + 5 + 2b(a+1)i + 2c(a+1)j = 0$ and thus has

$$\begin{cases} a^2 + 2a + 5 = 0, & (103) \\ 2b(a+1) = 0, & (104) \\ 2c(a+1) = 0. & (105) \end{cases}$$

Likewise, if $a = -1$, (104) and (105) hold, irrespective of b, c. But (103) doesn't, since $(-1)^2 + 2 \cdot (-1) + 5 = 4 \neq 0$. So $a \neq -1$, and we have $b = c = 0$, which means we are induced to 'forget' $\mathbb{T}_r$ again. (104) and (105) being trivial, we just solve (103) to get the roots $-1 \pm 2\imath$, as desired. Roots that have been obtained in terms of t_r's have so far coincided with those computed in a usual manner, which might give us some relief. But will this situation really continue?

---

[72] For computation of this, see [13, **Table 1**].

[73] See [13, **2.1**].

[74] Ditto.

9.2.3 The case roots are multiple.

Example 9.2.3.1. $x^2 + 2x + 1 = 0$. Its root is -1 only. Likewise, one gets $a^2 + 2a + 1 + 2b(a+1)i + 2c(a+1)j = 0$ and thus has

$$\begin{cases} a^2 + 2a + 1 = 0\,, & (106) \\ 2b(a+1) = 0\,, & (107) \\ 2c(a+1) = 0\,. & (108) \end{cases}$$

If $a = -1$, (107) and (108) hold, irrespective of b, c. This time, (106) does, too, since $(-1)^2 + 2 \cdot (-1) + 1 = 0$. Since -1 is the sole root, $a = -1$, and b, c are arbitrary. That is, we get the root $-1 + bi + cj$, $b, c \in \mathbb{R}$. In other words, the QE under consideration has been shown to have infinitely many solutions, since b, c are arbitrary real numbers. This seems rather unusual, and we further get interested in whether t_r's form a group in a normal sense, raising the following question.

10 t_r-related question 2: what about group formation under addition or multiplication?

It is known that for a nonempty set together with a binary operation to be called a group G, it needs to meet the following axioms [14]:

1. Closure [75]
2. Identity element
3. Associativity
4. Inverse element

In what follows, we check whether t_r's meet these axioms in a step-by-step manner.

10.1 Checking whether they form a group under addition

10.1.1 Closure

Axiom is that for all a, b in G, the result of the operation $a \cdot b$ is also in G ; we consider two elements in $\mathbb{T}_r$, *i.e.*,

$$\begin{cases} t_{r1} = a_1 + b_1 i + c_1 j, \qquad a_1, b_1, c_1 \in \mathbb{R}, & (109) \\ t_{r2} = a_2 + b_2 i + c_2 j, \qquad a_2, b_2, c_2 \in \mathbb{R}. & (110) \end{cases}$$

Then, we have

---

[75] This is sometimes omitted. See *e.g.,* here and [15].

$t_{r1} + t_{r2} = a_1 + b_1 i + c_1 j + a_2 + b_2 i + c_2 j = (a_1 + a_2) + (b_1 + b_2)i + (b_2 + c_2)j,$

which we rewrite as

$$t_{r3} = a_3 + b_3 i + c_3 j, \qquad a_3, b_3, c_3 \in \mathbb{R}. \tag{111}$$

So t_{r3}, the sum of t_{r1} and t_{r2}, is an element of $\mathbb{T}_r$, again, which means that this axiom holds.

10.1.2 Identity element

Axiom is that there is an element e in G such that for all elements a in G, one has $e \cdot a = a \cdot e = a$; first, we write

$$\begin{cases} t_{rid} = a_{rid} + b_{rid} i + c_{rid} j, \qquad a_{rid}, b_{rid}, c_{rid} \in \mathbb{R}, & (112) \\ t_r = a_r + b_r i + c_r j, \qquad a_r, b_r, c_r \in \mathbb{R}. & (113) \end{cases}$$

Then, we compute $t_{rid} + t_r = a_{rid} + b_{rid} i + c_{rid} j + a_r + b_r i + c_r j = a_{rid} + a_r + (b_{rid} + b_r)i + (c_{rid} + c_r)j$. For t_{rid} to be actually an identity element $t_{r\mathrm{ID}}$, this needs to amount to t_r. That is, we have

$$\begin{cases} a_{rid} + a_r = a_r, & (114) \\ b_{rid} + b_r = b_r, & (115) \\ c_{rid} + c_r = c_r. & (116) \end{cases}$$

It follows from $(114) - (116)$ that $a_{rid} = b_{rid} = c_{rid} = 0$. Likewise, computing $t_r + t_{rid}$, we get

$$\begin{cases} a_r + a_{rid} = a_r, & (117) \\ b_r + b_{rid} = b_r, & (118) \\ c_r + c_{rid} = c_r. & (119) \end{cases}$$

It follows from $(117) - (119)$ that $a_{rid} = b_{rid} = c_{rid} = 0$, again. Taken together, $t_{r\mathrm{ID}} = 0 + 0 \cdot i + 0 \cdot j = 0$ is the identity element, and thus, this axiom is satisfied.

10.1.3 Associativity

Axiom is that for all a, b and c in G, one has $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; using $(109) - (111)$, we compute

$(t_{r1} + t_{r2}) + t_{r3} = (a_1 + b_1 i + c_1 j + a_2 + b_2 i + c_2 j) + a_3 + b_3 i + c_3 j = a_1 + b_1 i + c_1 j + a_2 + b_2 i + c_2 j + a_3 + b_3 i + c_3 j.$

On the other hand,

$t_{r1} + (t_{r2} + t_{r3}) = a_1 + b_1 i + c_1 j + (a_2 + b_2 i + c_2 j + a_3 + b_3 i + c_3 j) = a_1 + b_1 i + c_1 j + a_2 + b_2 i + c_2 j + a_3 + b_3 i + c_3 j.$

We observe we have $(t_{r1} + t_{r2}) + t_{r3} = t_{r1} + (t_{r2} + t_{r3})$, and thus this axiom is met.

