On the spectral flow theorem of Robbin-Salamon for finite intervals

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Abstract

In this article we consider operators of the form $\partial_s \xi + A(s)\xi$ where s lies in an interval $[-T, T]$ and $s \mapsto A(s)$ is continuous. Without boundary conditions these operators are not Fredholm. However, using interpolation theory one can define suitable boundary conditions for these operators so that they become Fredholm. We show that in this case the Fredholm index is given by the spectral flow of the operator path A.

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1 Introduction

1.1 Main results

Definition 1.1. A pair $H = (H_0, H_1)$ is called a **Hilbert space pair** if H_0 and H_1 are both infinite dimensional Hilbert spaces such that $H_1 \subset H_0$ is a dense subset and the inclusion map $\iota\colon H_1 \to H_0$ is a compact linear map.
Both Hilbert spaces in a Hilbert space pair are separable by [FW24, Cor. A.5].

From now on (H_0, H_1) is a Hilbert space pair and $\mathcal{L}(H_0, H_1)$ is the set of all bounded linear operators $A\colon H_1 \to H_0$.

Definition 1.2 (H_0-symmetry)**.** An operator $A \in \mathcal{L}(H_1, H_0)$ is called ***H_0*-symmetric**, or simply **symmetric**, if

$$\langle Ax, y\rangle_0 = \langle x, Ay\rangle_0 , \quad \forall x, y \in H_1. \tag{1.1}$$

Note that while this notion of symmetry depends on the inner product on H_0 it only depends on the inner product on H_1 up to equivalence. Namely, we call two inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ on a Hilbert space H **equivalent** if there exists a constant c such that

$$\frac{1}{c}\|x\| \le \|x\|' \le c\|x\|, \qquad \|x\| := \sqrt{\langle x, x\rangle}, \quad \|x\|' := \sqrt{\langle x, x\rangle'},$$

for every $x \in H$. One calls $\|\cdot\|$ and $\|\cdot\|'$ the **induced norms**.

The condition of being symmetric is kind of asymmetric. While it depends on the H_0-inner product, it only depends on the H_1-inner product up to equivalence of norms. A more symmetric notion which only depends on the equivalence classes of the H_1- as well as the H_0-inner product is the following notion.

Definition 1.3. An element $A \in \mathcal{L}(H_1, H_0)$ is called **symmetrizable** if there exists an inner product $\langle\cdot,\cdot\rangle$ on H_0 equivalent to the given inner product $\langle\cdot,\cdot\rangle_0$ such that A is symmetric with respect to the new inner product $\langle\cdot,\cdot\rangle$.

We abbreviate by $\mathcal{F} = \mathcal{F}(H_1, H_0) \subset \mathcal{L}(H_1, H_0)$ the set of symmetrizable Fredholm operators of index zero from H_1 to H_0. We refer to the elements of $\mathcal{F}$ as **Hessians**. We endow the set $\mathcal{F}$ with the subset topology inherited from $\mathcal{L}(H_1, H_0)$. We define $\mathcal{F}^* := \{\mathbb{A} \in \mathcal{F} \mid \exists \mathbb{A}^{-1} \in \mathcal{L}(H_0, H_1)\}$. To indicate invertibility visually we shall use for the elements of $\mathcal{F}^*$ the font $\mathbb{A}$.

Taking adjoints gives rise to a bijection (see Lemma 2.7 for details)

$$*\colon \mathcal{F}(H_1, H_0) \to \mathcal{F}(H_0^*, H_1^*), \quad A \mapsto A^* \tag{1.2}$$

which has the property $** = \mathrm{Id}_{\mathcal{F}(H_1,H_0)}$ and maps invertibles to invertibles.

Remark 1.4. Note that (1.2) would not be true if one would replace symmetrizable by symmetric. In fact, the adjoint of a symmetric operator $A\colon H_1 \to H_0$ does not need to be symmetric. This is due to the asymmetric property of the symmetry condition mentioned above. Indeed the symmetry of $A\colon H_1 \to H_0$ depends on the inner product on H_0, while the symmetry of $A^*\colon H_0^* \to H_1^*$ depends on the inner product on H_1 which can be used to identify H_1 with H_1^*.

In the following we will consider paths of Hessians. Although in many applications one has paths of Hessians which are symmetric for a fixed inner product on H_0, and not for a time-dependent one as in the symmetrizable case, the advantage of relaxing the symmetry condition to symmetrizability is that it gives a *uniform* way to treat paths of Hessians and the path of its adjoints.

Let I be an interval of the form

$$\mathbb{R}, \quad I_- = \mathbb{R}_- = (-\infty, 0], \quad I_+ = \mathbb{R}_+ = [0, \infty), \quad I_T = [-T, T]. \tag{1.3}$$

Relevant **path spaces** are defined by

$$\begin{aligned} P_0(I) = P_0(I; H_0) &:= L^2(I, H_0), \\ P_1(I) = P_1(I; H_1, H_0) &:= L^2(I, H_1) \cap W^{1,2}(I, H_0), \end{aligned} \tag{1.4}$$

and these are Hilbert spaces with inner products

$$\langle v, w \rangle_{P_0} := \int_I \langle v(s), w(s) \rangle_0 \, ds$$

and

$$\langle v, w \rangle_{P_1} := \int_I \langle v'(s), w'(s) \rangle_0 \, ds + \int_I \langle v(s), w(s) \rangle_1 \, ds. \tag{1.5}$$

Definition 1.5. Denote the space of continuous paths of Hessians by

$$\mathcal{A}_I := \{A\colon I \to \mathcal{F} \text{ continuous}\}.$$

The Hessian path spaces are defined by

$$\begin{aligned} \mathcal{A}_{I_T}^* &:= \{A \in \mathcal{A}_{I_T} \mid \mathbb{A}_{-T} := A(-T) \text{ and } \mathbb{A}_T := A(T) \text{ are invertible}\} \\ \mathcal{A}_{I_+}^* &:= \{A \in \mathcal{A}_{I_+} \mid \ \mathbb{A}^+ := \lim_{s\to\infty} A(s) \text{ exists, } \mathbb{A}^+ \text{ and } A(0) \text{ invertible}\} \\ \mathcal{A}_{I_-}^* &:= \{A \in \mathcal{A}_{I_-} \mid \ \mathbb{A}^- := \lim_{s\to-\infty} A(s) \text{ exists, } \mathbb{A}^- \text{ and } A(0) \text{ invertible}\} \\ \mathcal{A}_{\mathbb{R}}^* &:= \{A \in \mathcal{A}_{I_-} \mid \ \mathbb{A}^\pm := \lim_{s\to\pm\infty} A(s) \text{ exist and are invertible}\}. \end{aligned}$$

For $A \in \mathcal{A}_I^*$, where I is one of the four interval types, we define the bounded linear operator

$$D_A\colon P_1(I) \to P_0(I), \quad \xi \mapsto \partial_s \xi + A\xi. \tag{1.6}$$

Definition 1.6 (Projections)**.** Assume that $\mathbb{A} \in \mathcal{F}^*$. Let $H_{1/2} = H_{1/2}(\mathbb{A})$ be the interpolation space between the domain and the co-domain of $\mathbb{A}$, namely between H_1 and H_0 in the case at hand. We denote by

$$\pi_+^{\mathbb{A}} \in \mathcal{L}(H_{\frac{1}{2}}), \qquad \pi_-^{\mathbb{A}} = \mathrm{Id} - \pi_+^{\mathbb{A}} \in \mathcal{L}(H_{\frac{1}{2}}),$$

the projection to the positive eigenspaces of $\mathbb{A}$ along the negative ones, respectively to the negative eigenspaces along the positive ones. The images

$$H^{\pm}_{\frac{1}{2}}(\mathbb{A}) := \pi^{\mathbb{A}}_{\pm}(H_{\frac{1}{2}})$$

are complementary closed subspaces of $H_{1/2}$, as explained in Section 2.3.

Definition 1.7. For each of the four interval types I we introduce **augmented operators** as follows. For $A \in \mathcal{A}^*_{I_T}$ we abbreviate $\mathbb{A}_{\pm T} := A(\pm T)$ and define

$$\begin{aligned} \mathfrak{D}_A \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T) \\ \xi &\mapsto \left(D_A\xi, \pi^{\mathbb{A}_{-T}}_+ \xi_{-T}, \pi^{\mathbb{A}_T}_- \xi_T \right). \end{aligned} \tag{1.7}$$

For $A \in \mathcal{A}^*_{I_+}$ we abbreviate $\mathbb{A}_0 := A(0)$ and define

$$\begin{aligned} \mathfrak{D}_A \colon P_1(I_+) &\to P_0(I_+) \times H^+_{\frac{1}{2}}(\mathbb{A}_0) =: \mathcal{W}(I_+; \mathbb{A}_0) \\ \xi &\mapsto \left(D_A\xi, \pi^{\mathbb{A}_0}_+ \xi_0 \right). \end{aligned}$$

For $A \in \mathcal{A}^*_{I_-}$ we abbreviate $\mathbb{A}_0 := A(0)$ and define

$$\begin{aligned} \mathfrak{D}_A \colon P_1(I_-) &\to P_0(I_-) \times H^-_{\frac{1}{2}}(\mathbb{A}_0) =: \mathcal{W}(I_-; \mathbb{A}_0) \\ \xi &\mapsto \left(D_A\xi, \pi^{\mathbb{A}_0}_- \xi_0 \right). \end{aligned}$$

For $A \in \mathcal{A}^*_{\mathbb{R}}$ we define

$$\begin{aligned} \mathfrak{D}_A \colon P_1(\mathbb{R}) &\to P_0(\mathbb{R}) \\ \xi &\mapsto D_A\xi. \end{aligned}$$

The main result of this article is the following theorem in which I is any of the four interval types. The proof uses [FW24, Thm. D]; see Theorem 4.23.

Theorem A. *For $A \in \mathcal{A}^*_I$ the augmented operator $\mathfrak{D}_A$ is Fredholm and* $\operatorname{index} \mathfrak{D}_A = \varsigma(A)$ *where $\varsigma(A)$ is the spectral flow of the path A of Hessians.*

Remark 1.8. In the case $I = \mathbb{R}$ Theorem A is the classical spectral flow theorem of Robbin and Salamon [RS95]. Strictly speaking, they proved the Fredholm property under an additional assumption on the path A, namely they required the existence of a weak derivative. It was later shown by Rabier [Rab04] that such a weak derivative is not needed to obtain the Fredholm property.

Although the special case $I = \mathbb{R}$ was known before our proof, even in this case it differs rather from the proofs of Robbin-Salamon and Rabier. While the Robbin-Salamon proof requires the rather involved infinite dimensional transversality theory [AR67] to perturb the path of Hessians to make it transverse in order to achieve only simple crossings of the eigenvalues at zero, our proof on concatenating finite intervals does not require these techniques. Instead we use elements λ in the resolvent set to shift the operators D_A to operators $\mathfrak{D}^\lambda_A$, defined by (4.56), for which the issue of non-simple crossings of eigenvalues at zero can be avoided.

Given $A \in \mathcal{A}^*_{I_T}(H_1, H_0)$, then $-A^* \in \mathcal{A}^*_{I_T}(H_0^*, H_1^*)$ by (1.2). We define the **adjoint** of $\mathfrak{D}_A$ in (1.7) to be the augmented operator associated to $-A^*$, i.e.

$$\mathfrak{D}_A^* := \mathfrak{D}_{-A^*} \colon P_1(I_T; H_0^*, H_1^*) \to \mathcal{W}(I_T; -\mathbb{A}^*_{-T}, -\mathbb{A}^*_T). \tag{1.8}$$

Since the spectrum of an operator and its adjoint coincide, see Lemma 2.6, we have for the spectral flow

$$\varsigma(A) = \varsigma(A^*) = -\varsigma(-A^*)$$

and therefore an immediate consequence of Theorem A is the index formula

$$\operatorname{index} \mathfrak{D}_A = -\operatorname{index} \mathfrak{D}_A^*.$$

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1.2 Motivation and general perspective

This article is part of our endeavor to find a general approach to Floer homology as outlined in the section "Motivation and general perspective" in [FW24].

Floer homology is a tool to detect periodic orbits. Periodic orbits are of importance in several problems like, for example, celestial mechanics, and in particular space mission design, where one is interested to find a periodic orbit where to position a satellite. They play an important role in the semi-classical approach to quantum mechanics where they enter Gutzwiler's celebrated trace formula [Gut90]. Another topic where periodic orbits are essential is in population dynamics like for example the famous Volterra models where small fishes are being eaten by big fishes.

In all these applications of periodic orbits it is of interest to go beyond periodic orbits of Hamiltonian ODEs, but also allow delay equations. This is obvious in population dynamics where the fishes first have to grow.
However, Hamiltonian delay equations also play an increasing role in celestial mechanics and dynamics of atoms. This is due to the fact that these systems are singular due to collisions. Although one definitely does not want to put a satellite on the collision orbit to understand the global picture of the intricate network of periodic orbits one also has to take into account collision orbits.
In the semi-classical approach to quantization using Gutzwiler's trace formula collision orbits have to be included. In fact, as is argued in [WRT93] a special class of collisional periodic orbits in Helium, so-called *eZe-orbits*, can be used to explain large portions of the spectrum of Helium.

All these applications of periodic orbits require new techniques in Floer homology and, in particular, a deeper understanding on the mechanisms which make Floer homology work. With this goal in mind we therefore provide in the present article a comprehensive study of the operators which play an important role in a uniform approach to Floer homology.

Operators of the form $\partial_s + A(s)$ for finite and half-infinite intervals appear in the Hardy-approach to Lagrangian Floer gluing of Tatjana Simčević [Sim14].

2 Preliminaries

Throughout this article let $H = (H_0, H_1)$ be a Hilbert space pair.

Notation. An operator is a bounded linear map. The font $\mathbb{A}$ indicates that an operator A is invertible. The Kronecker symbol δ_{ij} is 1 if $i = j$ and zero else. To ovoid excess of parentheses we write $f(x)$ often as f_x or, rarely, as $f|_x$.

2.1 *H*-self-adjoint operators

Definition 2.1. A bounded linear map $A\colon H_1 \to H_0$ is called ***H*-self-adjoint** or, more precisely, a **self-adjoint Hilbert space pair operator**, if it is

(H_0-sym) H_0-symmetric, see (1.1), and

(Fred-0) Fredholm of index zero.

The requirement Fredholm of index zero guarantees non-emptiness of the resolvent set $\mathcal{R}(A) := \mathbb{R} \setminus \operatorname{spec} A \neq \emptyset$, as we discuss right below. Non-emptiness will be used over and over again in Section 4 for perturbation arguments, see e.g. Step 4 in the proof of Theorem 4.2.

Remark 2.2 (Why Fredholm of index zero is important)**.** As opposed to an operator acting on a Hilbert space, say $H_0 \to H_0$, the Fredholm requirement arises from domain H_0 and co-domain H_1 being different in the case at hand.

Suppose a linear map $B\colon H_1 \to H_0$ is bounded, but not Fredholm of index zero. Then all reals lie in the spectrum

$$\mathbb{R} = \operatorname{spec} B := \{\lambda \in \mathbb{R} \mid B - \lambda\iota\colon H_1 \to H_0 \text{ is not bijective}\}.$$

Those $\lambda \in \operatorname{spec} B$ for which $B - \lambda\iota\colon H_1 \to H_0$ is not injective are called **eigenvalues** of B and non-zero kernel elements **eigenvectors**. To see equality suppose by contradiction that there is a real λ such that the bounded linear map $B - \lambda\iota\colon H_1 \to H_0$ is bijective and so, by the open mapping theorem, admits a bounded inverse $R_\lambda(B) := (B - \lambda\iota)^{-1}\colon H_0 \to H_1$ called the **resolvent of *B* at $\boldsymbol{\lambda}$**$\notin \operatorname{spec} B$. Hence $B - \lambda\iota$ is an isomorphism and thereby Fredholm of index zero. But so is then B, since ι is compact. Contradiction.

Remark 2.3 (H-self-adjoint spectrum is real, discrete, eigenvalues only)**.** In a Hilbert space pair both Hilbert spaces are separable by [FW24, Cor. A.5]. If interpreted as an unbounded operator on H_0, then an H-self-adjoint operator A is a self-adjoint operator $A\colon H_0 \supset H_1 \to H_0$ with dense domain H_1. This, and what follows, is detailed in Appendix D.

The spectrum of A consists of infinitely many discrete real eigenvalues a_ℓ, of finite multiplicity each, which accumulate either at $+\infty$, or at $-\infty$, or at both. By Theorem D.1, there is a countable **orthonormal basis** $\mathcal{V}(A)$ of H_0, see Definition A.2, composed of eigenvectors $v_\ell \in H_1$ of A. The complement $\mathcal{R}(A) := \mathbb{R} \setminus \operatorname{spec} A$ is called the **resolvent set** of A. It is dense in $\mathbb{R}$.

a) Invertible case. For invertible H-self-adjoint operators we use the boldface letter $\mathbb{A}\colon H_1 \to H_0$. Accounting for multiplicities we enumerate the eigenvalues of $\mathbb{A}$ in increasing order and write them as a list with finite repetitions

$$\cdots \le a_{-2} \le a_{-1} < 0 < a_1 \le a_2 \le \dots, \qquad \mathcal{S}(\mathbb{A}) = (a_\ell)_{\ell\in\Lambda}, \tag{2.9}$$

where the **eigenvalue index set** $\Lambda \subset \mathbb{Z}^*$, counting multiplicities, is of the form

	Morse	co-Morse	Floer
Λ	$-\Lambda_- \cup \mathbb{N}$	$-\mathbb{N} \cup \Lambda_+$	$-\mathbb{N} \cup \mathbb{N} =: \mathbb{Z}^*$
Λ_-	$\{\mu_-, \dots, 2, 1\}$ or $\emptyset$		$\mathbb{N}$
Λ_+		$\{1, 2, \dots, \mu_+\}$ or $\emptyset$	$\mathbb{N}$

(2.10)

The number of elements $\#\Lambda_-$ ($\#\Lambda_+$) is the **Morse** (**co-Morse**) **index** of $\mathbb{A}$. Using the same index set Λ we write the orthonormal basis of H_0 in the form

$$\mathcal{V}(\mathbb{A}) = \{v_\ell\}_{\ell\in\Lambda} \subset H_1, \qquad \mathbb{A} v_\ell = a_\ell v_\ell, \tag{2.11}$$

where the eigenvalues accumulate on the set $\{-\infty, +\infty\}$; see Theorem D.1.

b) Non-invertible case. In this case the only difference is that $A\colon H_1 \to H_0$ has a nontrivial, but finite dimensional kernel for which one chooses an ONB $\mathcal{V}(\ker A)$. In the notation $A = 0 \oplus \mathbb{A}$ of Appendix D, where $\mathbb{A}$ is invertible as in a), the eigenvalue list of A and the corresponding ONB of H_0 are the unions

$$\mathcal{V}(A) \overset{(2.11)}{=} \mathcal{V}(\mathbb{A}) \cup \mathcal{V}(\ker A), \qquad \mathcal{S}(A) \overset{(2.9)}{=} \mathcal{S}(\mathbb{A}) \cup \{0\}. \tag{2.12}$$

This concludes Remark 2.3.

