Calculating the spatial curvature of the universe. An equation that relates it to energy density

Fernando Salmon Iza *

* Madrid Complutense University UCM. APS member.

fernandosalmoniza@gmail.com

Abstracts

Understanding the spatial curvature of our universe is a very important topic in astrophysics. The FLRW metric that determines the evolution of the universe is based on the Cosmological Principle (the universe is homogeneous and isotropic on very large scales) and on Weyl's Postulate (the universe behaves like a perfect fluid whose components move as temporal geodesics without intersecting each other). This metric is specified in two equations, the Friedmann equations, in which the curvature term Ω_k plays an essential role in its resolution. Determining the value of this term with respect to the energy density term Ω_ρ may mean solving or not solving the equations in many cases. We do not have the solution to this important question, but we have begun to solve it. We have found an equation that relates, in the FLRW metric, the spatial curvature with the energy density and we have found that the spatial curvature is proportional to the energy density with a proportionality factor very similar to that which relates in Einstein's equations, the Einstein tensor with the energy-momentum tensor, that is, the curvature with the energy. This has important consequences, the first is that, in a universe with matter, the spatial curvature will never be zero, the second is that, for the density of matter in today's universe, the spatial curvature is very small.

Keywords: Spatial curvature constant, Friedmann´s equation, General Relativity, Cosmic spacetime

1. - The cosmic spacetime

We are going to study a uniform and isotropic spacetime from a physical point of view, this is equivalent from a geometric point of view to being invariant under translations and rotations.

According to Professor Fulvio Meliá in reference [1], we define “cosmic spacetime” as the set of points (t, r, Ø Φ) that satisfy the FLRW metric, that is, that satisfy the equation:

$$ds^2 = c^2dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right)$$

We define each of the "3D hypersurfaces" of cosmic spacetime as the set of points that have the same temporal coordinate. Thus, cosmic spacetime will have a different hypersurface for each time t. As we have defined them, these hypersurfaces do not intersect, that is, they have no common points and the set of all of them constitutes cosmic spacetime.

It is in these 3D hypersurfaces where we are going to calculate the spatial curvature that constitute the object of this report.

2.- Calculation of the spatial curvature of each of the 3D hypersurfaces of cosmic spacetime

First, we are going to calculate the curvature scalar of a 3D hypersurface of our homogeneous and isotropic cosmic spacetime with a matter density ρ_m.

2.1- Birkhoff–Jebsen theorem

We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity [2]. First, we summarize Professor Fulvio Melia in reference [3] to explain it.

“If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere”.

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity Birkhoff´s theorem states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means, that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be the Schwarzschild metric.

2.2- Calculating the spatial curvature constant

Let's consider our 3D hypersurface and a sphere of radius r inside, the Birkhoff–Jebsen theorem assures us that if we want to calculate the curvature at a point on its surface, we must consider only the interaction with the gravitational mass found inside, the gravitational mass inside for the sphere external point that we are considering behaves as a point mass of equal magnitude to that of the mass of the sphere and located at its central point. In this case we are already in the Schwarzschild model, and we can use its equations to calculate the corresponding curvature.

For all this, we can treat the problem of calculating the curvature scalar of each of the 3D hypersurfaces of our cosmic spacetime as a problem to be solved by the Schwarzschild model and calculate the curvature scalar from that model. In this model, spacetime is reduced to a 2D surface and so Gaussian curvatures are easily calculated; the scalar curvature in this case is twice the Gaussian curvature.

According to [4], we have found an equation that relates the Gaussian curvature K_{gauss} of the spacetime of the Schwarzschild model, with the cosmological parameters mass M and universal gravitation constant G. We are going to use this equation to solve our problem. This equation is the following:

$$K_{gauss}= -GM/c^2r^3$$

Since in our case it is a sphere, its mass will be given by

$$M = 4\pi r^3\rho/3$$

$$K_{gauss} = -4\pi G\ \rho/3c^2$$

having reduced the calculation to a two-dimensional problem, the curvature scalar R will be given by twice the Gaussian curvature and, in our case, it will also have the opposite sign. Thus:

$$R/\rho = 8\pi G/3c^{2,} = 0{,}62.10^{-26}$$

This curvature obtained here R. which is the same at each point of each one of 3D hypersurfaces and proportional to the energy density, $\rho(Kg/m^3)$, we will demonstrate in the discussion what the "spatial curvature K" is.