10.1.4 Inverse element

Axiom is that for each a in G, there exists an element b in G such that $a \cdot b = b \cdot a = e$, where e is the identity element. As shown in **10.1.2**, $t_{r\mathrm{ID}} = 0$ is the identity element. So for $t_{riv} = a_{riv} + b_{riv}i + c_{riv}j$, where $a_{riv}, b_{riv}, c_{riv} \in \mathbb{R}$, to be an inverse element, $t_r + t_{riv} = a_r + b_r i + c_r j + a_{riv} + b_{riv}i + c_{riv}j$ needs to amount to 0. That is, we have

$$\begin{cases} a_r + a_{riv} = 0, & (120) \\ b_r + b_{riv} = 0, & (121) \\ c_r + c_{riv} = 0. & (122) \end{cases}$$

It follows from $(120) - (122)$ that $a_{riv} = -a_r$, $b_{riv} = -b_r$, and $c_{riv} = -c_r$. Likewise, computing $t_{riv} + t_r$, we get $a_{riv} + a_r = b_{riv} + b_r = c_{riv} + c_r = 0$. Thus, $a_{riv} = -a_r$, $b_{riv} = -b_r$, and $c_{riv} = -c_r$, again. So $t_{riv} = -a_r - b_r i - c_r j$ is the inverse element $t_{r\mathrm{IV}}$.

Taken together, $\mathbb{T}_r$ meets the aforementioned four axioms. Hence, they form a group under addition.

10.2 Checking whether they form a group under multiplication

Paying attention to the relation $\frac{1}{t_r} = \frac{\bar{t_r}}{a^2}$, where $\bar{t_r} = a - bi - cj$ [13, (5) or (6)], we assume $a \neq 0$ in what follows [76].

10.2.1 Closure

We substitute [13, (1)] for showing that this axiom holds [77].

10.2.2 Identity element

Again, axiom is that there exists an element e in G such that for all elements a in G, one has $e \cdot a = a \cdot e = a$. [78] Using (112) and (113), we compute

$$\begin{aligned} t_{rid} \cdot t_r &= (a_{rid} + b_{rid}i + c_{rid}j) \cdot (a_r + b_r i + c_r j) \\ &= a_{rid} \cdot (a_r + b_r i + c_r j) + b_{rid}i \cdot (a_r + b_r i + c_r j) + c_{rid}j \cdot (a_r + b_r i + c_r j) \\ &= a_{rid}a_r + a_{rid}b_r i + a_{rid}c_r j + b_{rid}ia_r + b_{rid}ib_r i + b_{rid}ic_r j + c_{rid}ja_r + c_{rid}jb_r i + c_{rid}jc_r j \\ &= a_{rid}a_r + (a_{rid}b_r + b_{rid}a_r)i + (a_{rid}c_r + c_{rid}a_r)j. \end{aligned}$$

For t_{rid} to be an identity element, this needs to amount to t_r. That is,

---

[76] Incidentally, every nonzero quaternion has an inverse wrt the Hamilton product.

[77] Replacing a_1a_2, $a_1b_2 + a_2b_1$, and $a_1c_2 + a_2c_1$ therein by *e.g.*, d, e, and f, respectively yields $t_{r1} \cdot t_{r2} = d + ei + fj$, where d, e, $f \in \mathbb{R}$. Thus, $t_{r1} \cdot t_{r2} \in \mathbb{T}_r$.

[78] See **10.1.2**.

$$\begin{cases} a_{rid}a_r = a_r, & (123) \\ a_{rid}b_r + b_{rid}a_r = b_r, & (124) \\ a_{rid}c_r + c_{rid}a_r = c_r. & (125) \end{cases}$$

It follows from (123) that $(a_{rid} - 1)a_r = 0$. Since a_r is arbitrary, $a_{rid} = 1$. Substituting this into (124) and (125), one gets $b_{rid}a_r = c_{rid}a_r = 0$. Again, since a_r is arbitrary, $b_{rid} = c_{rid} = 0$.

We also check whether $a \cdot e = e$ holds. Likewise, we equate $t_r \cdot t_{rid}$ with t_r. Since t_r's are commutative under multiplication [79] , we have $t_r \cdot t_{rid} = t_{rid} \cdot t_r$. So we can say we have virtually done checking in our preceding arguments. That is, in either case, $a_{rid} = 1$, and $b_{rid} = c_{rid} = 0$; $t_{rid} = 1 + 0 \cdot i + 0 \cdot j = 1$ is the identity element $t_{r\mathrm{ID}}$. Hence, this axiom is satisfied.

10.2.3 Associativity

This axiom is met [12, **7.1**].

10.2.4 Inverse element

Again, axiom is that for each a in G, there exists an element b in G such that $a \cdot b = b \cdot a = e$, where e is the identity element. [80] As mentioned earlier,

$$\frac{1}{t_r} = \frac{a - bi - cj}{a^2}. \tag{126}$$

Multiplication of (126) by t_r on the left side yields

$$t_r \cdot \tfrac{1}{t_r} = t_r \cdot \tfrac{a-bi-cj}{a^2},$$

whose RHS amounts to

$$(a + bi + cj) \cdot \tfrac{a-bi-cj}{a^2} = \tfrac{(a+bi+cj)\cdot(a-bi-cj)}{a^2} = {}^{81}\ \tfrac{a^2}{a^2} = 1,$$

i.e., $t_{r\mathrm{ID}}$.