Definition 2.4 (H-self-adjoint operators come in three types)**.** We distinguish three **types** of H-self-adjoint operators $A\colon H_1 \to H_0$; cf. Remark 2.3.

1. **Morse.** Finitely many negative, infinitely many positive eigenvalues.
2. **Co-Morse.** Finitely many positive, infinitely many negative eigenvalues.
3. **Floer.** Infinitely many negative and positive eigenvalues.

2.2 The Banach adjoint

Definition 2.5. Let $A \in \mathcal{L}(H_1, H_0)$. For a dual space element $\eta \in H_0^* := \mathcal{L}(H_0, \mathbb{R})$, the image $A^*\eta \in H_1^* := \mathcal{L}(H_1, \mathbb{R})$ under the Banach **adjoint**

$$A^*\colon H_0^* \to H_1^*$$

is characterized by

$$(A^*\eta)\xi = \eta(A\xi), \quad \forall \xi \in H_1.$$

The inner products on the dual spaces are defined via the musical isomorphisms

$$\flat_0\colon H_0 \to H_0^*, \quad \xi \mapsto \langle \xi, \cdot\rangle_0, \qquad \flat_1\colon H_1 \to H_1^*, \quad \xi \mapsto \langle \xi, \cdot\rangle_1,$$

by

$$\langle\cdot,\cdot\rangle_0^* := \left\langle {\flat_0}^{-1}\cdot, {\flat_0}^{-1}\cdot\right\rangle_0, \qquad \langle\cdot,\cdot\rangle_1^* := \left\langle {\flat_1}^{-1}\cdot, {\flat_1}^{-1}\cdot\right\rangle_1.$$

Lemma 2.6. *Let $B \in \mathcal{L}(H_1, H_0)$. Then* $\operatorname{spec} B = \operatorname{spec} B^*$.

Proof. We first show that B is invertible iff B^* is invertible. Assume $B\colon H_1 \to H_0$ is invertible. This means that there is $C \in \mathcal{L}(H_0, H_1)$ such that $BC = \mathrm{Id}_{H_0}$ and $CB = \mathrm{Id}_{H_1}$. Applying $*$ to these equations we get $C^*B^* = \mathrm{Id}_{H_0^*}$ and $B^*C^* = \mathrm{Id}_{H_1^*}$. Hence we have shown that invertibility of B implies invertibility of B^*. Hence invertibility of B^* implies invertibility of B^{**}, but $B^{**} = B$. Therefore invertibility of B and B^* are equivalent.

Observe that $\lambda \in \operatorname{spec} B$ iff $B - \lambda\iota\colon H_1 \to H_0$ is not invertible. As we have just seen this is equivalent that $B^* - \lambda\iota^*\colon H_0^* \to H_1^*$ is not invertible. This shows that the spectrum of B coincides with the spectrum of B^*. □

Lemma 2.7. *Taking adjoints gives rise to a bijection*

$$*\colon \mathcal{F}(H_1, H_0) \to \mathcal{F}(H_0^*, H_1^*), \quad A \mapsto A^*$$

which has the property $** = \mathrm{Id}_{\mathcal{F}(H_1,H_0)}$ *and maps invertibles to invertibles.*

Proof. That $*$ maps invertibles to invertibles holds by Lemma 2.6. By [Mül07, §16 Thm. 4] an operator $A\colon H_1 \to H_0$ is Fredholm iff $A^*\colon H_0^* \to H_1^*$ is Fredholm. In our case $\operatorname{index} A^* = -\operatorname{index} A = 0$.

We first discuss the case when A is invertible: After replacing the inner product on H_1 and H_0 by equivalent ones, we can assume without loss of generality, that A is a symmetric isometry. Such inner products are referred to as A-adapted and the existence is discussed around (2.13). Using the A-adapted inner products we can naturally identify H_0^* with H_0 and H_1^* with H_{-1} and A^* becomes a symmetric isometry $H_0 \to H_{-1}$ as explained by Lemma A.8. This shows that A^* is in $\mathcal{F}(H_0^*, H_1^*)$ and is invertible as well.

It remains to discuss the case when A is not invertible: Choose λ in the resolvent set $\mathcal{R}(A)$. Then $\mathbb{A}_\lambda := A - \lambda\iota$ is an invertible element in $\mathcal{F}(H_1, H_0)$. By the discussion before $\mathbb{A}_\lambda^* \in \mathcal{F}(H_0^*, H_1^*)$. Using that $\mathbb{A}_\lambda^* = A^* - \lambda\iota^*$ we conclude that $\mathbb{A}^*$ is in $\mathcal{F}(H_0^*, H_1^*)$ as well.

Since H_0 and H_1 are Hilbert spaces, they are in particular reflexive so that we have the canonical isomorphisms $H_0 = H_0^{**}$ and $H_1 = H_1^{**}$ which does not depend on the choice of any inner product, so that A^{**} naturally becomes A. □

2.3 The interpolation classes $H^{\pm}_{\frac{1}{2}}(\mathbb{A})$

For a pair of Hilbert spaces $H = (H_0, H_1)$ let $H_{1/2}$ be the $\mathbb{R}$-scale interpolation space of H_0 and H_1 as defined in (A.79) for $r = 1/2$. The construction of the interpolation space $H_{1/2}$ uses the 0-inner product on H_0 and the 1-inner product on H_1 to get a $\frac{1}{2}$-inner product. A useful formula, in terms of a pair growth function and a scale basis, is (A.88).

A consequence of the Stein-Weiss interpolation theorem is that if we replace the inner products by equivalent ones, say a $0'$- and a $1'$-inner product, then we obtain on $H_{\frac{1}{2}}$ as well an equivalent $\frac{1}{2}'$-inner product. To see this abbreviate by H_0' the vector space H_0 endowed with the $0'$-inner product and analogously for H_1'. Interpret the identity map as a map $\mathrm{Id}\colon H_0 \to H_0'$. The identity map restricts to a map $H_1 \to H_1'$. Since the 0- and $0'$-inner products are equivalent there exists a constant c_0 such that $\|\mathrm{Id}\|_{\mathcal{L}(H_0,H_0')} \le c_0$. For the same reason there exists a constant c_1 such that $\|\mathrm{Id}\|_{\mathcal{L}(H_1,H_1')} \le c_1$. It follows from the Stein-Weiss interpolation theorem, see e.g. [BL76, 5.4.1 p.115, $U = V = \mathbb{N}$, $p = 2$, $\theta = \frac{1}{2}$] or [FW24, App. B], that the identity map maps $H_{\frac{1}{2}}$ to $H'_{\frac{1}{2}}$ and satisfies $\|\mathrm{Id}\|_{\mathcal{L}(H_{\frac{1}{2}},H'_{\frac{1}{2}})} \le \sqrt{c_0 c_1}$. Interchanging the roles of H_0 and H_0' shows that the restriction of the identity to $H_{\frac{1}{2}}$ actually is an isomorphism between $H_{\frac{1}{2}}$ and $H'_{\frac{1}{2}}$.

Definition 2.8. Assume that $\mathbb{A} \in \mathcal{F}^*(H_1, H_0)$ is a symmetrizable invertible bounded linear map from H_1 to H_0. We say that equivalent inner products $1'$ on H_1 and $0'$ on H_0 are **$\mathbb{A}$-adapted** if $\mathbb{A}$ is an isometry with respect to the inner products $1'$ and $0'$ and symmetric with respect to the inner product $0'$.

Existence. Note that $\mathbb{A}$-adapted inner products always exist: Indeed since $\mathbb{A}\colon H_1 \to H_0$ is symmetrizable there exists an inner product $0'$ on H_0 such that $\mathbb{A}$ is symmetric with respect to the inner product $0'$. Now define the $1'$ inner product on H_1 as the pull-back of the $0'$-inner product on H_0, i.e.

$$\langle \xi, \eta \rangle_{1'} = \langle \mathbb{A}\xi, \mathbb{A}\eta \rangle_{0'} \tag{2.13}$$

for all $\xi, \eta \in H_1$. Since $\mathbb{A}$ is invertible the $1'$-inner product is equivalent to the 1-inner product on H_1. By construction of the $1'$-inner product $\mathbb{A}$ becomes an isometry with respect to the $1'$- and $0'$-inner products.

Spectral decomposition. An operator $\mathbb{A} \in \mathcal{F}^*$ gives rise to a decomposition of interpolation space into two closed subspaces

$$H^{\pm}_{\frac{1}{2}}(\mathbb{A}) := \pi^{\mathbb{A}}_{\pm}(H_{\frac{1}{2}}), \quad H_{\frac{1}{2}} = H^{-}_{\frac{1}{2}}(\mathbb{A}) \oplus H^{+}_{\frac{1}{2}}(\mathbb{A}), \quad (\pi^{\mathbb{A}}_{\pm})^2 = \pi^{\mathbb{A}}_{\pm} \in \mathcal{L}(H_{\frac{1}{2}}), \tag{2.14}$$

corresponding to the negative and the positive eigenspaces of $\mathbb{A}$.

Case 1 (Symmetric isometry)**.** We first explain the construction of the spaces $H^{\pm}_{\frac{1}{2}}(\mathbb{A})$ in the special case where $\mathbb{A}\colon H_1 \to H_0$ is a symmetric isometry. In this case choose an orthonormal basis $\mathcal{V}(\mathbb{A}) = \{v_\ell\}_{\ell\in\Lambda}$ of H_0 as in (2.11), in particular consisting of eigenvectors, namely $\mathbb{A}v_\ell = a_\ell v_\ell$. The basis is orthogonal in H_1, indeed $\langle v_\ell, v_k \rangle_1 = \langle \mathbb{A}v_\ell, \mathbb{A}v_k \rangle_0 = a_\ell a_k \delta_{\ell k}$. So the H_1-lengths are given by

$$\|v_\ell\|_1 = |a_\ell|. \tag{2.15}$$

This basis is also orthogonal in $H_{\frac{1}{2}}$ and the $H_{\frac{1}{2}}$-lengths are given by[1]

$$\|v_\ell\|_{\frac{1}{2}} = |a_\ell|^{\frac{1}{2}}. \tag{2.16}$$

[1] By (A.80) for $\xi = \eta = v_\ell$ (the growth operator is $Tv_\ell = a_\ell^{-2} v_\ell$ since $\mathbb{A}$ is an isometry).

The projections $\pi_\pm^{\mathbb{A}}\colon H_{\frac{1}{2}} \to H_{\frac{1}{2}}$ to the positive/negative eigenspaces of $\mathbb{A}$ are defined by

$$\pi_+^{\mathbb{A}} v_\ell = \begin{cases} v_\ell & ,\ \ell > 0, \\ 0 & ,\ \ell < 0, \end{cases} \qquad \pi_-^{\mathbb{A}} v_\ell = \begin{cases} 0 & ,\ \ell > 0, \\ v_\ell & ,\ \ell < 0, \end{cases}$$

for every eigenvector $v_\ell \in \mathcal{V}(\mathbb{A})$. The definition shows that $(\pi_\pm^{\mathbb{A}})^2 = \pi_\pm^{\mathbb{A}}$, so image and fixed point set coincide. But the latter is a closed subspace by continuity.

Case 2 (Symmetrizable invertible)**.** Now consider the case where the bounded linear map $\mathbb{A}\colon H_1 \to H_0$ is still invertible, but only symmetrizable. In this case we can replace the 0- and 1-inner products on H_0 and H_1 respectively, by $\mathbb{A}$-adapted ones, say $0'$ and $1'$, as explained after Definition 2.8. The space $H_{\frac{1}{2}}$ gets endowed with an equivalent $\frac{1}{2}'$-inner product, too. For the new inner products $0'$, $1'$, and $\frac{1}{2}'$ we obtain projections $\pi_\pm^{\mathbb{A}}$ as above. By equivalence of the new and the original inner product on $H_{\frac{1}{2}}$ the projections

$$\pi_\pm^{\mathbb{A}}\colon H_{\frac{1}{2}} \to H_{\frac{1}{2}} \tag{2.17}$$

are still continuous with respect to the original inner product, although in general they will not be orthogonal any more. The images of $\pi_-^{\mathbb{A}}$ and $\pi_+^{\mathbb{A}}$ are complementary closed subspaces and we get (2.14).

3 Spectral flow

Inspired by Hofer, Wysocki, and Zehnder [HWZ98, p. 216] we define the spectral flow as follows. For the spaces $\mathcal{A}_I^*$ see Definition 1.5.

Finite intervals

Given $A \in \mathcal{A}_{I_T}^*$, we consider the invertible operator $\mathbb{A}_{-T} := A(-T)\colon H_1 \to H_0$ and we write its spectrum, similar to (2.9) but now denoting the non-eigenvalue 0 by a_0^{-T}, in the form

$$\dots \le a_{-2}^{-T} \le a_{-1}^{-T} < \underbrace{a_0^{-T}}_{:=0} < a_1^{-T} \le a_2^{-T} \le \dots, \qquad \mathcal{S}(\mathbb{A}_{-T}) = (a_\ell^{-T})_{\ell\in\Lambda}. \tag{3.18}$$

Extend these eigenvalues and $a_0^{-T} := 0$ to continuous functions $a_\ell\colon [-T,T] \to \mathbb{R}$ for $\ell \in \Lambda \cup \{0\}$ satisfying, for any $s \in [-T,T]$, the following conditions under inclusion of the zero function $[-T,T] \ni s \mapsto 0$, namely

(i) $\dots \le a_{-2}(s) \le a_{-1}(s) \le a_0(s) \le a_1(s) \le a_2(s) \le \dots$

(ii) $\mathcal{S}(A(s)) \cup \{0\} = (a_\ell(s))_{\ell\in\Lambda\cup\{0\}}$.

Since eigenvalues depend continuously on the operator, the functions a_ℓ exist and are uniquely determined by these two conditions.

Definition 3.1 (Spectral flow - finite interval)**.** Let $A \in \mathcal{A}^*_{I_T}$ be a path of Hessians. The **spectral flow** $\varsigma(A) \in \mathbb{Z}$ is defined by

$$\varsigma(A) := -i \text{ if } a_i(T) = 0. \tag{3.19}$$

This is the net count of eigenvalues that change from negative to positive.

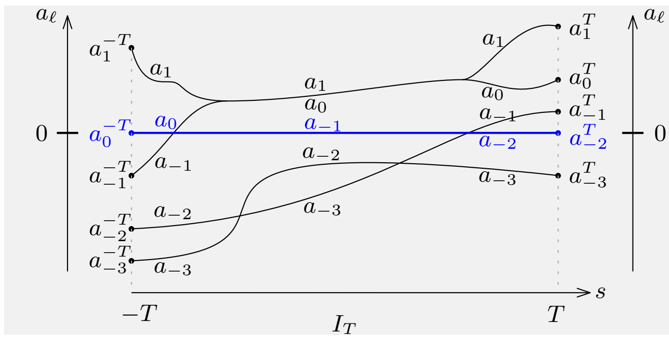

Figure 1: Spectral flow $\varsigma(A) = 2$ along $[-T, T]$

Lemma 3.2. *The map* $\mathcal{A}^*_{I_T} \to \mathbb{Z}$, $A \mapsto \varsigma(A)$, *has the following properties.*

`(Homotopy)` ς *is constant on the connected components of* $\mathcal{A}^*_{I_T}$.

`(Constant)` *If A is constant, then* $\varsigma(A) = 0$.

`(Direct Sum)` $\varsigma(A_1 \oplus A_2) = \varsigma(A_1) + \varsigma(A_2)$.

`(Normalization)` *For* $H_1 = H_0 = \mathbb{R}$ *and* $A(s) = \arctan(s)$, *it holds* $\varsigma(A) = 1$.

`(Catenation)` *If* $A = A_\ell \# A_r$, *then* $\varsigma(A) = \varsigma(A_\ell) + \varsigma(A_r)$.

The first four properties guarantee uniqueness and the first three imply `(Catenation)`, see [RS95, § 4]. Thus ς is the spectral flow as defined in [RS95].

Proof. `(Homotopy)` Assume that A_0 and A_1 lie in the same connected component of $\mathcal{A}^*_{I_T}$. This means that there exists a homotopy $\{A_r\}_{r\in[0,1]} \subset \mathcal{A}^*_{I_T}$ between them. Consider the map $[0, 1] \to \mathbb{Z}$, $r \mapsto \varsigma(A_r)$. By continuous dependence of the eigenvalues this map is continuous and since its image is discrete, the map is constant. In particular $\varsigma(A_0) = \varsigma(A_1)$ and therefore the homotopy property is proved.
`(Constant)` In this case $a_\ell(s) \equiv a_\ell(-T)$ $\forall s, \ell$, in particular we have $a_0(T) = a_0(-T) = 0$, hence $\varsigma(A) = 0$.
`(Direct Sum)` The net number of eigenvalues of the direct sum $A_1 \oplus A_2$ crossing zero corresponds to the sum of the net number of eigenvalues of A_1 crossing zero and A_2 crossing zero. Therefore the direct sum property holds.
`(Normalization)` Initially, since $a_0(-T) = 0$ and $\arctan(-T) < 0$, we have $\arctan(-T) = a_{-1}(-T)$. At the end, since $\arctan(T) > 0$, we have $0 = a_{-1}(T)$. Thus $\varsigma(A) = -(-1) = 1$. □

Half-infinite forward interval

Given $A \in \mathcal{A}^*_{I_+}$, we consider the invertible operator $\mathbb{A}_0 := A(0) \colon H_1 \to H_0$ and we write its spectrum, similar to (2.9) but now denoting the non-eigenvalue 0 by a^0_0, in the form

$$\cdots \le a^0_{-2} \le a^0_{-1} < \underbrace{a^0_0}_{:=0} < a^0_1 \le a^0_2 \le \ldots, \quad \mathcal{S}(\mathbb{A}_0) = (a^0_\ell)_{\ell\in\Lambda}. \tag{3.20}$$

Extend these eigenvalues and $a^0_0 := 0$ to continuous functions $a_\ell \colon [0,\infty) \to \mathbb{R}$ for $\ell \in \Lambda \cup \{0\}$ satisfying, for any $s \in [-T,T]$, the following conditions under inclusion of the zero function $[-T,T] \ni s \mapsto 0$, namely

(i) $\cdots \le a_{-2}(s) \le a_{-1}(s) \le a_0(s) \le a_1(s) \le a_2(s) \le \ldots$

(ii) $\mathcal{S}(A(s)) \cup \{0\} = (a_\ell(s))_{\ell\in\Lambda\cup\{0\}}$.

