Pi is Irrational Using One Trig Identity

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Abstract

We prove the contrapositive of π is rational implies $\lim_{n\to\infty} \sin n!$ converges to show π is irrational.

Introduction

There are many proofs of the irrationality of π, but none seem to use the periodicity of trigonometric functions [3, 4]. Here's a proof that uses a standard trigonometric identity, a difference to product [2], with an implication of periodicity to show π is irrational. Some rudimentary limit ideas are also used [1], but nothing that seems beyond the grasp of a beginning calculus student.

Proof

Lemma 1. *π is rational if and only if there exist natural numbers n and m, $n \neq m$ such that* $\sin n = \sin m$.

Proof. $\Rightarrow$
If $\pi = p/q$ then

$$q!\frac{p}{q} = 2k\pi$$

and

$$(q+1)!\frac{p}{q} = 2k'\pi,$$

where $k \neq k'$. This shows the existence of n and m, $n \neq m$ such that $\sin n = \sin m$.
$\Leftarrow$

If $\sin n = \sin m$ and $n \neq m$ then,

$$\sin n - \sin m = 2 \sin \frac{n+m}{2} \cos \frac{n-m}{2} = 0.$$

This implies that

$$\frac{n+m}{2} = k\pi \text{ or } \frac{n-m}{2} = k\frac{\pi}{2} \text{ with } k \text{ odd.}$$

In either case $k \neq 0$. Thus either

$$\pi = \frac{n+m}{2k} \text{ or } \pi = \frac{n-m}{2k}$$

giving that π is rational. □

Theorem 1. *π is irrational.*

Proof. We prove the contrapositive of $\lim_{n\to\infty} \sin n!$ converges implies π is rational. Assume π is irrational and

$$\lim_{n\to\infty} \sin n! = L.$$

Then

$$\lim_{n\to\infty} \sin(n+1)! = L \text{ and } \lim_{n\to\infty} \sin(n-1)! = L$$

and

$$\sin(n+1)! - \sin(n-1)! = 0$$

and using Lemma 1 this implies that π is rational, a contradiction. □

Conclusion

The periodicity of $\sin$ is embedded in Lemma 1.

References

[1] Apostol, T. M. (1974). *Mathematical Analysis*, 2nd ed. Reading, Massachusetts: Addison-Wesley.

[2] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.

[3] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[4] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.