On the other hand, multiplication of (126) by t_r on the right side yields

$$\tfrac{1}{t_r} \cdot t_r = \tfrac{a-bi-cj}{a^2} \cdot t_r,$$

whose RHS amounts to

$$\tfrac{a-bi-cj}{a^2} \cdot (a + bi + cj) = \tfrac{(a-bi-cj)\cdot(a+bi+cj)}{a^2} = {}^{82}\ \tfrac{a^2}{a^2} = 1,$$

i.e., $t_{r\mathrm{ID}}$. One thus can say $\frac{1}{t_r} \in \mathbb{T}_r$ [83] is an inverse element $t_{r\mathrm{IV}}$. Hence, $\mathbb{T}_r$ meets the aforementioned four axioms to form a group under multiplication, if $a \neq 0$ [84] .

---

[79] See [13, **3.1**].

[80] See **10.1.4**

[81] See [13, (3)].

[82] See [13, (4)].

[83] If one rewrites (126) as *e.g.*, $\frac{1}{t_r} = d + ei + fj$, where $d, e, f \in \mathbb{R}$, one easily sees it is an element of $\mathbb{T}_r$. *Cf.* [13, *Def.* 2.1.4].

[84] We might discuss the case $a = 0$ elsewhere.

11 Discussion

Since we have already mentioned CF's [85] , at the outset, we should like to consider the following.

$$f(x) = ax^3 + bx + c, \qquad a, b, c \in \mathbb{R}, \qquad a \neq 0. \tag{127}$$

This can be thought of as

$$g(x) = ax^3 + dx^2 + ex + f, \qquad a, d, e, f \in \mathbb{R}, \qquad a \neq 0$$

subjected to TT [86]. Then, we have

$$f'(x) = 3ax^2 + b, \qquad f''(x) = 6ax,$$

where the character $'$ stands for differentiation wrt x. Setting $f''(x) = 0$, we get $x = 0$, IP of $f(x)$, since $a \neq 0$ [87]. Putting $x = 0$ in $f(x)$ and $f'(x)$, one further gets

$$f(0) = c, \quad f'(0) = b,$$

which amount to some coefficients in the RHS of (127). Recalling that the RHS of (127) was regarded as something subjected to TT, we now understand the relationship between TT and IP [88].

As for Fig. 1, it seems interesting to point out that *env* is a solution of Clairaut's equation and can be a parabola like those seen in $(a)-(c)$. We also suggest the possibility that (a) has something to do with strip [16, FIG. 21(b)]. As for Fig. 2, if we are allowed to interpret the points -1 and 1 plotted on $\mathbb{R}^1$ as the set $\{-1, 1\}$, it *is* a group that is usually denoted $\mathbb{Z}_2$ [14]. As for φ, we raise a question similar to *Question* 3.2.2.

Question 11.1. What does computing φ mean at all?

We try to answer this question. We *a priori* consider

$$\frac{x}{y} - \frac{y}{x} = 1 \tag{128}$$

to replace its y's by 1's. Then, we get

$$\frac{x}{1} - \frac{1}{x} = 1. \tag{129}$$

Multiplying both sides of the above by x, we get $x^2 - 1 = x$, ***i.e.***, (44). This makes us aware that we have already been involved in computing φ and $1 - \varphi$, the roots of (44), while making such replacement.

By the way, multiplying both sides of (128) by xy, one gets $x^2 - y^2 = xy$ [89] . Regarding this as a QE wrt y, one solves it to get

---

[85] See *e.g.*, Figs. 8 and 11.

[86] *Cf.* **3.1**.

[87] See (127).

[88] By the way, we showed the relevance of TT to 'tracelessness' in **3.1**.

[89] One can derive this from (80) by setting $a+b+c=1$ and $c=1$. It is also derivable from (90) by setting $a+b+c=1$.

$$y = \frac{-x \pm \sqrt{5}|x|}{2}^{90}.$$

Since $x \neq 0$ in (129), we deal with the following cases.

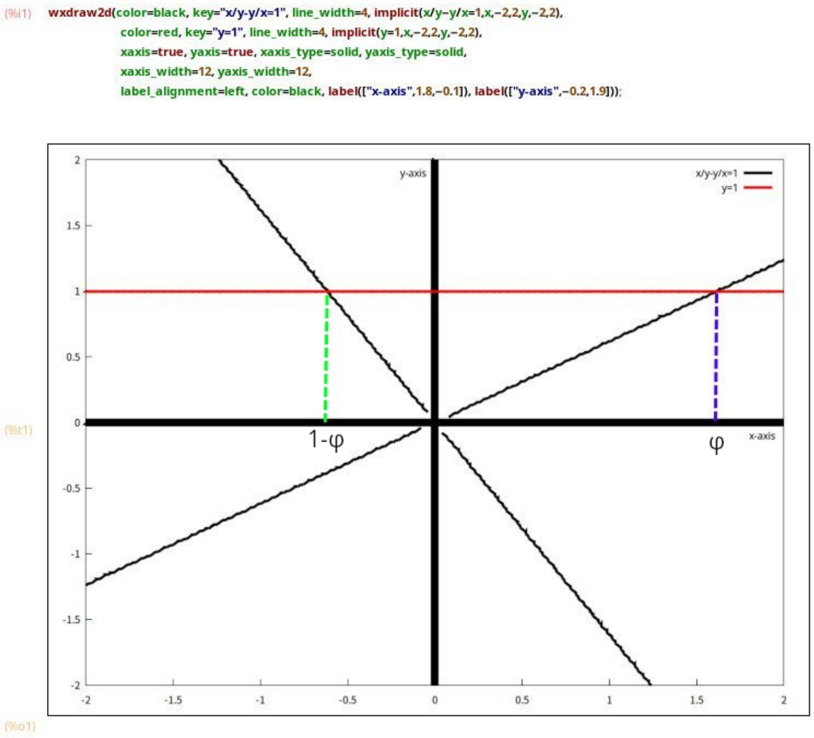
Case 1. $x > 0$. So $y = \frac{-x \pm \sqrt{5}x}{2}$.