Since the limit $\lim_{s\to\infty} A(s) =: \mathbb{A}^+$ exists so do the limits $\lim_{s\to\infty} a_\ell(s) =: a_\ell(\infty)$ for every $\ell \in \Lambda \cup \{0\}$ and they satisfy

(iii) $\cdots \le a_{-2}(\infty) \le a_{-1}(\infty) \le a_0(\infty) \le a_1(\infty) \le a_2(\infty) \le \ldots$

(iv) $\mathcal{S}(\mathbb{A}^+) = (a_\ell(\infty))_{\ell\in\Lambda\cup\{0\}}$.

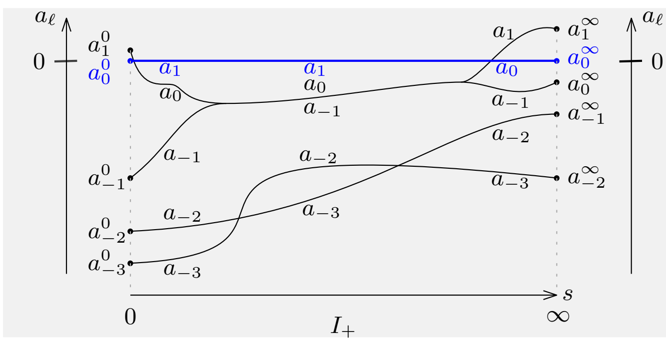

Figure 2: Spectral flow $\varsigma(A) = 0$ along $\mathbb{R}_+$

Definition 3.3 (Spectral flow - half-infinite forward interval)**.** The spectral flow of $A \in \mathcal{A}_{I_+}$ is defined as in Definition 3.1 just by replacing T by ∞.

Half-infinite backward interval

Definition 3.4 (Spectral flow - half-infinite backward interval)**.** Given a backward path $A \in \mathcal{A}^*_{I_-}$, we define a forward path $\tilde{A} \in \mathcal{A}^*_{I_+}$ by $\tilde{A}(s) := A(-s)$. Then we define the spectral flow of the backward path as the spectral flow of the negative forward path, in symbols $\varsigma(A) := \varsigma(-\tilde{A})$.

Real line

Similarly as in the case of half infinite intervals the spectral flow can be defined along the whole real line.

Note that since the asymptotics are invertible, no eigenvalues will cross zero any more whenever $|s| \geq T$ for some sufficiently large $T > 0$. Therefore, alternatively, one could also define the spectral flow of $A \in \mathcal{A}^*_{\mathbb{R}}$ as the spectral flow of A restricted to the finite interval $[-T, T]$.

4 Fredholm operators

Throughout (H_0, H_1) is a Hilbert space pair. Let $I \subset \mathbb{R}$ be connected, then[2]

$$\begin{aligned} P_1(I) = P_1(I; H_1, H_0) &:= L^2(I, H_1) \cap W^{1,2}(I, H_0) \subset C^0(I, H_0), \\ P_1^*(I) = P_1(I; H_0^*, H_1^*) &:= L^2(I, H_0^*) \cap W^{1,2}(I, H_1^*) \subset C^0(I, H_1^*). \end{aligned} \tag{4.22}$$

4.1 Real line

Definition 4.1 (Hessian path space $\mathcal{A}^*_{\mathbb{R}}$)**.** Let $\mathcal{A}^*_{\mathbb{R}}$ be the space of continuous maps $A\colon (-\infty, \infty) \to \mathcal{F}(H_1, H_0)$ such that both asymptotic limits exist and are invertible and symmetrizable, in symbols

$$\mathbb{A}^{\pm} := \lim_{s \to \pm\infty} A(s) \in \mathcal{F}^*(H_1, H_0).$$

4.1.1 Rabier's semi-Fredholm estimate for D_A

The following theorem is due to Rabier [Rab04]. In the case where $s \mapsto A(s)$ has a derivative the theorem is due to Robbin and Salamon [RS95, Lemma 3.9]. For the readers convenience we give a detailed explanation of Rabier's ingenious argument of how to *overcome the need of a derivative.* In fact, Rabier proved his theorem even more general for some Banach and not just Hilbert spaces which however requires additional arguments.

Theorem 4.2 (Rabier)**.** *Given $A \in \mathcal{A}^*_{\mathbb{R}}$, there are constants $c, T > 0$ such that*

$$\|\xi\|_{P_1(\mathbb{R})} \leq c \left(\|D_A \xi\|_{P_0(\mathbb{R})} + \|\xi\|_{P_0([-T,T])} \right)$$

for every $\xi \in P_1(\mathbb{R})$ where $D_A\colon P_1(\mathbb{R}) \to P_0(\mathbb{R})$ is defined by (1.6) for $I = \mathbb{R}$.

Remark 4.3 (Idea of proof)**.** One relates the operator D_A associated to a path of Hessians $s \mapsto A(s)$ to finitely many invertible operators $D_{\mathbb{A}^{\lambda(\sigma_j)}}$ associated to constant-in-s invertible paths $\mathbb{A}^{\lambda(\sigma_j)} := A(\sigma_j) - \lambda(\sigma_j)\iota$, as illustrated by Figure 3. We indicate invertibility of Hessians by using the font $\mathbb{A}$. Step 1: One shows that invertibility of a constant path $\mathbb{A}$ implies invertibility of $D_{\mathbb{A}}$.

[2] In the notation of Def. 2.10 and by Le. 2.15 in [Neu20], cf. [Rou13, Le. 7.1], we have that

$$\begin{aligned} P_1(\mathbb{R}_+; H_1, H_0) &= W^{1,(2,2)}(\mathbb{R}_+; H_1, H_0) \subset C^0(\mathbb{R}_+, H_0), \\ P_1(\mathbb{R}_+; H_0^*, H_1^*) &= W^{1,(2,2)}(\mathbb{R}_+; H_0^*, H_1^*) \subset C^0(\mathbb{R}_+, H_1^*). \end{aligned} \tag{4.21}$$

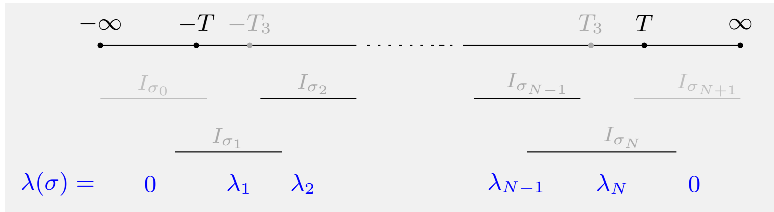

Figure 3: Approximate D_A via finitely many invertible $D_{\mathbb{A}^{\lambda(\sigma_j)}}$, (4.30), (4.31)

ASYMPTOTIC ENDS. Step 2: The asymptotic limits $\mathbb{A}^{\pm}$ are invertible by assumption. Hence so is, by continuity of the path $s \mapsto A(s)$, each member of the path outside a sufficiently large compact interval $[-T_2, T_2]$. Step 3: One derives the estimate for D_A outside a larger interval $[-T_3, T_3]$.
COMPACT CENTER. Step 4: Since at each time $\sigma \in [-T_3, T_3]$ the operator $A(\sigma)\colon H_1 \to H_0$ is H-self-adjoint, there exist non-eigenvalues μ_σ, see Remark 2.3. Pick one, then the shifted operator $\mathbb{A}^{\mu_\sigma} := A(\sigma) - \mu_\sigma \iota$ is invertible. Step 5: But invertibility is an open property, thus $\mathbb{A}^{\mu_\sigma}(\tau) := A(\tau) - \mu_\sigma \iota$ is still invertible in a sufficiently narrow interval about σ, in symbols $\forall \tau \in I_\sigma := (\sigma - \varepsilon_\sigma, \sigma + \varepsilon_\sigma)$. The compact interval $[-T_3, T_3]$ is covered by finitely many intervals $I_{\sigma_1}, I_{\sigma_2}, \dots, I_{\sigma_N}$, see Figure 3. Define $\lambda(\sigma_j) := \mu_{\sigma_j}$ for $j = 1, \dots, N$. Step 6. If $A(\tau)$ is sufficiently close to $A(\sigma)$, then one derives the desired estimate in a small neighborhood.
PUTTING THINGS TOGETHER. Step 7: One chooses $T > T_3$ and a suitable partition of unity for $\mathbb{R}$ to put the obtained estimates near $\pm\infty$ and such in the compact center together. The closeness condition in Step 6 is achieved by subdividing $[-T, T]$ in sufficiently small subintervals using that continuity of $s \mapsto A(s)$ along the compact $[-T, T]$ is uniform. This concludes Remark 4.3.

Proof. The proof is in seven steps. Let $A \in \mathcal{A}^*_{\mathbb{R}}$, notation $\mathbb{A}^{\pm} := \lim_{s\to\pm\infty} A(s)$.
Step 1 (Constant invertible path $\mathbb{A}$)**.** Let $A(s) \equiv \mathbb{A} \in \mathcal{F}^*(H_1, H_0)$ be constant in time. Then the following is true. There exists a constant C_1 such that the following injectivity estimate holds

$$\|\xi\|_{P_1(\mathbb{R})} \le C_1 \|D_{\mathbb{A}}\xi\|_{P_0(\mathbb{R})} \tag{4.23}$$

for every $\xi \in P_1(\mathbb{R})$ and $D_{\mathbb{A}}\colon P_1(\mathbb{R}) \to P_0(\mathbb{R})$ is an isomorphism with inverse bounded by

$$\|(D_{\mathbb{A}})^{-1}\|_{\mathcal{L}(P_0(\mathbb{R}), P_1(\mathbb{R}))} \le C_1. \tag{4.24}$$

1a – Proof of the injectivity estimate (4.23). We can assume without loss of generality that $\mathbb{A}\colon H_1 \to H_0$ is symmetric. Indeed replacing the inner product on H_0 by an equivalent one leads to equivalent norms on $P_1(\mathbb{R})$ and $P_0(\mathbb{R})$ and therefore (4.23) continues to hold after adapting the constant.

By definition (1.4) of the space $P_1(\mathbb{R})$ we get

$$\begin{aligned}\|\xi\|^2_{P_1(\mathbb{R})} &= \int_{-\infty}^{\infty} \left\|\mathbb{A}^{-1}\mathbb{A}\xi(s)\right\|^2_{H_1} + \|\partial_s\xi(s)\|^2_{H_0}\, ds \\ &\le (1+\|\mathbb{A}^{-1}\|^2_{\mathcal{L}(H_0,H_1)})\left(\|\mathbb{A}\xi\|^2_{P_0(\mathbb{R})} + \|\partial_s\xi\|^2_{P_0(\mathbb{R})}\right).\end{aligned} \tag{4.25}$$

On the other hand, by partial integration and symmetry of $\mathbb{A}$, the mixed term is zero and we get

$$\begin{aligned}\|D_\mathbb{A}\xi\|^2_{P_0(\mathbb{R})} &= \int_{-\infty}^{\infty}\left(\|\partial_s\xi(s)\|^2_{H_0} + 2\langle\partial_s\xi(s),\mathbb{A}\xi(s)\rangle_0 + \|\mathbb{A}\xi(s)\|^2_{H_0}\right) ds \\ &= \|\mathbb{A}\xi\|^2_{P_0(\mathbb{R})} + \|\partial_s\xi\|^2_{P_0(\mathbb{R})}.\end{aligned}$$

This identity, together with (4.25), proves the injectivity estimate in Step 1. □

1b – Proof of an injectivity estimate for $D_{-\mathbb{A}^*}$. There is a constant C_1^* with

$$\|\eta\|_{P_1(\mathbb{R};H_0^*,H_1^*)} \le C_1^*\,\|D_{-\mathbb{A}^*}\eta\|_{P_0(\mathbb{R}.H_1^*)}$$

for every $\eta \in P_1(\mathbb{R};H_0^*,H_1^*)$.
To see this note that the assumption $\mathbb{A}\in\mathcal{F}(H_1,H_0)$ implies $\mathbb{A}^*\in\mathcal{F}(H_0^*,H_1^*)$, by Lemma 2.7. Moreover, since $\mathbb{A}$ is invertible, the adjoint $\mathbb{A}^*$ is invertible as well. Therefore Step 1b follows from Step 1a. □

1c – Proof of surjectivity. To prove surjectivity we first show that the image of $D_\mathbb{A}$ is closed in $P_0(\mathbb{R})$. For this we use the obtained injectivity estimate. Suppose that η_ν is a sequence in the image of $D_\mathbb{A}$ which converges to some $\eta\in P_0(\mathbb{R})$. We need to show that $\eta\in \operatorname{im} D_\mathbb{A}$. Since $\eta_\nu\in\operatorname{im} D_\mathbb{A}$ there exists $\xi_\nu\in P_1(\mathbb{R})$ such that $\eta_\nu = D_\mathbb{A}\xi_\nu$. Since the sequence η_ν converges it is a Cauchy sequence in $P_0(\mathbb{R})$. By the injectivity estimate the sequence ξ_ν is as well a Cauchy sequence. Since $P_1(\mathbb{R})$ is complete the Cauchy sequence ξ_ν has a limit $\xi\in P_1(\mathbb{R})$. It follows that $\eta = D_\mathbb{A}\xi$ and therefore lies in the image of $D_\mathbb{A}$. So $\operatorname{im} D_\mathbb{A}$ is closed.
Hence to show that $D_\mathbb{A}$ is an isomorphism it suffices to check that the orthogonal complement of $\operatorname{im} D_\mathbb{A}$ is trivial. To see this pick $\eta\in(\operatorname{im} D_\mathbb{A})^\perp\subset P_0(\mathbb{R})$. This means that $\langle\eta, D_\mathbb{A}\xi\rangle_{P_0(\mathbb{R})} = 0$ for every $\xi\in P_1(\mathbb{R})$, hence

$$\begin{aligned}0 &= \int_{-\infty}^{\infty}\langle\eta(s),\partial_s\xi(s)+\mathbb{A}\xi(s)\rangle_0\, ds \\ &= \int_{-\infty}^{\infty}(\flat\eta(s))\,\partial_s\xi(s)\, ds + \int_{-\infty}^{\infty}\underbrace{(\mathbb{A}^*\flat\eta(s))}_{\mathbb{A}^*\colon H_0^*\to H_1^*}\underbrace{\xi(s)}_{\in H_1}\, ds\end{aligned} \tag{4.26}$$

for every $\xi\in P_1(\mathbb{R})$ and where $\flat\colon H_0\to H_0^*$ is the insertion isometry (A.89). Thus $\flat\eta\in L^2(\mathbb{R},H_0^*)$ has a weak derivative in H_1^* satisfying $\partial_s\flat\eta = \mathbb{A}^*\flat\eta$ where $\mathbb{A}^*\colon H_0^*\to H_1^*$ is the adjoint of $\mathbb{A}\colon H_1\to H_0$. In particular, $\flat\eta$ lies in the kernel of the operator $D_{-\mathbb{A}^*}\colon P_1(\mathbb{R};H_0^*,H_1^*)\to P_0(\mathbb{R};H_1^*)$. But $D_{-\mathbb{A}^*}$ is injective by part

1b of the proof, thus $\flat\eta = 0$, hence $\eta = 0$. This shows that $D_{\mathbb{A}}\colon P_1(\mathbb{R}; H_1, H_0) \to P_0(\mathbb{R}; H_0)$ is an isomorphism. Hence (4.24) follows from (4.23). This concludes the proof of Step 1. □

From now on we abbreviate

$$A_s := A(s)\colon H_1 \to H_0, \qquad \mathbb{A}_s := A(s) \text{ indicates invertibility.}$$

We enumerate the constants by the step where they appear, e.g. constant T_2 arises in Step 2.