Thus, the “spatial curvature K” at the points of each 3D hypersurface is the same and is proportional to the density of matter.

$$\mathbf{K = (8\pi G/3c^2)\ \rho = 0{,}62.10^{-26}\ \rho}$$

2.3- Discussion

Identification of the curvatures found, R = K:

We have found a curvature scalar R, which results from the relativistic gravitational interaction between the points that form the cosmic fluid. This curvature has the same value at each point of each 3D hypersurface corresponding to an instant of time in cosmic space-time. Moreover, this curvature depends only on the universal gravitational constant and the gravitational energy density. It is therefore very reasonable to identify this curvature with the -spatial curvature K- that determines the value of the parameter “k” in the Friedmann equation. According to the equation found we see that the proportionality factor between K and ρ is very similar to the proportionality factor that Einstein finds between the Einstein tensor and the energy-momentum tensor, that is, between curvature and energy, which further confirms our choice as the "spatial curvature".

Through the Friedmann equation, we can relate K with the curvature parameter "k" that appears in it, [5]:

$H^2 = (a'/a)^2 = (8\pi G\rho/3) - kc^2/a^2$

k = K/[K}. Where k = +1, -1, 0, according to the sign and value of K. If K is positive then k=+1, if K is negative then k=-1 and if K is zero, then k = 0.

Some consequences of the equation:

The first thing we can see is that zero spatial curvature is only possible if the energy density is zero. So, it does not seem that our universe has zero spatial curvature. What we do know is that the curvature term appearing in the Friedmann equation is very small, according to experimental data [4]. $\Omega_k = 0{,}001 \pm 0{,}002$, [6], this term $\Omega_k = kc^2/(Ha)^2$ is a function of k, the expansion parameter a(t), and the Hubble constant H, the small measured value of which has led some scientists to consider the possibility that $\Omega_k = 0$, being therefore k= 0. From what is stated here, our equation denies this hypothesis since k = 0 implies K = 0 and that is only possible if ρ = 0, which is not the case in our universe.

Furthermore, our equation will condition the possible physical existence of one of the most studied universes, the Milne universe. This is a universe with zero energy density ρ = 0 and curvature k = -1. It represents an expanding universe without matter. Our equation will condition its possible physical existence by the following. According to our equation, a universe with zero energy density implies a spatial curvature equal to zero K = 0 and therefore k = 0, therefore the Milne universe, with ρ = 0 and k = -1 would not be possible.

Calculation of the value of spatial curvature:

There are several experimental data available concerning the energy density due to mass ρ_m in our universe, [6], according to these data the value is $\rho_m = 0{,}3.10^{-26}$ kg/m^3. Substituting this value into our equation we can calculate the current spatial curvature of our universe:

$$\mathbf{K = (8\pi G/3c^2)\, \rho = (0{,}62.10^{-26})\, (0{,}3.10^{-26}) = 0{,}19.10^{-52}\ m^{-2}}$$

this is therefore an extremely small value.

The curvature equation we have obtained allows us to calculate the curvature scalar of each 3D surface of cosmic space-time. It should not be confused with the space-time curvature scalar of the FLRW metric, as this confusion can lead to serious errors.

3. – Conclusions

We have found a simple equation that relates, in the FLRW metric, a curvature scalar to the energy density. In the context of this metric, we have identified this curvature scalar with the spatial curvature K of each of the 3D hypersurfaces into which cosmic spacetime is divided. In this equation found, the spatial curvature is proportional to the energy density, with a proportionality constant equal to one third of the proportionality constant existing between the Einstein tensor and the energy-momentum tensor. Knowing the value of the energy density, we have calculated that the value of the current spatial curvature is extremely small. We have also come to the conclusion that a spatial curvature equal to zero is not possible in our universe because according to our equation, it only occurs if the energy density is equal to zero. Therefore, the Milne universe with $\rho=0$, and $k=-1$ is not physically possible. Our equation is valid in any FLRW metric universe.

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