Case 2. $x < 0$. So $y = \frac{-x \pm \sqrt{5}(-x)}{2} = \frac{-x \mp \sqrt{5}x}{2}$.

Since these cases are mathematically the same, it suffices to consider *Case* 1 only. Next, we separately rewrite $y = \frac{-x \pm \sqrt{5}x}{2}$ as

$$\begin{cases} y = \frac{-x+\sqrt{5}x}{2} = x \cdot \frac{-1+\sqrt{5}}{2} = (\varphi - 1)x, & (130) \\ \qquad\qquad \text{and} & \\ y = \frac{-x-\sqrt{5}x}{2} = -x \cdot \frac{1+\sqrt{5}}{2} = -\varphi x. & (131) \end{cases}$$

Thus, we have obtained φ. By the way, we have derived (129) from putting $y = 1$ in (128). What does this mean geometrically? We visualise (128) and $y = 1$:



```
(%i1) wxdraw2d(color=black, key="x/y-y/x=1", line_width=4, implicit(x/y−y/x=1,x,−2,2,y,−2,2),
               color=red, key="y=1", line_width=4, implicit(y=1,x,−2,2,y,−2,2),
               xaxis=true, yaxis=true, xaxis_type=solid, yaxis_type=solid,
               xaxis_width=12, yaxis_width=12,
               label_alignment=left, color=black, label(["x-axis",1.8,−0.1]), label(["y-axis",−0.2,1.9]));
```

Fig. 21. Visualisation of (128) and the line $y = 1$. Original wxMaxima image was retouched by GIMP ver. 2.10.38 and Pinta ver. 2.1.2.

In the above figure, the characters φ and $1 - \varphi$ denote the solutions to (44). Hence,

---

[90] Here we have used QF.

Answer 11.2. Adding a line to the graph of $\frac{x}{y} - \frac{y}{x} = 1$.

This is a rather 'geometric' answer.

Incidentally, since $(\varphi - 1) \cdot (-\varphi) = -(\varphi^2 - \varphi) = -1$, and $(x, y) = (0, 0)$ is a solution to the system of equations $(130) - (131)$, some might be ready to say that they intersect vertically at O. But actually, those lines are 'punctured' at O as shown in Fig. 21. Moreover, it is clear from (128) that $(x, y) \neq (0, 0)$. So it seems hard to think they do so in a normal sense.

Remark 11.3. That said, one can imagine that they do so by ignoring such a 'puncture', if one is (unnecessarily) imaginative [91] .

What about the following question, then?

Question 11.4. What if we imagine

$$\frac{x}{1} + \frac{1}{x} \tag{132}$$

instead of the LHS of (129)?

We try to answer this in terms of EMC. If one puts $x =$EMC in (132), one gets

$$\frac{\text{EMC}}{1} + \frac{1}{\text{EMC}}. \tag{133}$$

The value of (133) is known to be $2.3096\ldots$. What about its (ir)rationality? If it is a rational, then, using $p, q \in \mathbb{Z}$, one can write

$$\frac{\text{EMC}}{1} + \frac{1}{\text{EMC}} = \frac{p}{q}, \qquad q \neq 0.$$

That is, we deal with $q\text{EMC}^2 - p\text{EMC} + q = 0$, which leads us to consider

$$qx^2 - px + q = 0,$$

a QE wrt x whose root is EMC. Since this is an algebraic equation, its roots include rational numbers and irrational numbers . Hence,

Answer 11.5. We might get faced with the (ir)rationality of EMC.

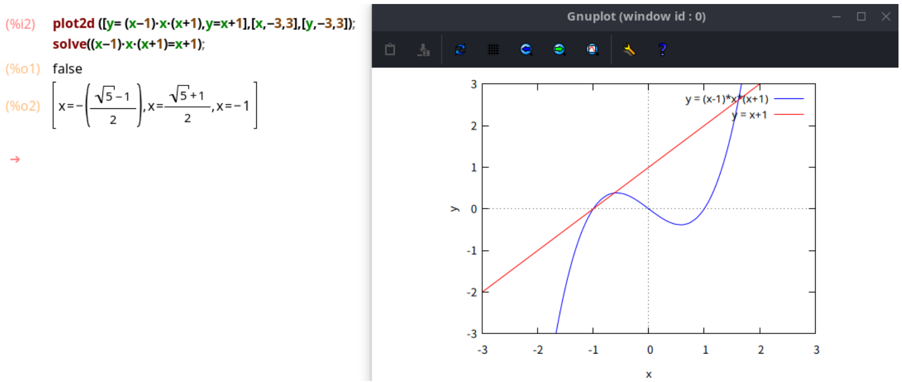
This is the 'tidbit' implicated in **Abstract**. Now let us get back to the matter of φ and raise a question.

Question 11.6. Can CF play a role similar to (128) in Fig. 21?

In order to answer this question, we draw the following:

---

[91] Some might recall vector field corresponding to a differential form on the punctured plane .



```
(%i2) plot2d ([y= (x−1)·x·(x+1),y=x+1],[x,−3,3],[y,−3,3]);
      solve((x−1)·x·(x+1)=x+1);
(%o1) false
```

(%o2) $\left[x=-\left(\frac{\sqrt{5}-1}{2}\right),x=\frac{\sqrt{5}+1}{2},x=-1\right]$

Fig. 22. Visualisation of the CF $y = x^3 - x$ and the line $y = x + 1$

In Fig. 22, one can easily ‘see’ $1 - \varphi$ and φ (plus -1) on the x-axis. Hence,

Answer 11.7. Yes.

Again, addition of a line revealed ‘latent’ ϕ and $1 - \phi$ [92]. In addition to dealing with QE, this might also serve as a propaedeutic to revisiting CE, since a CE appeared and was solved in Fig. 22.