Step 2 (Invertible near $\mathbb{A}^{\pm}$)**.** There are constants $T_2, C_2 > 0$ such that for any fixed time $\sigma \in (-\infty, -T_2] \cup [T_2, \infty)$ the operators $\mathbb{A}_\sigma$ and $D_{\mathbb{A}_\sigma}$ are invertible and

$$\left\|(D_{\mathbb{A}_\sigma})^{-1}\right\|_{\mathcal{L}(P_0(\mathbb{R}), P_1(\mathbb{R}))} \le C_2.$$

Proof. To prove Step 2 we show that the map

$$T\colon \mathcal{F} \to \mathcal{L}(P_1, P_0), \quad A \mapsto D_A$$

is continuous. To see this, given $A, \tilde{A} \in \mathcal{F}$ and using (1.6), we calculate

$$\begin{aligned}
\|D_A - D_{\tilde{A}}\|_{\mathcal{L}(P_1,P_0)} &:= \sup_{\|\xi\|_{P_1}=1} \|(D_A - D_{\tilde{A}})\xi\|_{P_0 = L^2(\mathbb{R}, H_0)} \\
&= \sup_{\|\xi\|_{P_1}=1} \left(\int_{\mathbb{R}} \|(A - \tilde{A})\xi(s)\|_{H_0}^2 \, ds\right)^{\frac{1}{2}} \\
&\le \|A - \tilde{A}\|_{\mathcal{L}(H_1,H_0)} \sup_{\|\xi\|_{P_1}=1} \underbrace{\left(\int_{\mathbb{R}} \|\xi(s)\|_{H_1}^2 \, ds\right)^{\frac{1}{2}}}_{= \|\xi\|_{L^2(\mathbb{R},H_1)} \le \|\xi\|_{P_1} = 1} \\
&\le \|A - \tilde{A}\|_{\mathcal{L}(H_1,H_0)}.
\end{aligned} \tag{4.27}$$

Step 2 follows now from Step 1 (invertibility of asymptotic operators $D_{\mathbb{A}^\pm}$) and with the help of Lemma B.1 since the path $\mathbb{R} \ni s \mapsto A(s)$ converges at the ends to $D_{\mathbb{A}^\pm}$. This proves Step 2. □

Step 3 (Asymptotic estimate)**.** Let $T_2 > 0$ be the constant of Step 2. There exists $T_3 \ge T_2$ such that the following is true. Suppose $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies

$$\text{either } \operatorname{supp} \beta \subset (-\infty, -T_3) \qquad \text{or } \operatorname{supp} \beta \subset (T_3, \infty).$$

Then for every $\xi \in P_1(\mathbb{R})$ we have the estimate

$$\|\beta\xi\|_{P_1(\mathbb{R})} \le 2C_2 \left(\|\beta D_A \xi\|_{P_0(\mathbb{R})} + \|\beta'\xi\|_{P_0(\mathbb{R})}\right)$$

where $C_2 > 0$ is the constant from Step 2.

Proof. Since $\lim_{s\to\pm\infty} A(s) = \mathbb{A}^\pm$ there exists $T_3 \geq T_2$ such that

$$\begin{aligned} \left\|\mathbb{A}_\sigma - \mathbb{A}^+\right\|_{\mathcal{L}(H_1,H_0)} &\leq \tfrac{1}{4C_2} \quad \forall \sigma \geq T_3 \\ \left\|\mathbb{A}_\sigma - \mathbb{A}^-\right\|_{\mathcal{L}(H_1,H_0)} &\leq \tfrac{1}{4C_2} \quad \forall \sigma \leq -T_3 \end{aligned}$$

where Step 2 provides the constant $C_2 > 0$ and invertibility of $\mathbb{A}_\sigma := A(\sigma)$.

In the following we only discuss the case where $\operatorname{supp}\beta \subset (T_3, \infty)$, the other case is analogous. Assume that $\sigma \in \operatorname{supp}\beta \subset (T_3, \infty)$. We calculate

$$\begin{aligned} D_{\mathbb{A}_\sigma}\beta\xi &= \partial_s(\beta\xi) + \mathbb{A}_\sigma\beta\xi \\ &= \beta'\xi + \beta\left(\partial_s\xi + (\mathbb{A}_\sigma - A + A)\xi\right) \\ &= \beta'\xi + \beta D_A\xi + (\mathbb{A}_\sigma - A)\beta\xi. \end{aligned} \tag{4.28}$$

By Step 2 the operator $D_{\mathbb{A}_\sigma}$ is invertible, we multiply with $(D_{\mathbb{A}_\sigma})^{-1}$ to get

$$\beta\xi = (D_{\mathbb{A}_\sigma})^{-1}\big(\beta'\xi + \beta D_A\xi + (\mathbb{A}_\sigma - A)\beta\xi\big).$$

Taking norms we estimate

$$\begin{aligned} \|\beta\xi\|_{P_1(\mathbb{R})} &\leq \left\|(D_{\mathbb{A}_\sigma})^{-1}\right\|_{\mathcal{L}(P_0(\mathbb{R}),P_1(\mathbb{R}))} \|\beta'\xi + \beta D_A\xi + (\mathbb{A}_\sigma - A)\beta\xi\|_{P_0(\mathbb{R})} \\ &\leq C_2\left(\|\beta D_A\xi\|_{P_0(\mathbb{R})} + \|\beta'\xi\|_{P_0(\mathbb{R})} + \|(\mathbb{A}_\sigma - A)\beta\xi\|_{P_0(\mathbb{R})}\right). \end{aligned}$$

It remains to estimate the difference

$$\begin{aligned} &\|(\mathbb{A}_\sigma - A)\beta\xi\|^2_{P_0(\mathbb{R})} \\ &= \int_{-\infty}^{\infty} \|(\mathbb{A}_\sigma - A(s))\,\beta(s)\xi(s)\|^2_{H_0}\, ds \\ &= \int_{T_3}^{\infty} \left\|\mathbb{A}_\sigma - \mathbb{A}^+ + \mathbb{A}^+ - A(s)\right\|^2_{\mathcal{L}(H_1,H_0)} \|\beta(s)\xi(s)\|^2_{H_1}\, ds \\ &\leq \int_{T_3}^{\infty} 2\left(\left\|\mathbb{A}_\sigma - \mathbb{A}^+\right\|^2_{\mathcal{L}(H_1,H_0)} + \left\|\mathbb{A}^+ - A(s)\right\|^2_{\mathcal{L}(H_1,H_0)}\right) \|\beta(s)\xi(s)\|^2_{H_1}\, ds \\ &\leq \frac{1}{4C_2^2}\int_{T_3}^{\infty} \|\beta(s)\xi(s)\|^2_{H_1}\, ds \\ &\leq \frac{1}{4C_2^2}\|\beta\xi\|^2_{P_1(\mathbb{R})}. \end{aligned}$$

The last two estimates together show that

$$\|\beta\xi\|_{P_1(\mathbb{R})} \leq 2C_2\left(\|\beta D_A\xi\|_{P_0(\mathbb{R})} + \|\beta'\xi\|_{P_0(\mathbb{R})}\right).$$

This proves Step 3. □

Step 4 (Invertibility perturbation)**.** For any $\sigma \in \mathbb{R}$ there is $\mu_\sigma \in \mathbb{R}$ such that the **shifted operator**

$$\mathbb{A}^{\mu_\sigma} := A_\sigma - \mu_\sigma\iota\colon H_1 \to H_0$$

is invertible where $\iota\colon H_1 \hookrightarrow H_0$ is inclusion.

Proof. Since A_σ is symmetric and inclusion $\iota\colon H_1 \hookrightarrow H_0$ is compact, the spectrum of A_σ, as unbounded operator on H_0 with dense domain H_1, is a discrete unbounded subset of $\mathbb{R}$ with no (finite) accumulation point, see Remarks 2.2 and 2.3. Pick μ_σ in the complement of the spectrum of A_σ, that is $\mu_\sigma \in \mathcal{R}(A)$. □

Step 5 (Invertibilizing A along $[-T_3, T_3]$ by finitely many shifts $\lambda_1, \dots, \lambda_N$)**.** Let $T_3 > 0$ be from Step 3. There is a finite set $\Lambda' = \{\lambda_1, \dots, \lambda_N\} \subset \mathbb{R}$ and a constant $C_5 > 0$ such the following holds. Fix any $\sigma \in [-T_3, T_3]$. Then there exists an element $\lambda(\sigma) \in \{\lambda_1, \dots, \lambda_N\}$, such that the operator $D_{\mathbb{A}^{\lambda(\sigma)}} = \partial_s + \mathbb{A}^{\lambda(\sigma)}$ is invertible and there is the estimate

$$\|(D_{\mathbb{A}^{\lambda(\sigma)}})^{-1}\|_{\mathcal{L}(P_0(\mathbb{R}),P_1(\mathbb{R}))} \le C_5.$$

Proof. Invertibility is an open property. Given any $\sigma \in \mathbb{R}$ there exists, by Step 4, a real number μ_σ, pick $\mu_\sigma \in \mathcal{R}(A_\sigma)$, such that $\mathbb{A}^{\mu_\sigma} = A_\sigma - \mu_\sigma \iota$ is invertible. Hence $D_{\mathbb{A}^{\mu_\sigma}}$ is invertible by Step 1. For τ near σ we vary $\mathbb{A}^{\mu_\sigma}$ in the form

$$\mathbb{A}^{\mu_\sigma}(\tau) := A_\tau - \mu_\sigma \iota, \qquad \mathbb{A}^{\mu_\sigma}(\sigma) = \mathbb{A}^{\mu_\sigma}.$$

By continuity (4.27) of $s \mapsto D_{A(s)}$ there exists, by Lemma B.1, a constant $\varepsilon_\sigma > 0$ with the following significance. At any time $\tau \in I_\sigma := (\sigma - \varepsilon_\sigma, \sigma + \varepsilon_\sigma)$ the operator $D_{\mathbb{A}^{\mu_\sigma}(\tau)}$ is invertible and the inverse is bounded by

$$\|\left(D_{\mathbb{A}^{\mu_\sigma}(\tau)}\right)^{-1}\|_{\mathcal{L}(P_0,P_1)} \le 2\|(D_{\mathbb{A}^{\mu_\sigma}})^{-1}\|_{\mathcal{L}(P_0,P_1)}. \tag{4.29}$$

Since $[-T_3, T_3]$ is compact there exist $N \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_N \in [-T_3, T_3]$ with

$$[-T_3, T_3] \subset \bigcup_{i=1}^N I_{\sigma_i}, \qquad I_{\sigma_i} := (\sigma_i - \varepsilon_{\sigma_i}, \sigma_i + \varepsilon_{\sigma_i}). \tag{4.30}$$

Now define

$$\Lambda' := \{\lambda_i := \mu_{\sigma_i} \mid i = 1, \dots, N\}, \quad C_5 := 2 \max_{i=1,\dots,N} \|(D_{\mathbb{A}^{\mu_{\sigma_i}}})^{-1}\|_{\mathcal{L}(P_0,P_1)}.$$

Suppose now that $\sigma \in [-T_3, T_3]$, then by the finite covering property (4.30) there exists $i \in \{1, \dots, N\}$ such that $\sigma \in I_{\sigma_i}$. We choose such i and define

$$\lambda(\sigma) := \mu_{\sigma_i}.$$

For this choice $\mathbb{A}^{\lambda(\sigma)} = A_\sigma - \mu_{\sigma_i} \iota =: \mathbb{A}^{\mu_{\sigma_i}}(\sigma)$ and there is the estimate

$$\|(D_{\mathbb{A}^{\lambda(\sigma)}})^{-1}\|_{\mathcal{L}(P_0,P_1)} = \|(D_{\mathbb{A}^{\mu_{\sigma_i}}(\sigma)})^{-1}\|_{\mathcal{L}} \overset{(4.29)}{\le} 2\|(D_{\mathbb{A}^{\mu_{\sigma_i}}})^{-1}\|_{\mathcal{L}} \le C_5$$

where $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathcal{L}(P_0,P_1)}$. This proves Step 5. □

Step 6 (Small intervals)**.** Let $C_6 := \max\{C_2, C_5\}$. Let $\lambda^* := \max|\Lambda'|$ be the maximal absolute value of the elements of the finite set $\Lambda' \subset \mathbb{R}$ in Step 5. Then for every $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ with the property

$$\sup_{\sigma,\tau\in\operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \frac{1}{2C_6}$$

it holds that

$$\|\beta\xi\|_{P_1(\mathbb{R})} \le 2C_6 \left(\|\beta D_A\xi\|_{P_0(\mathbb{R})} + \|\beta'\xi\|_{P_0(\mathbb{R})} + \lambda^* \|\beta\xi\|_{P_0(\mathbb{R})}\right)$$

for every $\xi \in P_1(\mathbb{R})$.

Proof. By Step 2 and Step 5 there exists a map $\lambda\colon \mathbb{R} \to \mathbb{R}$ satisfying $\lambda(\sigma) \in \Lambda'$ if $\sigma \in [-T_3, T_3]$ and $\lambda(\sigma) = 0$ if $\sigma \in (-\infty, -T_3) \cup (T_3, \infty)$ and such that

$$\|(D_{\mathbb{A}^{\lambda(\sigma)}})^{-1}\|_{\mathcal{L}(P_0(\mathbb{R}),P_1(\mathbb{R}))} \le C_5 \le C_6, \qquad \mathbb{A}^{\lambda(\sigma)} := A_\sigma - \lambda(\sigma)\iota.$$

Now the proof of Step 6 proceeds similarly as the proof of Step 3. Suppose that σ lies in the support of β. Computing as in (4.28) we get

$$D_{\mathbb{A}^{\lambda(\sigma)}}\beta\xi = \beta'\xi + \beta D_A\xi + (A_\sigma - A)\beta\xi - \lambda(\sigma)\beta\xi. \tag{4.31}$$

By construction of λ the operator $D_{\mathbb{A}^{\lambda(\sigma)}}$ is invertible so that we can write

$$\beta\xi = (D_{\mathbb{A}^{\lambda(\sigma)}})^{-1}\big(\beta'\xi + \beta D_A\xi + (A_\sigma - A)\beta\xi - \lambda(\sigma)\beta\xi\big).$$

Taking norms we estimate

$$\begin{aligned} &\|\beta\xi\|_{P_1(\mathbb{R})} \\ &\le C_6 \left(\|\beta'\xi\|_{P_0(\mathbb{R})} + \|\beta D_A\xi\|_{P_0(\mathbb{R})} + \|(A_\sigma - A)\beta\xi\|_{P_0(\mathbb{R})} + |\lambda(\sigma)|\,\|\beta\xi\|_{P_0(\mathbb{R})}\right). \end{aligned}$$

Now we use the hypothesis $\sup_{\sigma,\tau\in\operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \frac{1}{2C_6}$ to estimate

$$\begin{aligned} \|(A_\sigma - A)\beta\xi\|^2_{P_0(\mathbb{R})} &= \int_{-\infty}^{\infty} \|(A_\sigma - A(s))\,\beta(s)\xi(s)\|^2_{H_0}\, ds \\ &= \int_{\operatorname{supp}\beta} \|A_\sigma - A(s)\|^2_{\mathcal{L}(H_1,H_0)}\, \|\beta(s)\xi(s)\|^2_{H_1}\, ds \\ &\le \frac{1}{4C_6^2} \int_{\operatorname{supp}\beta} \|\beta(s)\xi(s)\|^2_{H_1}\, ds \\ &\le \frac{1}{4C_6^2}\, \|\beta\xi\|^2_{P_1(\mathbb{R})}. \end{aligned}$$

The last two estimates together imply Step 6. □

Step 7 (Partition of unity)**.** We prove Theorem 4.2.

Proof. Choose $T > T_3$ and a finite partition of unity $\{\beta_j\}_{j=0}^{M+1}$ for $\mathbb{R}$, where each $\beta_j \colon [0,1] \to \mathbb{R}$ is smooth, with the properties

$$\operatorname{supp} \beta_0 \subset (-\infty, -T_3), \qquad \operatorname{supp} \beta_{M+1} \subset (T_3, \infty),$$

and for $j = 1, \dots, M$ it holds

$$\sup_{\sigma,\tau \in \operatorname{supp} \beta_j} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \frac{1}{2C_6}, \qquad \operatorname{supp} \beta_j \subset (-T, T).$$

That such a partition exists follows from the continuity of $s \mapsto A(s)$ and the fact that on the compact set $[-T_3, T_3]$ continuity becomes uniform continuity. Let $\xi \in P_1(\mathbb{R})$. Then by the asymptotic estimate Step 3 we have

$$\begin{aligned}
\|\beta_0 \xi\|_{P_1(\mathbb{R})} &\le 2C_6 \left(\|\beta_0 D_A \xi\|_{P_0(\mathbb{R})} + \|\beta_0' \xi\|_{P_0(\mathbb{R})} \right), \\
\|\beta_{M+1} \xi\|_{P_1(\mathbb{R})} &\le 2C_6 \left(\|\beta_{M+1} D_A \xi\|_{P_0(\mathbb{R})} + \left\|\beta_{M+1}' \xi\right\|_{P_0(\mathbb{R})} \right).
\end{aligned}$$

By Step 6 (small intervals) we have for each $j = 1, \dots, M$ an estimate

$$\|\beta_j \xi\|_{P_1(\mathbb{R})} \le 2C_6 \left(\|\beta_j D_A \xi\|_{P_0(\mathbb{R})} + \left\|\beta_j' \xi\right\|_{P_0(\mathbb{R})} + \lambda^* \|\beta_j \xi\|_{P_0(\mathbb{R})} \right).$$

Let $B := \max\{\|\beta_0'\|_\infty, \|\beta_1'\|_\infty, \dots, \|\beta_{M+1}'\|_\infty\}$. Put the estimates together to get

$$\begin{aligned}
&\|\xi\|_{P_1(\mathbb{R})} \\
&\le \sum_{j=0}^{M+1} \|\beta_j \xi\|_{P_1(\mathbb{R})} \\
&\le 2C_6 \sum_{j=0}^{M+1} \left(\|\beta_j D_A \xi\|_{P_0(\mathbb{R})} + \left\|\beta_j' \xi\right\|_{P_0([-T,T])} \right) + 2C_6 \lambda^* \sum_{j=1}^{M} \|\beta_j \xi\|_{P_0([-T,T])} \\
&\le 2C_6 (M+2) \|D_A \xi\|_{P_0(\mathbb{R})} + 2C_6 \left(B(M+2) + \lambda^* M \right) \|\xi\|_{P_0([-T,T])}
\end{aligned}$$

where in the second inequality we replaced the $P_0(\mathbb{R})$ norm by the $P_0([-T,T])$ norm due to the supports of the β_j's and their derivatives.[3] Setting

$$c := \max\{2C_6(M+2), 2C_6 \left(B(M+2) + \lambda^* M \right)\}$$

proves Step 7. □

The proof of Theorem 4.2 is complete. □

4.1.2 Semi-Fredholm estimate for the adjoint D_A^*

Let $A \in \mathcal{A}_{\mathbb{R}}^*$. We call the following operator the **adjoint of D_A**, namely

$$D_A^* := D_{-A^*} \colon P_1(\mathbb{R}; H_0^*, H_1^*) \to P_0(\mathbb{R}; H_1^*), \quad \xi \mapsto \partial_s \xi - A(s)^* \xi.$$

[3] $\beta_0 \equiv 1$ on $(-\infty, -T]$ and $\beta_{M+1} \equiv 1$ on $[T, \infty)$, so the derivatives vanish.