As for verifications performed in **5.3**, since our chief interest consisted in math underlying NM and our method, we introduced only mathematical difference to two kinds of codes used in our verifications — the difference between the equations to which we appealed [93] —, expecting that allowing for such difference would sharpen the contrast between them. In other words, other parameters like initial value and iteration number were basically identical.

Incidentally, it is known that convergence in NM hinges upon the choice of the initial value x_0 [17]. So we wish to turn our attention to ‘initial value problem’, by which we mean we are interested in how the difference in x_0 can affect convergence. We start from an example related to (47) and (48).

Example 11.8. $a = 1, b = -1, c = -6, x_0 = \sqrt{6}$.

N.B. The following Clojure computation is carried out in conformity to *Steps* $1 - 3$ in **5.3**.

```
% more cubic_eq_part_1_discuss_1.clj
(defn discuss_1[i]
(if(< i 2)(Math/sqrt 6)
(Math/sqrt(+ 6(discuss_1(dec i))))))
(doseq [x (range 1 26)]
(prn(discuss_1 x)))
% time clojure cubic_eq_part_1_discuss_1.clj
2.449489742783178
2.9068006025152773
```

---

[92] *Cf.* Fig. 21.

[93] Compare *e.g.*, *Step* 3 of *Example* 5.3.3 with (50).

```
2.984426343958798
2.9974032668226007
2.9995671799148957
2.999927862451845
2.9999879770512154
2.9999979961745336
2.9999996660290704
2.999999944338178
2.9999999907230297
2.999999998453838
2.999999999742306
2.999999999957051
2.9999999999928417
2.9999999999988067
2.999999999999801
2.9999999999999667
2.9999999999999942
2.999999999999999
3.0
3.0
3.0
3.0
3.0
1.288u 0.114s 0:00.54 257.4% 0+0k 0+0io 0pf+0w
```

Double-checking is done by ECL:

```
% more cubic_eq_part_1_discuss_2.lsp
(do((n 1 (incf n)))((>= n 26))
(defun discuss_2(n)(if(< n 2)(sqrt 6)
(sqrt (+ (discuss_2(- n 1)) 6))))
(format t "~2d~%" (discuss_2 n)))
(quit)
% time ecl --load cubic_eq_part_1_discuss_2.lsp
2.4494898
2.9068005
2.9844263
2.9974034
2.9995673
2.9999278
2.9999878
2.9999979
2.9999998
3.0
3.0
```

```
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
0.066u 0.006s 0:00.05 120.0% 0+0k 6840+0io 22pf+0w
```

Hence, we have confirmed that the recurrence converges to 3. In this example, the recurrence started from $\sqrt{6}$. What if it starts from, say 6? Here is another one:

Example 11.9. $a = 1, b = -1, c = -6, x_0 = 6.$

```
% more cubic_eq_part_1_discuss_3.clj
(defn discuss_3 [i]
(if(< i 2)6
(Math/sqrt(+ 6(discuss_3 (dec i))))))
(doseq [x (range 1 26)]
(prn(discuss_3 x)))
% time clojure cubic_eq_part_1_discuss_3.clj
6
3.4641016151377544
3.076378002641703
3.0127027736970176
3.0021163824370665
3.0003527096721587
3.000058784369426
3.000009797378906
3.00000163289604
3.000000272149328
3.000000045358221
3.0000000075597035
3.000000001259951
3.0000000002099916
3.0000000000349987
3.000000000005833
3.000000000000972
```

```
3.000000000000162
3.000000000000027
3.0000000000000044
3.0000000000000004
3.0
3.0
3.0
3.0
1.276u 0.130s 0:00.54 259.2% 0+0k 0+0io 0pf+0w
```

This is double-checked in a similar fashion.

```
% more cubic_eq_part_1_discuss_4.lsp
(do((n 1 (incf n)))((>= n 26))
(defun discuss_4(n)(if(< n 2)6
(sqrt (+ (discuss_4(- n 1)) 6))))
(format t "~2d~%" (discuss_4 n)))
(quit)
% time ecl --load cubic_eq_part_1_discuss_4.lsp
 6
3.4641016
3.076378
3.0127027
3.0021164
3.0003526
3.000059
3.0000098
3.0000017
3.0000002
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
3.0
0.059u 0.011s 0:00.04 150.0% 0+0k 0+0io 0pf+0w
```

Thus, we have confirmed the recurrence converges to 3. Taken together, x_0 doesn't matter at least in these examples.

Here we recall FS and its 'numbness'. But what is that 'numbness'? The sequence is either

$$[0, 1], 1, 2, 3, 5, 8, \cdots$$

or

$$[1, 1], 2, 3, 5, 8, 13, \cdots ,$$

where 'initial values' are emphasised by brackets. Since whether the 'initial value' is $[0, 1]$ doesn't matter in the long run, in a sense, FS seems 'callous' to that kind of condition. And we dare to call such a property 'Fibonacci numbness'.

Remark 11.10. By contrast, roughly speaking, chaotic systems exhibit a great sensitivity to initial conditions.

But does such 'Fibonacci numbness' prevail? Specifically, what about the recurrence other than (45)? We take

$$a_{n+2} = 2a_{n+1} - 4a_n \tag{134}$$

for example.

Remark 11.11. (134) can be regarded as a part of

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 2 \end{pmatrix}^{94} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}.$$

First, we run '[0,1]' Ruby code.

```
% which ruby
/usr/bin/ruby
% ruby -v
ruby 3.3.4 (2024-07-09 revision be1089c8ec) [x86_64-linux-gnu]
% more qe_starting_from_0_1.rb
#!/usr/bin/ruby
def qe_starting_from_0_1(n)
f1,f2=0,1
while f1<=n
puts f1
f1,f2=f2,-4*f1+2*f2
end end
puts qe_starting_from_0_1(4444)
```

---

[94] This 2×2 matrix is a kind of T_2. *Cf. Example* 7.1.

```
% ruby qe_starting_from_0_1.rb
0
1
2
0
-8
-16
0
64
128
0
-512
-1024
0
4096
```

Remark 11.12. These values coincide with a part of this sequence .

Next, '[1, 1]' code is run.

```
% more qe_starting_from_1_1.rb
#!/usr/bin/ruby
def qe_starting_from_1_1(n)
f1,f2=1,1
while f1<=n
puts f1
f1,f2=f2,-4*f1+2*f2
end end
puts qe_starting_from_1_1(4444)
% ruby qe_starting_from_1_1.rb
1
1
-2
-8
-8
16
64
64
-128
-512
-512
1024
4096
4096
-8192
```

```
-32768
-32768
```

Remark 11.13. These values coincide with a part of this sequence .

That the difference between $[0, 1]$ and $[1, 1]$ gave rise to a non-negligible difference in values is noteworthy; the recurrence (134) seems ‘not-so-numb’ to the difference in ‘initial values’ unlike (45).

Talking of closed form , one seems to prefer a closed-form expression to a recurrence [18]. But are closed-forms *always* superior to recurrences? Recalling **6.1.2**, we compare a closed-form expression with a recurrence, both of which are written in ALGOL and yield FS.

First, we run closed-form version.

```
% a68g -v
Algol 68 Genie 3.1.2
Copyright 2001-2023 Marcel van der Veer <algol68g@xs4all.nl>.
% more fib-cl.a68
PROC closed fibonacci = (INT n)INT:(
  REAL sqrt 5=sqrt(5);
  REAL p=(1+sqrt 5)/2;
CO
p is GR.
CO
  REAL q=1-p;
  ENTIER((p**n-q**n)/sqrt 5));
FOR i FROM 0 TO 16 WHILE
  print(whole(closed fibonacci(i),0));
# WHILE # i /= 16 DO
  print(", ")
OD;
print(new line)
% a68g -heap=2048M fib-cl.a68
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 54, 89, 143, 232, 377, 610, 986
```

In the above program, we appealed to the closed-form expression $\frac{\phi^n-(1-\phi)^n}{\sqrt{5}}$; the values 54, 143, 232, and 986 should be 55, 144, 233, and 987, respectively [95] .

---

[95] See *e.g.*, here .

We now run recurrence-version:

```
% more fib-rec.a68
PROC rec fibonacci= (INT n)INT:
  ( n < 2 | n | rec fibonacci(n-1) + rec fibonacci(n-2));
  FOR i FROM 0 TO 16 WHILE
  print(whole(rec fibonacci(i),0));
# WHILE # i /= 16 DO
  print(", ")
OD;
print(new line)
CO
Cf.https://rosettacode.org/wiki/Fibonacci_sequence#Recursive_3
CO
% a68g -heap=2048M fib-rec.a68
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987
```

This time, one gets correct values . By the way, R can also yield such values as shown below.

```
% more fibo.R
fibo<-function(n){
if(n<2)n else Recall(n-1)+Recall(n-2)}
print.table(lapply(0:16,fibo))
# Cf.
# https://rosettacode.org/wiki/Fibonacci_sequence#R
# https://rosettacode.org/wiki/Comments#R
% Rscript --version
Rscript (R) version 4.4.1 (2024-06-14)
% Rscript fibo.R
[1] 0   1   1   2   3   5   8   13  21  34  55  89  144 233 377
610 987
```

So it seems use of a closed-form expression can cause a practical problem. Ironically, FS betrayed some subtlety math and computer programming entail.

We move on to t_r's to discuss matrix representation of i and j [13, **Table 1**]. They can be represented as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively, since $i^2 = ij = O_3$, etc. hold [96] .

Remark 11.14. Let $\theta = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$. Then, $\theta^2 = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) = j$. Moreover, we have

$(\theta_i)^2 = 0$. So in addition to *Claim* 9.1.4 and *Remark* 9.1.5, these might make one recall Grassmann number .

Now it is time we answered 't_r-related question 1' [97] .

Answer 11.15. There can be infinitely many solutions, which are seen in *Example* 9.2.3.1.

And it is natural that the following question should ensue.

Question 11.16. Do such solutions pose a (serious) drawback of $\mathbb{T}_r$?

Answer 11.17. Yes. They are problematic especially in terms of the uniqueness of QE solutions.

Though we are somewhat embarrassed at this answer, circumstances are not so futile. Thinkable reasons include

1. We recall that every Lie group gives rise to a Lie algebra, which is the tangent space at the identity . We apply this to our case. Then, regarding $bi + cj$ as (b, c), where $b, c \in \mathbb{R}$ (like a point in $\mathbb{R}^2 = \mathbb{C}$), we can imagine a TP;
2. Equations like Diophantine equations can have infinitely many solutions ;

etc.

By regarding a TP in Reason 1 as a three-dimensional object like this , we can 'higher-dimensionalise' the complex plane, which we mentioned in *e.g.*, **8**. Taking this opportunity, we should like to apply this 'higher-dimensionalisation' to Fig.'s 8 and 11 as follows.

N.B. 'O's' in the following figures are interpreted as the origins $(0, 0, 0)$.

---

[96] *Cf.* [12, footnote 28].

[97] See **9**.