Corollary 4.4. *For $A \in \mathcal{A}^*_{\mathbb{R}}$ there are constants c and $T > 0$ with*

$$\|\xi\|_{P_1(\mathbb{R};H_0^*,H_1^*)} \le c\left(\|D_A^*\xi\|_{P_0(\mathbb{R};H_1^*)} + \|\xi\|_{P_0([-T,T];H_1^*)}\right)$$

for every $\xi \in P_1(\mathbb{R}; H_0^*, H_1^*)$.

Proof. Theorem 4.2 and Lemma 2.7; see also Remark 1.4. □

4.1.3 Fredholm property of D_A

An immediate consequence of Theorem 4.2 is that the operator $D_A\colon P_1(\mathbb{R}) \to P_0(\mathbb{R})$ is **semi-Fredholm**, i.e. the kernel of D_A is of finite dimension and the range is closed. Indeed the restriction map $P_1(\mathbb{R}) \to P_0([-T,T])$ is compact, see e.g. [RS95, Lemma 3.8]. Hence the semi-Fredholm property follows from [MS04, Lemma A.1.1].

To see that D_A is actually a Fredholm operator we need to examine its cokernel. For that purpose let $\eta \in (\operatorname{im} D_A)^\perp \subset P_0(\mathbb{R}) = L^2(\mathbb{R}, H_0)$, that is

$$\langle \eta, D_A\xi\rangle_{P_0(\mathbb{R})} = 0, \quad \forall \xi \in P_1(\mathbb{R}) = L^2(\mathbb{R}, H_1) \cap W^{1,2}(\mathbb{R}, H_0).$$

To put it differently

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \langle \eta(s), \partial_s\xi(s) + A(s)\xi(s)\rangle_0 \, ds \\ &= \int_{-\infty}^{\infty} (\flat\eta(s))\, \partial_s\xi(s)\, ds + \int_{-\infty}^{\infty} \underbrace{(A(s)^*\flat\eta(s))}_{A(s)^*\colon H_0^*\to H_1^*} \xi(s)\, ds \end{aligned} \tag{4.32}$$

for every $\xi \in P_1(\mathbb{R})$ and where $\flat\colon H_0 \to H_0^*$ is the insertion isometry (A.89). Interpreting the ξ's as test functions this means that $\flat\eta \in L^2(\mathbb{R}, H_0^*)$ has a weak derivative in H_1^*, namely $\partial_s\flat\eta = A^*\flat\eta$. Hence $\flat\eta$ lies in $L^2(\mathbb{R}, H_0^*) \cap W^{1,2}(\mathbb{R}, H_1^*) = P_1(\mathbb{R}; H_0^*, H_1^*)$ and satisfies $D_A^*\flat\eta = \partial_s\flat\eta - A(s)^*\flat\eta = 0$. Observe that D_A^* is a map

$$D_A^* = \partial_s - A(s)^*\colon \underbrace{L^2(\mathbb{R}, H_0^*) \cap W^{1,2}(\mathbb{R}, H_1^*)}_{P_1(\mathbb{R};H_0^*,H_1^*) \overset{(4.22)}{\subset} C^0(\mathbb{R},H_1^*)} \to \underbrace{L^2(\mathbb{R}, H_1^*)}_{P_0(\mathbb{R};H_1^*)}. \tag{4.33}$$

We proved $\flat\,(\operatorname{im} D_A)^\perp \subset \ker D_A^*$. Vice versa, fix $\flat\eta \in \ker D_A^*$ Then $(D_A^*\flat\eta)(s) = 0_{H_1^*}$ for every $s \in \mathbb{R}$. Pick $\xi \in P_1(\mathbb{R})$ and integrate $(D_A^*\flat\eta)(s)\xi(s) = 0$ over $s \in \mathbb{R}$ to get back to $\langle \eta, D_A\xi\rangle_{P_0(\mathbb{R})} = 0$. We proved

Lemma 4.5. *Given $A \in \mathcal{A}^*_{\mathbb{R}}$, consider $D_A\colon P_1(\mathbb{R}; H_1, H_0) \to P_0(\mathbb{R}; H_0)$ and $D_A^*\colon P_1(\mathbb{R}; H_0^*, H_1^*) \to P_0(\mathbb{R}; H_1^*)$, then*

$$(\operatorname{im} D_A)^\perp \overset{\flat}{\simeq} \ker D_A^* \subset L^2(\mathbb{R}, H_0^*) \cap W^{1,2}(\mathbb{R}, H_1^*).$$

Corollary 4.6. *$D_A = \mathfrak{D}_A\colon P_1(\mathbb{R}; H_1, H_0) \to P_0(\mathbb{R}; H_0)$ is Fredholm $\forall A \in \mathcal{A}^*_{\mathbb{R}}$.*

Proof. By Theorem 4.2 the operator D_A is semi-Fredholm (finite dimensional kernel and closed image). Since D_A has closed image, it follows that $\operatorname{coker} D_A = (\operatorname{im} D_A)^\perp$, but $(\operatorname{im} D_A)^\perp \simeq \ker D_A^*$ by Lemma 4.5. By Corollary 4.4 the operator (4.33) is semi-Fredholm as well. This proves Corollary 4.6. □

4.2 Finite interval

Pick a Hessian path $A \in \mathcal{A}^*_{I_T} := \{A \in \mathcal{A}_{I_T} \mid A(-T) \text{ and } A(T) \text{ are invertible}\}$ along the finite interval $I_T = [-T, T]$. Then $A\colon [-T,T] \to \mathcal{F} = \mathcal{F}(H_1, H_0)$ takes values in the symmetrizable Fredholm operators of index zero; cf. Remarks 1.4 and 2.3. In order to eventually get to Fredholm operators, it is not enough that the Hessians at the interval ends are invertible, notation

$$\mathbb{A}_{-T} := A(-T), \qquad \mathbb{A}_T := A(T).$$

In addition, one must impose boundary conditions formulated in terms of the spectral projections $\pi_+^{\mathbb{A}_{-T}}$ sitting at time $-T$ and $\pi_-^{\mathbb{A}_T}$ at time T; see (2.14).

4.2.1 Estimate for D_A

In this section we study the linear operator $\partial_s + A$ as a map

$$D_A\colon P_1(I_T) \to P_0(I_T), \quad \xi \mapsto \partial_s \xi + A(s)\xi$$

and the augmented operator $\mathfrak{D}_A\colon P_1(I_T) \to \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T)$ in (1.7). Define Hilbert spaces $P_0(I_T) = P_0(I_T; H_0)$ and $P_1(I_T) = P_1(I_T; H_1, H_0)$ by (1.4). These operators are *not* Fredholm: although D_A has closed image and finite dimensional co-kernel, the kernel is infinite dimensional in the Floer and Morse case; see Figure 4 and (2.10).

Theorem 4.7. *For $A \in \mathcal{A}^*_{I_T}$ there exists a constant $c > 0$ such that*

$$\|\xi\|_{P_1(I_T)} \le c \left(\|D_A \xi\|_{P_0(I_T)} + \|\xi\|_{P_0(I_T)} + \|\pi_+^{\mathbb{A}_{-T}} \xi(-T)\|_{\frac12} + \|\pi_-^{\mathbb{A}_T} \xi(T)\|_{\frac12} \right)$$

for every $\xi \in P_1(I_T)$. *we abbreviate $\|\cdot\|_{\frac12} := \|\cdot\|_{H_{\frac12}}$*

Orthogonal projections are bounded by 1, hence the theorem reduces to

Corollary 4.8. *For $A \in \mathcal{A}^*_{I_T}$ there exists a constant $c > 0$ such that*

$$\|\xi\|_{P_1(I_T)} \le c \left(\|D_A \xi\|_{P_0(I_T)} + \|\xi\|_{P_0(I_T)} + \|\xi(-T)\|_{\frac12} + \|\xi(T)\|_{\frac12} \right)$$

for every $\xi \in P_1(I_T)$.

Proof of Theorem 4.7. We prove the theorem in five steps. It is often convenient to abbreviate $\xi_s := \xi(s)$ and $A_s := A(s)$. We enumerate the constants by the step where they appear, e.g. constant C_1 arises in Step 1.

Step 1 (Constant invertible case)**.** Let $A(s) \equiv \mathbb{A} = \mathbb{A}_T = \mathbb{A}_{-T}$ be constant in time and invertible. Then there is a constant $C_1 > 0$ such that

$$\|\xi\|_{P_1(I_T)} \le C_1 \left(\|D_{\mathbb{A}}\xi\|_{P_0(I_T)} + \|\pi_+^{\mathbb{A}}\xi_{-T}\|_{\frac{1}{2}} + \|\pi_-^{\mathbb{A}}\xi_T\|_{\frac{1}{2}} \right) \tag{4.34}$$

for every $\xi \in P_1(I_T)$. Moreover, for the constant path $\mathbb{A}$ the augmented operator

$$\begin{aligned} \mathfrak{D}_{\mathbb{A}} \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}) \times H^-_{\frac{1}{2}}(\mathbb{A}) =: \mathcal{W}(I_T; \mathbb{A}, \mathbb{A}) \\ \xi &\mapsto \left(D_A\xi, \pi_+^{\mathbb{A}}\xi_{-T}, \pi_-^{\mathbb{A}}\xi_T \right) \end{aligned} \tag{4.35}$$

is bijective.

Proof. Step 1 was proved in [Sim14, Thm. 3.1.6 i)]. We follow her proof. By changing the constant C_1 if necessary, we can assume without loss of generality, as explained in Section 2.3, that

$$\mathbb{A} \colon H_1 \to H_0 \text{ is a symmetric isometry.} \tag{4.36}$$

In this case (4.34) actually holds with unit constant $C_1 = 1$. We abbreviate $\pi_\pm := \pi_\pm^{\mathbb{A}}$. Since $D_{\mathbb{A}}\xi = \partial_s\xi + \mathbb{A}\xi$, integration by parts yields

$$\begin{aligned} &\|D_{\mathbb{A}}\xi\|^2_{P_0(I_T)} \\ &= \int_{-T}^{T} \left(\|\partial_s\xi_s\|^2_{H_0} + 2\left\langle \partial_s\xi_s, \mathbb{A}\xi_s \right\rangle_0 + \|\mathbb{A}\xi_s\|^2_{H_0} \right) ds \\ &= \|\mathbb{A}\xi\|^2_{P_0(I_T)} + \|\partial_s\xi\|^2_{P_0(I_T)} + \left\langle \xi_T, \mathbb{A}\xi_T \right\rangle_0 - \left\langle \xi_{-T}, \mathbb{A}\xi_{-T} \right\rangle_0 \end{aligned} \tag{4.37}$$

for every $\xi \in P_1(I_T)$. To see this note that

$$\begin{aligned} \int_{-T}^{T} \left\langle \partial_s\xi_s, \mathbb{A}\xi_s \right\rangle_0 ds &= \int_{-T}^{T} \left(\partial_s \left\langle \xi_s, \mathbb{A}\xi_s \right\rangle_0 - \left\langle \xi_s, \mathbb{A}\partial_s\xi_s \right\rangle_0 \right) ds \\ &= \int_{-T}^{T} \left(\partial_s \left\langle \xi_s, \mathbb{A}\xi_s \right\rangle_0 - \left\langle \mathbb{A}\xi_s, \partial_s\xi_s \right\rangle_0 \right) ds \end{aligned}$$

where in the last step we used H_0-symmetry of $\mathbb{A}$. Thus

$$\begin{aligned} 2\int_{-T}^{T} \left\langle \partial_s\xi_s, \mathbb{A}\xi_s \right\rangle_0 ds &= \int_{-T}^{T} \partial_s \left\langle \xi_s, \mathbb{A}\xi_s \right\rangle_0 ds \\ &= \left\langle \xi_T, \mathbb{A}\xi_T \right\rangle_0 - \left\langle \xi_{-T}, \mathbb{A}\xi_{-T} \right\rangle_0 . \end{aligned}$$

This proves (4.37). The opposite signs will be crucial soon. As $\mathbb{A}$ is an isometry we get

$$\begin{aligned} -\left\langle \pi_-\xi_T, \mathbb{A}\pi_-\xi_T \right\rangle_0 &\stackrel{(2.16)}{=} \|\pi_-\xi_T\|^2_{\frac{1}{2}} \\ \left\langle \pi_+\xi_{-T}, \mathbb{A}\pi_+\xi_{-T} \right\rangle_0 &\stackrel{(2.16)}{=} \|\pi_+\xi_{-T}\|^2_{\frac{1}{2}} . \end{aligned}$$

So we get

$$
\begin{aligned}
\langle \xi_T, \mathbb{A}\xi_T\rangle_0 &= \overbrace{\langle \pi_+\xi_T, \mathbb{A}\pi_+\xi_T\rangle_0}^{\ge 0} + \langle \pi_-\xi_T, \mathbb{A}\pi_-\xi_T\rangle_0 \ge -\|\pi_-\xi_T\|^2_{\frac{1}{2}} \\
\langle \xi_{-T}, \mathbb{A}\xi_{-T}\rangle_0 &= \langle \pi_+\xi_{-T}, \mathbb{A}\pi_+\xi_{-T}\rangle_0 + \underbrace{\langle \pi_-\xi_{-T}, \mathbb{A}\pi_-\xi_{-T}\rangle_0}_{\le 0} \le \|\pi_+\xi_{-T}\|^2_{\frac{1}{2}}
\end{aligned}
\tag{4.38}
$$

where the identities use that the mixed terms vanish by H_0-orthogonality; see Case 1 in Section 2.3. Since $\mathbb{A}$ is an isometry we get identity one in the following

$$
\begin{aligned}
\|\xi\|^2_{P_1(I_T)} &\overset{(1.5)}{=} \|\mathbb{A}\xi\|^2_{P_0(I_T)} + \|\partial_s\xi\|^2_{P_0(I_T)} \\
&\overset{(4.37)}{=} \|D_{\mathbb{A}}\xi\|^2_{P_0(I_T)} + \langle \xi_{-T}, \mathbb{A}\xi_{-T}\rangle_0 - \langle \xi_T, \mathbb{A}\xi_T\rangle_0 \\
&\le \|D_{\mathbb{A}}\xi\|^2_{P_0(I_T)} + \|\pi_+\xi_{-T}\|^2_{H_{\frac{1}{2}}} + \|\pi_-\xi_T\|^2_{H_{\frac{1}{2}}} .
\end{aligned}
$$

The last inequality is by the previous estimates (4.38). This proves (4.34). In particular, this implies that the augmented operator $\mathfrak{D}_{\mathbb{A}}$ in (4.35) is injective.

Claim 1. $\mathfrak{D}_{\mathbb{A}}$ is surjective.

To see this consider the orthonormal basis $\mathcal{V}(\mathbb{A}) = (v_\ell)_{\ell\in\Lambda} \subset H_1$ of H_0 from (2.11) enumerated by the ordering (2.9) of the eigenvalues a_ℓ of $\mathbb{A}$. Pick $\zeta = (\eta, x, y) \in \mathcal{W}(I_T; \mathbb{A}, \mathbb{A})$. Given $s \in [-T, T]$, with respect to the common basis $\mathcal{V}(\mathbb{A})$ of H_0 and $H^-_{\frac{1}{2}}(\mathbb{A}) \oplus H^+_{\frac{1}{2}}(\mathbb{A})$ we write

$$
\eta(s) = \sum_{\ell\in\Lambda} \eta_\ell(s) v_\ell \in H_0, \qquad x = \sum_{\nu\in\Lambda_+} x_\nu v_\nu, \quad y = \sum_{\nu\in\Lambda_-} y_{-\nu} v_{-\nu}.
$$

We are looking for a map $\xi \in P_1(I_T)$ of the form $s \mapsto \xi(s) = \sum_{\ell\in\Lambda} \xi_\ell(s) v_\ell$ which, for each $\ell \in \Lambda$, satisfies a linear inhomogeneous ODE of the form

$$
\partial_s \xi_\ell(s) + a_\ell \xi_\ell(s) = \eta_\ell(s) \tag{4.39}
$$

with the mixed boundary condition

$$
\xi_\nu(-T) = x_\nu, \quad \forall \nu \in \Lambda_+, \qquad \xi_{-\nu}(T) = y_{-\nu}, \quad \forall \nu \in \Lambda_-. \tag{4.40}
$$

We make the variation of constant Ansatz $\xi_\ell(s) = c_\ell(s) e^{-a_\ell s}$. Apply $\frac{d}{ds}$ to both sides and use (4.39) to get

$$
\partial_s c_\ell(s) = \eta_\ell(s) e^{a_\ell s}.
$$

Positive eigenvalue a_ν. Then we get $x_\nu = \xi_\nu(-T) = c_\nu(-T) e^{a_\nu T}$, so $c_\nu(-T) = x_\nu e^{-a_\nu T}$. Integrate $\partial_s c_\nu(s)$ from $-T$ to s to get

$$
c_\nu(s) = x_\nu e^{-a_\nu T} + \int_{-T}^{s} \eta_\nu(t) e^{a_\nu t}\, dt
$$

and therefore

$$\xi_\nu(s) = \underbrace{x_\nu e^{-a_\nu(T+s)}}_{=:\xi_\nu^1(s)} + \underbrace{\int_{-T}^{s} \eta_\nu(t) e^{a_\nu(t-s)}\, dt}_{=:\xi_\nu^2(s)}. \tag{4.41}$$

Negative eigenvalue $a_{-\nu}$. Then we get $y_{-\nu} = \xi_{-\nu}(T) = c_{-\nu}(T)e^{-a_{-\nu}T}$, so $c_{-\nu}(T) = y_{-\nu}e^{a_{-\nu}T}$. Integrate $\partial_s c_{-\nu}(s)$ from s to T to get

$$c_{-\nu}(s) = y_{-\nu}e^{a_{-\nu}T} - \int_s^T \eta_{-\nu}(t)e^{a_{-\nu}t}\, dt$$

and therefore

$$\xi_{-\nu}(s) = y_{-\nu}e^{a_{-\nu}(T-s)} - \int_s^T \eta_{-\nu}(t)e^{a_{-\nu}(t-s)}\, dt.$$

To finish the proof of Claim 1 it suffices to show

Claim 2. ξ lies in $P_1(I_T) = L^2(I_T, H_1) \cap W^{1,2}(I_T, H_0)$.

To we see this consider the case of a positive eigenvalue a_ν and write $\xi_\nu(s) = \xi_\nu^1(s) + \xi_\nu^2(s)$, see (4.41). To estimate ξ_ν^2 define a function $g_\nu \colon \mathbb{R} \to \mathbb{R}$ by

$$g_\nu(s) := \begin{cases} e^{-a_\nu s} & , s \geq 0, \\ 0 & , s < 0. \end{cases}$$

We estimate the $L^1(I_T) := L^1([-T, T], \mathbb{R})$ norm of g_ν by

$$\|g_\nu\|_{L^1(I_T)} = \int_{-T}^{T} g(s)\, ds = \int_0^T e^{-a_\nu s}\, ds = \frac{1 - e^{-a_\nu T}}{a_\nu} \leq \frac{1}{a_\nu}.$$

Thus, writing $\xi_\nu^2(s) = (\eta_\nu * g_\nu)(s)$ and by Young's inequality, we obtain

$$\|\xi_\nu^2\|_{L^2(I_T)} = \|\eta_\nu * g_\nu\|_{L^2(I_T)} \leq \|\eta_\nu\|_{L^2(I_T)} \|g_\nu\|_{L^1(I_T)} \leq \tfrac{1}{a_\nu}\|\eta_\nu\|_{L^2(I_T)}.$$

We estimate ξ_ν^1 as follows

$$\|\xi_\nu^1\|_{L^2(I_T)}^2 = \int_{-T}^{T} x_\nu^2 e^{-2a_\nu(T+s)}\, ds = x_\nu^2 \frac{1 - e^{-4a_\nu T}}{2a_\nu} \leq \frac{x_\nu^2}{2a_\nu}.$$

Now we use these estimates to obtain for the sum $\xi_\nu = \xi_\nu^1 + \xi_\nu^2$ that

$$\|\xi_\nu\|_{L^2(I_T)}^2 \leq 2\|\xi_\nu^1\|_{L^2(I_T)}^2 + 2\|\xi_\nu^2\|_{L^2(I_T)}^2 \leq \frac{x_\nu^2}{a_\nu} + \frac{2}{a_\nu^2}\|\eta_\nu\|_{L^2(I_T)}^2. \tag{4.42}$$

In the case of a negative eigenvalue $a_{-\nu}$ we observe that

$$\xi_{-\nu}(-s) = y_{-\nu}e^{a_{-\nu}(T+s)} - \int_{-T}^{s} \eta_{-\nu}(-\tau)e^{-a_{-\nu}(\tau-s)}\, d\tau.$$

Comparing this expression with (4.41) shows that we get the analogous estimate

$$\|\xi_{-\nu}\|^2_{L^2(I_T)} \le \frac{y^2_{-\nu}}{-a_{-\nu}} + \frac{2}{a^2_{-\nu}} \|\eta_{-\nu}\|^2_{L^2(I_T)}. \tag{4.43}$$

Now we show that $\xi \in L^2(I_T, H_1)$, namely

$$\begin{aligned}
\int_{-T}^{T} \|\xi(s)\|^2_{H_1}\, ds &\overset{1}{=} \int_{-T}^{T} \sum_{\ell\in\Lambda} a_\ell^2 \xi_\ell(s)^2\, ds \\
&= \sum_{\ell\in\Lambda} a_\ell^2 \|\xi_\ell\|^2_{L^2(I_T)} \\
&\overset{3}{=} \sum_{\nu\in\Lambda_-} a^2_{-\nu} \|\xi_{-\nu}\|^2_{L^2(I_T)} + \sum_{\nu\in\Lambda_+} a_\nu^2 \|\xi_\nu\|^2_{L^2(I_T)} \\
&\overset{4}{\le} 2 \sum_{\nu\in\Lambda_-} \|\eta_{-\nu}\|^2_{L^2(I_T)} + \sum_{\nu\in\Lambda_-} -a_{-\nu} y^2_{-\nu} \\
&\quad + 2 \sum_{\nu\in\Lambda_+} \|\eta_\nu\|^2_{L^2(I_T)} + \sum_{\nu\in\Lambda_+} a_\nu x_\nu^2 \\
&\overset{5}{=} 2 \sum_{\ell\in\Lambda} \|\eta_\ell\|^2_{L^2(I_T)} + \|y\|^2_{\frac{1}{2}} + \|x\|^2_{\frac{1}{2}} \\
&= 2 \int_{-T}^{T} \|\eta(s)\|^2_{H_0}\, ds + \|y\|^2_{\frac{1}{2}} + \|x\|^2_{\frac{1}{2}} \\
&= 2 \|\eta\|^2_{P_0(I_T)} + \|y\|^2_{\frac{1}{2}} + \|x\|^2_{\frac{1}{2}}.
\end{aligned}$$

Equality 1 uses that $\mathbb{A}$ is an isometry and the fact that the basis $\mathcal{V}(\mathbb{A})$ consists of eigenvectors of $\mathbb{A}$. Equality 3 uses the decomposition (2.10) of Λ. Inequality 4 uses (4.42) and (4.43). To see equality 5 go backwards and write $x = \sum_{\nu\in\Lambda_+} x_\nu v_\nu$. Then use that the basis is 1/2-orthogonal and that the 1/2-length of v_ν is $\sqrt{a_\nu}$ by (2.16). Similarly for y.

It remains to show that $\partial_s \xi \in L^2(I_T, H_0)$. To see this we consider the case of a positive eigenvalue a_ν. Use (4.39) in 1 and (4.42) in 3 to obtain

$$\begin{aligned}
\|\partial_s \xi_\nu\|^2_{L^2(I_T)} &\overset{1}{=} \|\eta_\nu - a_\nu \xi_\nu\|^2_{L^2(I_T)} \\
&\le 2\|\eta_\nu\|^2_{L^2(I_T)} + 2a_\nu^2 \|\xi_\nu\|^2_{L^2(I_T)} \\
&\overset{3}{\le} 6\|\eta_\nu\|^2_{L^2(I_T)} + 2a_\nu x_\nu^2.
\end{aligned}$$

Similarly, by using (4.43) instead of (4.42) we obtain

$$\|\partial_s \xi_{-\nu}\|^2_{L^2(I_T)} \le 6\|\eta_{-\nu}\|^2_{L^2(I_T)} - 2a_{-\nu} y^2_{-\nu}.$$

Similarly as above we obtain the estimate

$$\int_{-T}^{T} \|\partial_s \xi(s)\|^2_{H_0}\, ds = \sum_{\ell\in\Lambda} \|\partial_s \xi_\ell\|^2_{L^2(I_T)} \le 6\|\eta\|^2_{P_0(I_T)} + 2\|y\|^2_{\frac{1}{2}} + 2\|x\|^2_{\frac{1}{2}}.$$

We have shown that $\xi \in L^2(I_T, H_1) \subset L^2(I_T, H_0)$ and $\partial_s \xi \in L^2(I_T, H_0)$. Thus $\xi \in P_1(I_T; H_1, H_0)$ and

$$\|\xi\|^2_{P_1(I_T;H_1,H_0)} \leq 10\|\eta\|^2_{P_0(I_T)} + 4\|y\|^2_{\frac{1}{2}} + 4\|x\|^2_{\frac{1}{2}}.$$

This concludes the proof of Claim 2, hence of Claim 1 and Step 1. □

From now on we abbreviate $A_\sigma := A(\sigma) \in \mathcal{L}(H_1, H_0)$.

Step 2 (Small interior interval)**.** There is a finite subset $\Lambda' \subset \mathbb{R}$ and a constant $C_2 > 0$ such that for every $\beta \in C^\infty(I_T, \mathbb{R})$ which vanishes on the interval boundary, in symbols $\beta(-T) = \beta(T) = 0$, and has the property

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \leq \frac{1}{C_2}$$

it holds that

$$\|\beta\xi\|_{P_1(I_T)} \leq C_2 \left(\|\beta D_A \xi\|_{P_0(I_T)} + \|\beta'\xi\|_{P_0(I_T)} + \|\beta\xi\|_{P_0(I_T)}\right)$$

for every $\xi \in P_1(I_T)$.

Proof. This is Step 6 in the proof of the Rabier Theorem 4.2. □

Step 3 (Small interval at right boundary)**.** There exist constants $\varepsilon_3 > 0$ and $C_3 > 0$ such that for every $\beta \in C^\infty(I_T, \mathbb{R})$ which vanishes on the left interval boundary, in symbols $\beta(-T) = 0$, and has the property

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \leq \varepsilon_3$$

it holds that

$$\|\beta\xi\|_{P_1(I_T)} \leq C_3 \left(\|\beta D_A \xi\|_{P_0(I_T)} + \|\beta'\xi\|_{P_0(I_T)} + \|\beta\xi\|_{P_0(I_T)} + \|\pi_-\beta_T\xi_T\|_{\frac{1}{2}}\right)$$

for every $\xi \in P_1(I_T)$.

Proof. By continuity of the map $s \mapsto A(s) =: A_s$ there exists the limit

$$\lim_{\sigma\to T} A_\sigma = \mathbb{A}_T.$$

By Step 1 the augmented operator associated to the constant path $\mathbb{A}_T$, namely

$$\begin{aligned} \mathfrak{D}_{\mathbb{A}_T} \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}_T) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: \mathcal{W}(I_T; \mathbb{A}_T, \mathbb{A}_T) \\ \xi &\mapsto (D_{\mathbb{A}_T}\xi, \pi_+\xi_{-T}, \pi_-\xi_T) \end{aligned}$$

is bijective, in particular invertible. Together these two facts imply that there is $\varepsilon_3 > 0$ such that the following is true. If $\|A_\sigma - \mathbb{A}_T\|_{\mathcal{L}(H_1,H_0)} \leq \varepsilon_3$, then $\mathfrak{D}_{A_\sigma}$ is still invertible with inverse bound

$$\left\|(\mathfrak{D}_{A_\sigma})^{-1}\right\|_{\mathcal{L}(\mathcal{W}(I_T;\mathbb{A}_t,\mathbb{A}_T),P_1(I_T))} \leq \frac{1}{2\varepsilon_3}.$$

This follows from the fact that the map

$$T\colon \mathcal{F}^* \to \mathcal{L}(P_1, P_0), \quad A \mapsto D_A$$

is continuous, by (4.27), in view of the injectivity Lemma B.1.

Now pick $\sigma \in \operatorname{supp}\beta$. Then, in particular, the previous estimate for the inverse is valid. By formula (4.31) in Step 6 in the proof of the Rabier Theorem 4.2 we get a formula for the first component $D_{A_\sigma}\beta\xi$, namely

$$\begin{aligned}\mathfrak{D}_{A_\sigma}\beta\xi &= (D_{A_\sigma}\beta\xi\,,\, \pi_+\beta_{-T}\xi_{-T}\,,\, \pi_-\beta_T\xi_T)\\ &= (\beta'\xi + \beta D_A\xi + (A_\sigma - A)\beta\xi\,,\, 0\,,\, \pi_-\beta_T\xi_T)\,.\end{aligned}$$

In the second equality we use the assumption $\beta_{-T} = 0$. Since the operator $\mathfrak{D}_{A_\sigma}$ is invertible we can write

$$\beta\xi = (\mathfrak{D}_{A_\sigma})^{-1}\,(\beta'\xi + \beta D_A\xi + (A_\sigma - A)\beta\xi\,,\, 0\,,\, \pi_-\beta_T\xi_T)\,.$$

Taking norms we estimate

$$\begin{aligned}&\|\beta\xi\|_{P_1(I_T)}\\ &\le \frac{1}{\varepsilon_3}\left(\|\beta'\xi\|_{P_0(I_T)} + \|\beta D_A\xi\|_{P_0(I_T)} + \|(A_\sigma - A)\beta\xi\|_{P_0(I_T)} + \|\pi_-\beta_T\xi_T\|_{\frac{1}{2}}\right).\end{aligned}$$

Now we estimate

$$\begin{aligned}\|(A_\sigma - A)\beta\xi\|^2_{P_0(I_T)} &= \int_{-T}^{T} \|(A_\sigma - A_s)\,\beta_s\xi_s\|^2_{H_0}\, ds\\ &\le \int_{\operatorname{supp}\beta} \|A_\sigma - A_s\|^2_{\mathcal{L}(H_1,H_0)}\,\|\beta_s\xi_s\|^2_{H_1}\, ds\\ &\le \varepsilon_3^2\,\|\beta\xi\|^2_{P_1(I_T)}\,.\end{aligned}$$

The last two estimates together imply Step 5 with $C_3 := \frac{2}{\varepsilon_3}$. ☐

Step 4 (Small interval at left boundary)**.** There exist constants $\varepsilon_4 > 0$ and $C_4 > 0$ such that for every $\beta \in C^\infty(I_T, \mathbb{R})$ which vanishes on the right interval boundary, in symbols $\beta(T) = 0$, and has the property

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \varepsilon_4$$

it holds that

$$\|\beta\xi\|_{P_1(I_T)} \le C_4\left(\|\beta D_A\xi\|_{P_0(I_T)} + \|\beta'\xi\|_{P_0(I_T)} + \|\beta\xi\|_{P_0(I_T)} + \|\pi_+\beta_{-T}\xi_{-T}\|_{\frac{1}{2}}\right)$$

for every $\xi \in P_1(I_T)$.

Proof. Same argument as in Step 3. ☐

Step 5 (Partition of unity)**.** We prove Theorem 4.7.

Proof. Let $\varepsilon := \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and $C := \max\{C_2, C_3, C_4\}$. Choose a finite partition of unity $\{\beta_j\}_{j=0}^{M+1}$ for $I_T = [-T, T]$ with the properties

$$\beta_0(-T) = 1, \quad \beta_0(T) = 0, \qquad \beta_{M+1}(-T) = 0, \quad \beta_{M+1}(T) = 1,$$

and

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta_i} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \varepsilon, \qquad \operatorname{supp}\beta_j \subset (-T, T),$$

for $i = 0, 1, \dots, M, M+1$ and $j = 1, \dots, M$. That such a partition exists follows from the continuity of $s \mapsto A(s)$ and since on the compact set $[-T, T]$ continuity becomes uniform continuity. Let $\xi \in P_1(I_T)$. By Steps 4 and 3 we get

$$\begin{aligned}
\|\beta_0\xi\|_{P_1(I_T)} &\le C\Big(\|\beta_0 D_A\xi\|_{P_0(I_T)} + \|\beta_0'\xi\|_{P_0(I_T)} \\
&\qquad + \|\beta_0\xi\|_{P_0(I_T)} + \|\pi_+\xi_{-T}\|_{\frac{1}{2}}\Big) \\
\|\beta_{M+1}\xi\|_{P_1(I_T)} &\le C\Big(\|\beta_{M+1} D_A\xi\|_{P_0(I_T)} + \left\|\beta_{M+1}'\xi\right\|_{P_0(I_T)} \\
&\qquad + \|\beta_{M+1}\xi\|_{P_0(I_T)} + \|\pi_-\xi_T\|_{\frac{1}{2}}\Big).
\end{aligned}$$

By Step 2 we have

$$\|\beta_j\xi\|_{P_1(I_T)} \le C\left(\|\beta_j D_A\xi\|_{P_0(I_T)} + \left\|\beta_j'\xi\right\|_{P_0(I_T)} + \|\beta_j\xi\|_{P_0(I_T)}\right)$$

for $j = 1, \dots, M$. We abbreviate $B := \max\{\|\beta_0'\|_\infty, \|\beta_1'\|_\infty, \dots, \|\beta_{M+1}'\|_\infty\}$. Putting these estimates together we obtain

$$\begin{aligned}
\|\xi\|_{P_1(I_T)} &\le \sum_{j=0}^{M+1} \|\beta_j\xi\|_{P_1(I_T)} \\
&\le C\sum_{j=0}^{M+1}\left(\|\beta_j D_A\xi\|_{P_0(I_T)} + \left\|\beta_j'\xi\right\|_{P_0(I_T)} + \|\beta_j\xi\|_{P_0(I_T)}\right) \\
&\quad + C\|\pi_+\xi_{-T}\|_{\frac{1}{2}} + C\|\pi_-\xi_T\|_{\frac{1}{2}} \\
&\le C(M+2)\,\|D_A\xi\|_{P_0(I_T)} + C(B+1)(M+2)\,\|\xi\|_{P_0(I_T)} \\
&\quad + C\|\pi_+\xi_{-T}\|_{\frac{1}{2}} + C\|\pi_-\xi_T\|_{\frac{1}{2}}.
\end{aligned}$$

Setting $c := C(B+1)(M+2)$ proves Step 5. □

The proof of Theorem 4.7 is complete. □

4.2.2 Estimate for the adjoint D_A^*

Let $A \in \mathcal{A}^*_{I_T}$. We call the following operator the **adjoint of D_A**, namely

$$D_A^* := D_{-A^*}\colon P_1(I_T; H_0^*, H_1^*) \to P_0(I_T; H_1^*), \quad \eta \mapsto \partial_s \eta - A(s)^* \eta.$$

Corollary 4.9. *For $A \in \mathcal{A}^*_{I_T}$ there exists a constant $c > 0$ such that*

$$\begin{aligned} &\|\eta\|_{P_1(I_T;H_0^*,H_1^*)} \\ &\le c\Big(\|D_A^*\eta\|_{P_0(I_T;H_1^*)} + \|\eta\|_{P_0(I_T;H_1^*)} + \|\pi_+^{-\mathbb{A}^*_{-T}}\eta(-T)\|_{\frac{1}{2}} + \|\pi_-^{-\mathbb{A}^*_T}\eta(T)\|_{\frac{1}{2}} \Big) \end{aligned}$$

for every $\eta \in P_1(I_T; H_0^*, H_1^*)$.