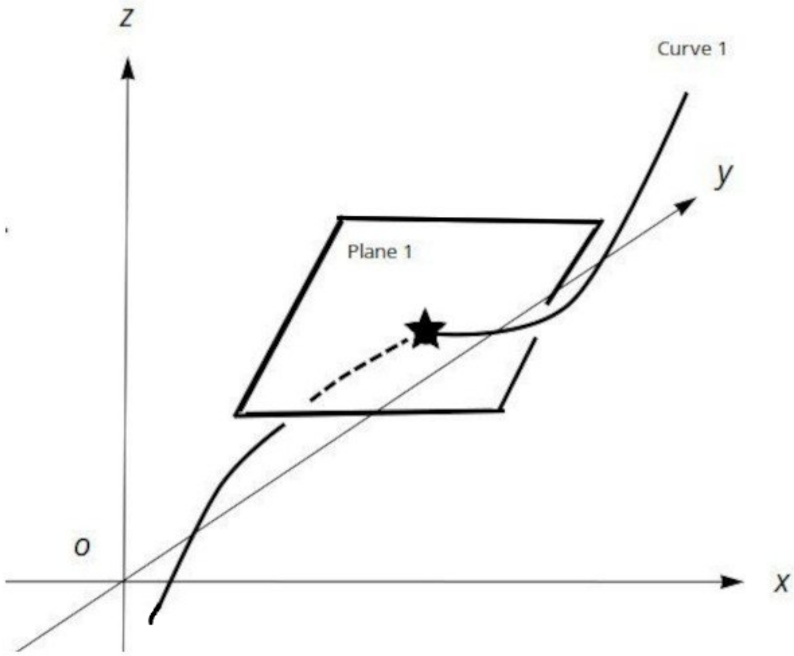


Fig. 23. ‘3D-isation’ of Fig. 8. Original figure, which was drawn by Pinta ver. 2.1.2, has been slightly retouched by GIMP ver. 2.10.38.

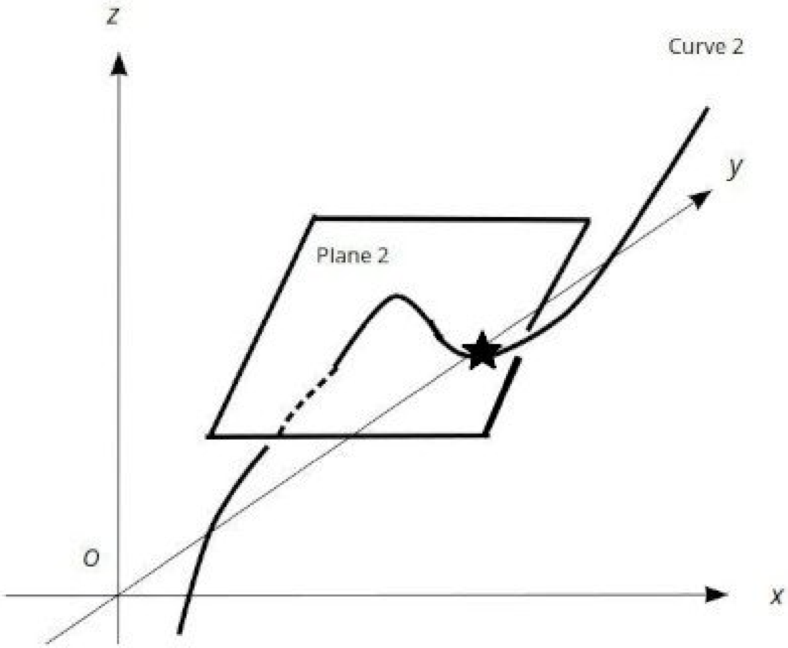


Fig. 24. ‘3D-isation’ of Fig. 11. This figure was prepared like Fig. 23.

In either case, starting from a root-finding algorithm for QE's, one ends up with CF yielding trajectory -like stuff in 3D space. So the transition from QE to CE which will be dealt with in a full-fledged fashion in the sequel is smooth (again hopefully). Furthermore, consider *e.g.*,

$$f(x) \longrightarrow xf(x) + \text{some constant.} \tag{135}$$

Then, starting from $f(x) = 0$, we get the 'sequence'

$$\begin{aligned} &0 \\ &\downarrow \\ &a \\ &\downarrow \\ &ax + b \qquad (136) \\ &\downarrow \\ &ax^2 + bx + c \qquad (137) \\ &\downarrow \\ &ax^3 + bx^2 + cx + d \qquad (138) \\ &\downarrow \\ &\cdots \end{aligned}$$

So we can imagine that if things go neatly and/or sequentially like $(136) - (138)$ dictated by (135), we are to deal with the CE $ax^3 + bx^2 + cx + d = 0$ elsewhere.

In summary, we hope questions, answers, etc. described herein will serve as a propaedeutic to revisiting CE, since many of the deepest questions in mathematics still involve questions about cubics [19].

Finally, we can also imagine fibre bundle [20, Figure 3.2]-like structure by '3D-ising' *e.g.*, Fig. 3, but wonder if it is premature to put forward the notion of 'fibre algebra/group'. . ..

Acknowledgment. We should like to thank the developers of ALGOL, AXIOM , Clojure , ECL, Erlang , FriCAS , GIMP, Inkscape , Julia , Pinta , R , Ruby , SBCL, SVG, SWI-Prolog , and wxMaxima for their indirect help, which enabled us to carry out visualisations and some computations.

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12 Appendix

12.1 How (2) is computed

Replacing x in the polynomial $x^2+\frac{b}{a}x+\frac{c}{a}$ by $x-\frac{b}{2a}$, one computes $(x-\frac{b}{2a})^2+\frac{b}{a}\cdot(x-\frac{b}{2a})+\frac{c}{a}=x^2-\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{b}{a}x-\frac{b^2}{2a^2}+\frac{c}{a}=x^2+\frac{4ac-b^2}{4a^2}$, which is rewritten as x^2+d.

12.2 Another way to get D of (3)

First, we differentiate both sides of (3) wrt t to get $2t - 2\sqrt{a}x = 0$, from which it follows that $t = \sqrt{a}x$. Next, we substitute this into the LHS of (3) to get $(\sqrt{a}x)^2 - 2\sqrt{a}x \cdot \sqrt{a}x + y - bx - c$. After some computation, this becomes $y - ax^2 - bx - c$. Multiplication of this by -4 yields $4(ax^2 + bx + c - y)$, or D of (3).