Proof. Theorem 4.7 and Lemma 2.7; see also Remark 1.4. □

4.2.3 Fredholm under boundary conditions: D_A^{+-}

Let $A \in \mathcal{A}^*_{I_T}$. For the spectral projections $\pi_+^{\mathbb{A}_{-T}}$ and $\pi_-^{\mathbb{A}_T}$ see (2.14). To turn Theorem 4.7 to a semi-Fredholm estimate we restrict the domain of the operator

$$D_A\colon P_1(I_T; H_1, H_0) \to P_0(I_T; H_0), \quad \xi \mapsto \partial_s \xi + A(s)\xi$$

by imposing appropriate boundary conditions that cut down the operator kernel to finite dimension. To this end we define a subspace of the domain as follows

$$\begin{aligned} &P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0) \\ &:= \{\xi \in P_1(I_T; H_1, H_0) \mid \pi_+^{\mathbb{A}_{-T}}\xi_{-T} = 0 \ \wedge \ \pi_-^{\mathbb{A}_T}\xi_T = 0\}. \end{aligned} \tag{4.44}$$

The associated restriction of D_A we denote by

$$\boxed{D_A^{+-} = \partial_s + A\colon P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0) \to P_0(I_T; H_0).} \tag{4.45}$$

To $A \in \mathcal{A}^*_{I_T}$ we associate the path $-A^* \in \mathcal{A}^*_{I_T}$, namely $s \mapsto -A(s)^*\colon H_0^* \to H_1^*$. Define a Hilbert space $P_1(I_T; H_0^*, H_1^*) := L^2(I_T, H_0^*) \cap W^{1,2}(I_T, H_1^*)$, analogous to (1.4). A closed linear subspace is defined by imposing boundary conditions

$$\begin{aligned} &P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*) \\ &:= \{\eta \in P_1(I_T; H_0^*, H_1^*) \mid \pi_+^{-\mathbb{A}^*_{-T}}\eta_{-T} = 0 \ \wedge \ \pi_-^{-\mathbb{A}^*_T}\eta_T = 0\}. \end{aligned} \tag{4.46}$$

It includes into $P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*) \subset P_1(I_T; H_0^*, H_1^*) \overset{(4.22)}{\subset} C^0(I_T, H_1^*)$. The restriction of the linear operator

$$D_{-A^*}\colon P_1(I_T; H_0^*, H_1^*) \to P_0(I_T; H_1^*), \quad \eta \mapsto \partial_s \eta - A(s)^*\eta$$

to the closed linear subspace (4.46) is denoted by

$$\boxed{D_{-A^*}^{+-} = \partial_s - A^*\colon P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*) \to P_0(I_T; H_1^*).} \tag{4.47}$$

Corollary 4.10 (Semi-Fredholm)**.** *For any $A \in \mathcal{A}^*_{I_T}$ the operators*

$$D_A^{+-} \colon P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0) \to P_0(I_T; H_0)$$

and

$$D_{-A^*}^{+-} \colon P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*) \to P_0(I_T; H_1^*)$$

are semi-Fredholm (finite dimensional kernel and closed image).

Proof. Theorem 4.7 provides for D_A^{+-} the semi-Fredholm estimate[4]

$$\|\xi\|_{P_1(I_T;H_1,H_0)} \le c\left(\|D_A^{+-}\xi\|_{P_0(I_T;H_0)} + \|\xi\|_{P_0(I_T;H_0)}\right)$$

$\forall \xi \in P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0)$. Corollary 4.9 provides the semi-Fredholm estimate

$$\|\eta\|_{P_1(I_T;H_0^*,H_1^*)} \le c\left(\|D_{-A^*}^{+-}\eta\|_{P_0(I_T;H_1^*)} + \|\eta\|_{P_0(I_T;H_1^*)}\right)$$

for the operator $D_{-A^*}^{+-}$ and every $\eta \in P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*)$. □

Theorem 4.11 (Fredholm)**.** *For any Hessian path $A \in \mathcal{A}^*_{I_T}$ the operator $D_A^{+-} \colon P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0) \to P_0(I_T; H_0)$ is Fredholm.*

Corollary 4.12. *The operator $D_A \colon P_1(I_T) \to P_0(I_T)$ in (1.6) has closed image of finite co-dimension for any Hessian path $A \in \mathcal{A}^*_{I_T}$.*

Proof. By Theorem 4.11 the image of D_A^{+-} is closed and of finite co-dimension. Since D_A^{+-} is a restriction of D_A we have inclusion $\operatorname{im} D_A^{+-} \subset \operatorname{im} D_A$. So $\operatorname{im} D_A$ is of finite co-dimension. Thus $\operatorname{im} D_A$ is closed by [Bre11, Prop. 11.5]. □

	$D_A \colon P_1 \to P_0$		$D_A^{+-} \colon P_1^{+-} \to P_0$
dim ker	∞		$k < \infty$
dim coker	$\le \ell$	$\Leftarrow$	$\ell < \infty$
	co-semi-Fredholm		Fredholm
image	closed	$\Leftarrow$	closed
coker			$\operatorname{coker} D_A^{+-} \simeq \ker D_{-A^*}^{+-}$
ker	huge		$\ker D_A^{+-} \simeq \operatorname{coker} D_{-A^*}^{+-}$

Figure 4: $D_A = \partial_s + A(s)$ on $P_1(I_T)$ and its restriction D_A^{+-} to P_1^{+-}

Proof of Theorem 4.11. Pick $A \in \mathcal{A}^*_{I_T}$, then $\mathbb{A}_{-T} := A(-T)$ and $\mathbb{A}_T := A(T)$ are invertible. By Corollary 4.10 the operator D_A^{+-} (and also $D_{-A^*}^{+-}$) has finite dimensional kernel and closed image. It remains to show that D_A^{+-} has finite dimensional co-kernel. This is proved in the following Proposition 4.13. The proof of Theorem 4.11 is complete. □

[4] The inclusion map $P_1(I_T) \to P_0(I_T)$ is compact, see e.g. [RS95, Lemma 3.8]. Hence the semi-Fredholm property follows from [MS04, Lemma A.1.1].

To prove that D_A^{+-} has finite dimensional co-kernel we show how the annihilator of D_A^{+-} can be identified with the kernel of the semi-Fredholm operator $D_{-A^*}^{+-}$. The **annihilator** of D_A^{+-} consists of all linear functionals on the co-domain which vanish along the image

$$\mathrm{Ann}(D_A^{+-}) := \{\zeta \in P_0(I_T; H_0)^* \mid \zeta(D_A\xi) = 0 \ \ \forall \xi \in P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0)\}.$$

Since $\zeta(D_A\xi) = \langle \zeta, D_A\xi\rangle_0$ the annihilator identifies naturally with the orthogonal complement of the image of D_A^{+-}, which is the cokernel, in symbols

$$\mathrm{Ann}(D_A^{+-}) \simeq \operatorname{coker} D_A^{+-}.$$

We have a natural map

$$\mathcal{K}\colon P_0(I_T; H_0^*) \to P_0(I_T; H_0)^*$$

defined by

$$(\mathcal{K}\eta)\chi := \int_{I_T} \eta(s)\chi(s)\, ds$$

for every $\chi \in P_0(I_T; H_0) = L^2(I_T, H_0)$. As shown by Kreuter [Kre15, Thm. 2.22] the map $\mathcal{K}$ is an isometry.

Proposition 4.13. *The vector spaces* $\mathcal{K}^{-1}\mathrm{Ann}(D_A^{+-}) = \ker D_{-A^*}^{+-}$ *coincide as vector subspaces of* $P_0(I_T; H_0^*)$.

Proof. Note that both are subspaces of $P_0(I_T; H_0^*) = L^2(I_T, H_0^*)$, indeed

$$\mathcal{K}^{-1}\mathrm{Ann}(D_A^{+-}) \subset \underline{P_0(I_T; H_0^*)} \supset P_1^*(I_T) \supset P_1^{*,+-}(I_T) \supset \ker D_{-A^*}^{+-}.$$

For equality $\mathcal{K}^{-1}\mathrm{Ann}(D_A^{+-}) = \ker D_{-A^*}^{+-}$ we show inclusions I. $\subset$ and II. $\supset$.

I. The inclusion "$\subset$". Pick $\eta \in \mathcal{K}^{-1}\mathrm{Ann}(D_A^{+-}) \subset L^2(I_T, H_0^*)$. For $\xi \in P_1^{+-}(I_T)$ we calculate

$$\begin{aligned} 0 &= (\mathcal{K}\eta)D_A\xi \\ &= \int_{I_T} \eta(D_A\xi)\, ds \\ &\stackrel{3}{=} \int_{I_T} \eta(\partial_s\xi)\, ds + \int_{I_T} \eta(A\xi)\, ds \\ &= \int_{I_T} \eta(\partial_s\xi)\, ds + \int_{I_T} (A^*\eta)\xi\, ds. \end{aligned} \tag{4.48}$$

This shows that η admits a weak derivative in H_1^*, notation $\partial_s\eta$, with

$$\partial_s\eta - A^*\eta = 0.$$

We first show that this equation implies that $\partial_s\eta \in L^2(I_T, H_1^*)$. To see this we first note that since $A \in \mathcal{A}_{I_T}^*$ and I_T is compact there is a constant c such that

$\|A(s)\|_{\mathcal{L}(H_1,H_0)} \le c$ for every $s \in I_T$. The map $*\colon \mathcal{L}(H_1,H_0) \to \mathcal{L}(H_0^*,H_1^*)$, $A \mapsto A^*$, is an isometry. Hence we also have $\|A(s)^*\|_{\mathcal{L}(H_0^*,H_1^*)} \le c$ for every $s \in I_T$. Using this we obtain finiteness of

$$\|\partial_s \eta\|^2_{L^2(I_T,H_1^*)} = \|A^*\eta\|^2_{L^2(I_T,H_1^*)} = \int_{I_T} \|A(s)^*\eta(s)\|^2_{H_1^*}\, ds \le c^2 \|\eta\|^2_{L^2(I_T,H_0^*)}.$$

Indeed, since $\eta \in L^2(I_T,H_0^*)$, it follows that $\partial_s\eta \in L^2(I_T,H_1^*)$. To summarize, we proved that

$$\eta \in P_1(I_T;H_0^*,H_1^*) \;\wedge\; \eta \in \ker D_{-A^*}.$$

It remains to show that η satisfies the boundary conditions (4.46). We check the <u>boundary condition at $-T$</u>, namely

$$\underline{0 = \pi_+^{-\mathbb{A}^*_{-T}} \eta(-T)} = \pi_-^{\mathbb{A}^*_{-T}} \eta(-T),$$

while the boundary condition at T one checks analogously. By Section 2.3 the boundary condition does not depend on the choice of the inner products on H_1 and H_0, therefore we can assume without loss of generality that the inner products are $\mathbb{A}_{-T}$-adapted, i.e. from now on

$$\mathbb{A}_{-T}\colon H_1 \to H_0 \text{ is a symmetric isometry.}$$

We pick an orthonormal basis $\mathcal{V}(\mathbb{A}_{-T}) = \{v_\ell\}_{\ell\in\Lambda} \subset H_1$ of H_0, see (2.11), which consists of eigenvectors of $\mathbb{A}_{-T}$, more precisely $\mathbb{A}_{-T} v_\ell = a_\ell v_\ell$. From now on we identify $\mathbb{A}^*_{-T}\colon H_0^* \to H_1^*$ isometrically with $A_{-T}\colon H_1 \to H_0$ according to Lemma A.8.

We have that

$$\eta \in L^2(I_T;H_0) \cap W^{1,2}(I_T;H_{-1}) = P_1(I_T;H_{-1},H_0) \subset C^0(I_T,H_{-1}).$$

Here the last inclusion follows from [Rou13, Le. 7.1]; see also (4.21).

Since η is continuous, it makes sense to consider η pointwise at any time s and use the orthogonal basis $\mathcal{V}(\mathbb{A}_{-T})$ of H_{-1} to write

$$\eta(s) = \sum_{\ell\in\Lambda} \eta_\ell(s) v_\ell \in H_{-1}$$

where the coefficients $\eta_\ell(s) \in \mathbb{R}$ depends continuously on s. Moreover, the norm of η is related to the norms of the coefficients as follows

$$\|\eta\|^2_{P_1(I_T;H_{-1},H_0)} = \sum_{\ell\in\Lambda} \left(\|\eta_\ell\|^2_{L^2(I_T,\mathbb{R})} + \frac{1}{a_\ell^2} \|\eta_\ell\|^2_{W^{1,2}(I_T,\mathbb{R})} \right).$$

Since $\pi_-^{\mathbb{A}_{-T}}\xi(-T)$ is arbitrary and $\pi_+^{\mathbb{A}_{-T}}\xi(-T) = 0$, see (4.44), we prove next that (4.48) implies

Claim. $\eta_{-\nu}(-T) = 0$ for every $\nu \in \Lambda_- = \Lambda_-(\mathbb{A}_{-T})$.

Proof of Claim. We pick a smooth cut-off function $\beta\colon [-T,T] \to [0,1]$ such that $\beta(-T) = 1$, that $\beta \equiv 0$ outside a small interval $[-T, -T+\delta]$, and that $\|\beta'\|_{L^\infty} \le \frac{2}{\delta}$. Pick $\nu \in \Lambda_-$ and consider the corresponding eigenvector $v_{-\nu}$. Let us define a map $\zeta\colon [-T,T] \to \mathbb{R}v_{-\nu}$ and conclude the following two properties

$$\boxed{\zeta := \beta\eta_{-\nu}(-T)v_{-\nu}}, \quad \pi_+^{\mathbb{A}_{-T}}\zeta(-T) = 0, \quad \zeta \in P_1^{+-}(I_T, \mathbb{A}_{\pm T}; H_1, H_0). \tag{4.49}$$

Recall that $\mathbb{A}_{-T}v_{-\nu} = a_{-\nu}v_{-\nu}$ where $a_{-\nu} < 0$. Given $\varepsilon > 0$, pick a parameter $0 < \delta \le \min\{1, (8|a_{-\nu}|)^{-1}\}$ so small that

$$\begin{aligned}
&\|\eta\|^2_{L^2(I_T,H_0)} \sup_{s\in[-T,-T+\delta]} \|A(s) - \mathbb{A}_{-T}\|^2_{\mathcal{L}(H_1,H_0)} (a_{-\nu})^2 4 \\
&+ \max\{16, 2\,|a_{-\nu}|\,, 4(a_{-\nu})^2\} \sup_{s\in[-T,-T+\delta]} |\eta_{-\nu}(s) - \eta_{-\nu}(-T)|^2 \\
&\le \tfrac{1}{4}\varepsilon^2.
\end{aligned} \tag{4.50}$$

This will be used together with $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ on terms 1-4 below. We calculate

$$\begin{aligned}
&\eta_{-\nu}(-T)^2 \\
&= -\int_{-T}^{T} \eta_{-\nu}(-T)^2 \beta'(s)\, ds \\
&= -\int_{-T}^{T} \langle \eta(-T), \partial_s\zeta(s)\rangle_0\, ds \\
&\stackrel{3}{=} -\int_{-T}^{T} \Big\langle \underline{\eta(s)}, \partial_s\zeta(s) \Big\rangle_0\, ds + \int_{-T}^{T} \Big\langle \underline{\eta(s)} - \eta(-T), \partial_s\zeta(s) \Big\rangle_0\, ds \\
&\stackrel{4}{=} \int_{-T}^{-T+\delta} \Big\langle \eta(s), (A(s) - \underline{\mathbb{A}_{-T}})\zeta(s) \Big\rangle_0\, ds + \int_{-T}^{-T+\delta} \langle \eta(s) - \eta(-T), \partial_s\zeta(s)\rangle_0\, ds \\
&\quad + \int_{-T}^{-T+\delta} \Big\langle \eta(s), \underline{\mathbb{A}_{-T}}\zeta(s) \Big\rangle_0\, ds \\
&\stackrel{5}{=} \int_{-T}^{-T+\delta} \langle \eta(s), (A(s) - \mathbb{A}_{-T})\zeta(s)\rangle_0\, ds + \int_{-T}^{-T+\delta} \langle \eta(s) - \eta(-T), \partial_s\zeta(s)\rangle_0\, ds \\
&\quad + \int_{-T}^{-T+\delta} \Big\langle \eta(s) - \underline{\eta(-T)}, \mathbb{A}_{-T}\zeta(s) \Big\rangle_0\, ds + \int_{-T}^{-T+\delta} \Big\langle \underline{\eta(-T)}, \mathbb{A}_{-T}\zeta(s) \Big\rangle_0\, ds
\end{aligned}$$

where in equalities 3, 4, and 5 we added zero, equality 4 is by line 3 in (4.48) for $\xi := \zeta$ and since $\operatorname{supp}\zeta \subset \operatorname{supp}\beta \subset [-T, -T+\delta]$. Now we discuss each of the four terms in the sum individually using the estimate $ab \le \frac{a^2}{2} + \frac{b^2}{2}$.

Term 1. By Cauchy-Schwarz and definition (4.49) of ζ we obtain

$$\begin{aligned}
&\int_{-T}^{-T+\delta} \langle \eta(s), (A(s) - \mathbb{A}_{-T})\zeta(s)\rangle_0 \, ds \\
&\le \int_{-T}^{-T+\delta} \|\eta(s)\|_0 \|A(s) - \mathbb{A}_{-T}\|_{\mathcal{L}(H_1,H_0)} \|\zeta(s)\|_1 ds \\
&\overset{2}{\le} \|\eta\|_{L^2(I_T,H_0)} \sup_{s\in[-T,-T+\delta]} \|A(s) - \mathbb{A}_{-T}\|_{\mathcal{L}(H_1,H_0)} |a_{-\nu}| \, 2\cdot \tfrac{1}{2} |\eta_{-\nu}(-T)| \\
&\overset{3}{\le} \tfrac{\varepsilon^2}{8} + \tfrac{1}{8}\eta_{-\nu}(-T)^2.
\end{aligned}$$

Inequality 2 uses that $\|v_{-\nu}\|_1 = |a_{-\nu}|$, by (2.15), inequality 3 is by (4.50).

Term 2. By definition of ζ and (v_ℓ) being an orthonormal basis of H_0 we get

$$\begin{aligned}
&\int_{-T}^{-T+\delta} \langle \eta(s) - \eta(-T), \partial_s \zeta(s)\rangle_0 \, ds \\
&= \int_{-T}^{-T+\delta} (\eta_{-\nu}(s) - \eta_{-\nu}(-T))\beta'(s)\eta_{-\nu}(-T) \langle v_{-\nu}, v_{-\nu}\rangle_0 \, ds \\
&\overset{2}{\le} \delta \sup_{s\in[-T,-T+\delta]} |\eta_{-\nu}(s) - \eta_{-\nu}(-T)| \, \tfrac{2}{\delta} 2\cdot \tfrac{1}{2} |\eta_{-\nu}(-T)| \\
&\overset{3}{\le} \tfrac{\varepsilon^2}{8} + \tfrac{1}{8}\eta_{-\nu}(-T)^2.
\end{aligned}$$

Inequality 2 pulls out the supremum, uses $\|\beta'\|_{L^\infty} \le \frac{2}{\delta}$, inequality 3 is by (4.50).