Remark 12.2.1. Elimination of t in this fashion is not seen in **3.2**.

12.3 Another way of φ derivation

We *a priori* consider

$$x^5 = 1, \tag{139}$$

which is factored as

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0.$$

So one root of (139) is 1; other roots are obtained by solving the QuE

$$x^4 + x^3 + x^2 + x + 1 = 0. \tag{140}$$

Since it is clear that $x = 0$ is not a root of (140), we can divide both sides of it by $x^2 (\neq 0)$ to get

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

After some manipulation, the above becomes

$$(x + \frac{1}{x})^2 + (x + \frac{1}{x}) - 1 = 0.$$

Setting $x + \frac{1}{x} = t$, we rewrite the above as the QE

$$t^2 + t - 1 = 0,$$

which we solve to get the roots

$$\begin{cases} t_+ = \frac{-1+\sqrt{5}}{2} = \varphi - 1, & (141) \\ t_- = \frac{-1-\sqrt{5}}{2} = -\varphi. & (142) \end{cases}$$

Adding 1 to the relation (141) (or multiplying the relation (142) by -1), one gets φ. Hence, we have derived φ from (139).

Remark 12.3.1. The fifth roots of unity in the complex plane might help us understand (139) (rather) geometrically.

12.4 Why (46) holds

We rewrite (46) as

$$A = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}},$$

where A is some unknown. Then, squaring both sides, one gets

$$A^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}},$$

which can be further rewritten as $A^2 = 1 + A$. Solving this for A yields $A = \frac{1 \pm \sqrt{5}}{2}$. However, since it is clear from the RHS of (46) that A is a positive real number, we drop $\frac{1-\sqrt{5}}{2}$. So we get $A = \frac{1+\sqrt{5}}{2} = \varphi$. Hence, we have

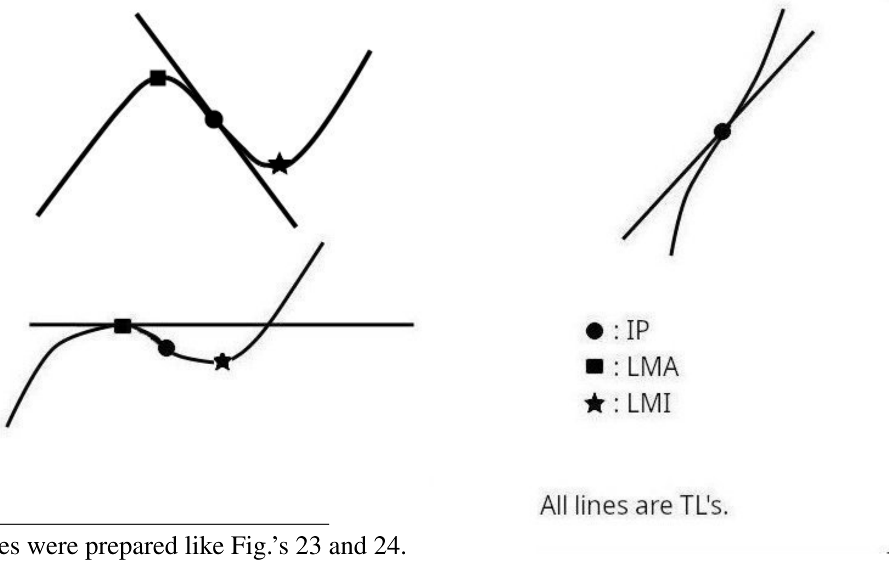
$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}.$$

Remark 12.4.1. If we set $x_1 = \sqrt[3]{b + a\sqrt[3]{b + a\sqrt[3]{b + \cdots}}}$, we get $x_1^3 = b + a\sqrt[3]{b + a\sqrt[3]{b + \cdots}}$ in a similar fashion. So $x_1^3 - b = ax_1$, which means that x_1 is a root of the CE $x^3 - ax - b = 0$.

Remark 12.4.2. The above CE, which can be regarded as an equation subjected to TT , might also serve as a propaedeutic to revisiting CE.

12.5 Other CF-related stuff

We add the following [98] to Fig.'s 8 and 11.



---

[98] These figures were prepared like Fig.'s 23 and 24.

Remark 12.5.1. These might help one understand approximations used in those Fig.'s[99] .

12.6 Slight modification in an SBCL code and its outcome

We slightly modify the SBCL code used in *Verification* 5.3.12 and observe how things change:

```
% more cubic_eq_part_1_v_5_3_12m.lsp
(defun v_5_3_12m(n)(if(= n 0)(/ 3 2)
(/(+ (* 4(* (v_5_3_12m(- n 1))(v_5_3_12m(- n 1))))9)
(- (* 8 (v_5_3_12m(- n 1)))5.0))))
(loop for i from 0 to 20 do(format t "~D~%"(v_5_3_12m i)))
% time sbcl --script cubic_eq_part_1_v_5_3_12m.lsp
3/2
2.5714285
2.27654
2.2502131
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
2.25
219.463u 0.208s 3:39.74 99.9% 0+0k 0+0io 0pf+0w
```

One repeatedly sees convergence to get $X_1 = 2.25$. This is not surprising, since $a\hat{}2$ ('ˆ' is '`expt`' in the original code) is mathematically the same as $a \times a$ ('$\times$' is '`*`' in the modified code). However, we observe the last lines of each output, *i.e.,* `0.151u`... and `219.463u`..., are rather different. So a change that does not affect mathematical content can cause a (significant) change. This phenomenon *itself* might be well-known [100] , but it seems of some interest.

---

[99] *Cf.* [10, Fig. 4.6(a), (c)].

[100] *Cf.* footnote 1.