Term 3. By definition of ζ and (v_ℓ) being an orthonormal basis of H_0 we get

$$\begin{aligned}
&\int_{-T}^{-T+\delta} \langle \eta(s) - \eta(-T), \mathbb{A}_{-T}\zeta(s)\rangle_0 \, ds \\
&\overset{1}{\le} \int_{-T}^{-T+\delta} \Big(\underline{\eta_{-\nu}(s) - \eta_{-\nu}(-T)}\Big) \beta(s) a_{-\nu} \Big(\underline{\eta_{-\nu}(s) - \eta_{-\nu}(-T)} + \eta_{-\nu}(-T)\Big) \, ds \\
&\overset{2}{\le} \delta \sup_{s\in[-T,-T+\delta]} |\eta_{-\nu}(s) - \eta_{-\nu}(-T)|^2 \, |a_{-\nu}| \\
&\quad + \delta \sup_{s\in[-T,-T+\delta]} |\eta_{-\nu}(s) - \eta_{-\nu}(-T)| \, |a_{-\nu}| \, 2\cdot \tfrac{1}{2} |\eta_{-\nu}(-T)| \\
&\overset{3}{\le} \tfrac{\varepsilon^2}{8} + \tfrac{\varepsilon^2}{8} + \tfrac{1}{8}\eta_{-\nu}(-T)^2.
\end{aligned}$$

Inequality 1 uses the eigenvalue $a_{-\nu}$ of $\mathbb{A}_{-T}$ and we added zero. Inequality 2 pulls out the supremum, uses $\|\beta\|_{L^\infty} \le 1$. Inequality 3 uses $\delta < 1$ and (4.50).

Term 4. By definition of ζ and (v_ℓ) being an orthonormal basis of H_0 we get

$$\begin{aligned}
\int_{-T}^{-T+\delta} \langle \eta(-T), \mathbb{A}_{-T}\zeta(s)\rangle_0 \, ds &\overset{1}{=} \int_{-T}^{-T+\delta} \eta_{-\nu}(-T)^2 \beta(s) a_{-\nu} ds \\
&\le \delta \eta_{-\nu}(-T)^2 \, |a_{-\nu}| \\
&\le \tfrac{1}{8}\eta_{-\nu}(-T)^2.
\end{aligned}$$

Equality 1 uses the eigenvalue $a_{-\nu}$ of $\mathbb{A}_{-T}$. Then we use that $\|\beta\|_{L^\infty} \le 1$ and the final inequality exploits the choice of δ.

The analysis of terms 1-4 shows $\eta_{-\nu}(-T)^2 \le 4\frac{\varepsilon^2}{8} + 4\frac{1}{8}\eta_{-\nu}(-T)^2$. Thus $\eta_{-\nu}(-T)^2 \le \varepsilon^2$ for every $\varepsilon > 0$. So $\eta_{-\nu}(-T) = 0$. This proves the claim. □

II. The inclusion $\mathcal{K} \ker D^{+-}_{-A^*} \subset \mathrm{Ann}(D^{+-}_A)$. Pick $\eta \in \ker D^{+-}_{-A^*} \subset P_1^{+-}(I_T, -\mathbb{A}^*_{\pm T}; H_0^*, H_1^*)$. For $\xi \in P_1^{+-}(I_T; H_1, H_0)$ we calculate

$$
\begin{aligned}
&(\mathcal{K}\eta) D_A \xi \\
&= \int_{I_T} \eta(D_A\xi)\, ds \\
&= \int_{I_T} \eta(\partial_s \xi)\, ds + \int_{I_T} \eta(A\xi)\, ds \\
&\overset{3}{=} -\int_{I_T} (\partial_s \eta)\xi\, ds + \int_{I_T} (A^*\eta)\xi\, ds \\
&= (D_{-A^*}\eta)\xi \\
&= 0.
\end{aligned}
\tag{4.51}
$$

Equation 1 is by definition of $\mathcal{K}$. Equation 3 is integration by parts together with the fact that η and ξ satisfy mutually orthogonal boundary conditions at $-T$ as well as at T. This proves that $\mathcal{K}\eta \in \mathrm{Ann}(D^{+-}_A)$.

This concludes the proof of Proposition 4.13. □

4.2.4 Theorem A – Fredholm property

Corollary 4.14 (to Theorem 4.11, Fredholm)**.** *For any $A \in \mathcal{A}^*_{I_T}$ the operator*

$$
\begin{aligned}
\mathfrak{D}_A = \mathfrak{D}_A^{I_T} \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac12}(\mathbb{A}_{-T}) \times H^-_{\frac12}(\mathbb{A}_T) =: \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T) \\
\xi &\mapsto \left(D_A\xi, \pi_+^{\mathbb{A}_{-T}}\xi_{-T}, \pi_-^{\mathbb{A}_T}\xi_T\right)
\end{aligned}
$$

is Fredholm, where $\mathbb{A}_{\pm T} := A(\pm T)$, and of the same index as D^{+-}_A. More precisely, the kernels coincide and the co-kernels are of equal dimension.

Proof. By Theorem 4.7 the operator $\mathfrak{D}_A$ is semi-Fredholm. So it has finite dimensional kernel and closed image. We shall show that kernel and image of $\mathfrak{D}_A$ are equal, respectively isomorphic, to those of the Fredholm operator D^{+-}_A from Theorem 4.11.

Step 1. $\ker \mathfrak{D}_A = \ker D^{+-}_A$.

Proof. Clearly $\xi \in P_1(I_T)$ and $(0,0,0) = \mathfrak{D}_A\xi := (D_A\xi, \pi_+^{\mathbb{A}_{-T}}\xi_{-T}, \pi_-^{\mathbb{A}_T}\xi_T)$ is equivalent to $D_A\xi = 0$ and $\xi \in P_1^{+-}$; see (4.44). □

Step 2. D^{+-}_A is surjective iff $\mathfrak{D}_A$ is surjective.

Proof. "⇒" Given $(\eta, x, y) \in \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T)$, we need to find a $\xi \in P_1$ such that $\mathfrak{D}_A\xi = (\eta, x, y)$. To this end, pick $\xi_1 \in P_1(I_T)$ which satisfies the given boundary conditions $\xi_1(-T) = x$ and $\xi_1(T) = y$; see Corollary C.4. Since D_A^{+-} is surjective, there exists $\xi_0 \in P_1^{+-}$ such that $D_A\xi_0 = \eta - D_A\xi_1 \in P_0(I_T)$. We define $\xi := \xi_0 + \xi_1 \in P_1(I_T)$. Then $D_A\xi = \eta$ and $\pi_+^{\mathbb{A}_{-T}}\xi(-T) = \pi_+^{\mathbb{A}_{-T}}\xi_0(-T) + \pi_+^{\mathbb{A}_{-T}}\xi_1(-T) = 0 + \pi_+^{\mathbb{A}_{-T}}x = x$, similarly $\pi_-^{\mathbb{A}_T}\xi(T) = y$. Hence $\mathfrak{D}_A\xi = (\eta, x, y)$.

"⇐" Pick $\eta \in P_0(I_T)$ and consider $(\eta, 0, 0) \in \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T)$. Since $\mathfrak{D}_A$ is surjective there exists $\xi \in P_1(I_T)$ such that $(\eta, 0, 0) = \mathfrak{D}_A\xi = (D_A\xi, \pi_+^{\mathbb{A}_{-T}}\xi_{-T}, \pi_-^{\mathbb{A}_T}\xi_T)$. Since $\pi_+^{\mathbb{A}_{-T}}\xi_{-T} = 0$ and $\pi_-^{\mathbb{A}_T}\xi_T = 0$ we conclude that $\xi \in P_A^{+-}$ so that $D_A^{+-}\xi = \eta$. This shows that D_A^{+-} is surjective and concludes the proof of Step 2. □

Step 3. $\dim\operatorname{coker}\mathfrak{D}_A \le \dim\operatorname{coker} D_A^{+-} < \infty$ since D_A^{+-} is Fredholm.

Proof. Suppose $n := \dim\operatorname{coker} D_A^{+-} \ge 1$. Let $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ be a basis of the orthogonal complement $(\operatorname{im} D_A^{+-})^\perp$. Define an image filling operator by

$$\widetilde{D}_A^{+-}\colon P_1^{+-} \times \mathbb{R}^n \to P_0, \quad (\xi, a) \mapsto D_A\xi + \sum_{i=1}^n a_i\beta_i$$

and define a candidate to be image filling by

$$\widetilde{\mathfrak{D}}_A\colon P_1 \times \mathbb{R}^n \to \mathcal{W}, \quad (\xi, a) \mapsto \left(D_A\xi + \sum_{i=1}^n a_i\beta_i, \pi_+^{\mathbb{A}_{-T}}\xi(-T), \pi_-^{\mathbb{A}_T}\xi(T)\right).$$

We now show that $\widetilde{\mathfrak{D}}_A$ is surjective as well. Pick $(\eta, x, y) \in \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T)$. We need to find $(\xi, a) \in P_1 \times \mathbb{R}^n$ such that $\widetilde{\mathfrak{D}}_A(\xi, a) = (\eta, x, y)$. To this end, pick $\xi_1 \in P_1(I_T)$ which satisfies the given boundary conditions $\xi_1(-T) = x$ and $\xi_1(T) = y$; see Corollary C.4. Since $\widetilde{D}_A^{+-}$ is surjective, there exists $(\xi_0, a) \in P_1^{+-} \times \mathbb{R}^n$ such that

$$\widetilde{D}_A^{+-}(\xi, a) := D_A\xi_0 + \sum_{i=1}^n a_i\beta_i = \eta - D_A\xi_1 \in P_0(I_T).$$

This identity applied to $\xi := \xi_0 + \xi_1 \in P_1(I_T)$ yields

$$\widetilde{\mathfrak{D}}_A(\xi, a) = \left(D_A\xi + \sum_{i=1}^n a_i\beta_i, \pi_+^{\mathbb{A}_{-T}}\xi(-T), \pi_-^{\mathbb{A}_T}\xi(T)\right) = (\eta, x, y).$$

This proves that $\widetilde{\mathfrak{D}}_A$ is surjective. Hence $\dim\operatorname{coker}\mathfrak{D}_A \le n = \dim\operatorname{coker} D_A^{+-}$. This proves Step 3. □

Step 4. $\dim\operatorname{coker} D_A^{+-} \le \dim\operatorname{coker}\mathfrak{D}_A < \infty$ by Step 3.

Proof. We choose a finite basis $\mathcal{B} = \{\beta_1, \dots, \beta_n\} \subset P_0$ of the orthogonal complement of the image of D_A^{+-}. Suppose by contradiction that $n > \dim\operatorname{coker} \mathfrak{D}_A$. Then the span of $\{(\beta_1, 0, 0), \dots, (\beta_n, 0, 0)\} \in \mathcal{W}$ has non-trivial intersection with $\operatorname{im} \mathfrak{D}_A \subset \mathcal{W}$. Otherwise, the span would form a complement of $\operatorname{im} \mathfrak{D}_A$ and so $\dim\operatorname{coker} \mathfrak{D}_A \geq n$. Contradiction. Hence some non-zero element in the span lies in the image of $\mathfrak{D}_A$, in symbols $\exists a \in \mathbb{R}^n \setminus \{0\}$ such that

$$\left(\sum_{j=1}^{n} a_j \beta_j, 0, 0 \right) = \sum_{j=1}^{n} a_j (\beta_j, 0, 0) \in \operatorname{im} \mathfrak{D}_A.$$

Thus there exists $\xi \in P_1$ such that

$$\left(\sum_{j=1}^{n} a_j \beta_j, 0, 0 \right) = \mathfrak{D}_A \xi = \left(D_A \xi, \pi_+^{\mathbb{A}_{-T}} \xi_{-T}, \pi_-^{\mathbb{A}_T} \xi_T \right).$$

Hence $\xi \in P_1^{+-}$ and $D_A^{+-} \xi = \sum_{j=1}^{n} a_j \beta_j$. Now the left hand side lies in the image of D_A^{+-} and the right hand side in the orthogonal complement, hence it is the zero vector. Since $\mathcal{B}$ is a basis all coefficients a_j are zero. Contradiction. This proves Step 4. □

By Steps 2–4 the dimension of $\operatorname{coker} D_A^{+-}$ equals the one of $\operatorname{coker} \mathfrak{D}_A$. Hence, by Step 1, the Fredholm indices are equal. This proves Corollary 4.14. □

4.2.5 Path concatenation

Let $A \in \mathcal{A}^*_{I_T}$ be a Hessian path such that not only $\mathbb{A}_{-T} := A(-T)$ and $\mathbb{A}_T := A(T)$ are invertible, but also the Hessian operator at time zero $\mathbb{A}_0 := A(0)$ is. Decomposing the time interval at time zero

$$I_T := [-T, T] = I_T^- \cup I_T^+, \qquad I_T^- := [-T, 0], \quad I_T^+ := [0, T],$$

gives rise to three augmented operators, one along each of the three intervals. Firstly, there is the operator along I_T, defined by

$$\begin{aligned} \mathfrak{D}_A \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: \mathcal{W}(I_T; \mathbb{A}_{-T}, \mathbb{A}_T) \\ \xi &\mapsto \left(D_A \xi, \pi_+^{\mathbb{A}_{-T}} \xi_{-T}, \pi_-^{\mathbb{A}_T} \xi_T \right) \end{aligned}$$

where $\xi_{\pm T} := \xi(\pm T)$. We define the operator along $I_T^- = [-T, 0]$ by

$$\begin{aligned} \mathfrak{D}_{A|_{[-T,0]}} \colon P_1(I_T^-) &\to P_0(I_T^-) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}_0) =: \mathcal{W}(I_T^-; \mathbb{A}_{-T}, \mathbb{A}_0) \\ \xi &\mapsto \left(D_A \xi \,,\, \pi_+^{\mathbb{A}_{-T}} \xi_{-T} \,,\, \pi_-^{\mathbb{A}_0} \xi_0 \right) \end{aligned}$$

and the operator along $I_T^+ = [0, T]$ is defined by

$$\begin{aligned} \mathfrak{D}_{A|_{[0,T]}} \colon P_1(I_T^+) &\to P_0(I_T^+) \times H^+_{\frac{1}{2}}(\mathbb{A}_0) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: \mathcal{W}(I_T^+; \mathbb{A}_0, \mathbb{A}_T) \\ \xi &\mapsto \left(D_A \xi \,,\, \pi_+^{\mathbb{A}_0} \xi_0 \,,\, \pi_-^{\mathbb{A}_T} \xi_T \right). \end{aligned}$$

Theorem 4.15 (Path concatenation)**.** *Suppose $A \in \mathcal{A}^*_{I_T}$ is such that $\mathbb{A}_0 := A(0)$ is invertible. Then the Fredholm index is additive under concatenation*

$$\operatorname{index} \mathfrak{D}_A = \operatorname{index} \mathfrak{D}_{A|_{[-T,0]}} + \operatorname{index} \mathfrak{D}_{A|_{[0,T]}}.$$

A main proof ingredient is a domain-homotopy of Fredholm operators. For paths $\xi \in P_1(I_T^-)$ and $\eta \in P_1(I_T^+)$ we abbreviate

$$\xi_\pm(0) := \pi_\pm^{\mathbb{A}_0} \xi(0), \qquad \eta_\pm(0) := \pi_\pm^{\mathbb{A}_0} \eta(0). \tag{4.52}$$

For $r \in [0, 1]$ we define a family of spaces by

$$\mathfrak{P}_r := \{(\xi, \eta) \in P_1(I_T^-) \times P_1(I_T^+) \mid \xi_-(0) = r\eta_-(0) \;\wedge\; r\xi_+(0) = \eta_+(0)\}.$$

Proposition 4.16. *Consider the family of operators defined, for $r \in [0, 1]$, by*

$$\begin{aligned} \mathfrak{D}_{A,r} \colon X \supset \mathfrak{P}_r &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: Y \\ (\xi, \eta) &\mapsto \Big((D_A\xi)\#(D_A\eta), \pi_+^{\mathbb{A}_{-T}} \xi_{-T}, \pi_-^{\mathbb{A}_T} \eta_T\Big) \end{aligned} \tag{4.53}$$

where $X = P_1(I_T^-) \times P_1(I_T^+)$ and

$$(D_A\xi) \# (D_A\eta)\ (s) := \begin{cases} D_A\xi(s) & , s \in [-T, 0), \\ D_A\eta(s) & , s \in [0, T]. \end{cases}$$

Then the following is true. a) Each member of the family is a Fredholm operator and b) the Fredholm index is constant along the family.

Proof of Theorem 4.15. Let us present right away the proof in a nutshell

$$\begin{aligned} \operatorname{index} \mathfrak{D}_A &\overset{1.}{=} \operatorname{index} \mathfrak{D}_{A,1} && \text{, equal operators (Step 1)} \\ &\overset{2.}{=} \operatorname{index} \mathfrak{D}_{A,0} && \text{, homotopy, Prop. 4.16} \\ &\overset{3.}{=} \operatorname{index} \Big(\mathfrak{D}^{\bullet -}_{A|_{[-T,0]}} \oplus \mathfrak{D}^{+\bullet}_{A|_{[0,T]}}\Big) && \text{, decompose } P_0(I_T) \text{ (Step 3)} \\ &\overset{4.}{=} \operatorname{index} \mathfrak{D}^{\bullet -}_{A|_{[-T,0]}} + \operatorname{index} \mathfrak{D}^{+\bullet}_{A|_{[0,T]}} && \text{, direct sum (obvious)} \\ &\overset{5.}{=} \operatorname{index} \mathfrak{D}_{A|_{[-T,0]}} + \operatorname{index} \mathfrak{D}_{A|_{[0,T]}} && \text{, summand-wise equality.} \end{aligned}$$

Before filling in the details of steps 1-5 we need to define the $+-$ operators appearing individually in step 3 and as direct sum in step 4. The operators[5]

$$\begin{aligned} \mathfrak{D}^{\bullet -}_{A|_{[-T,0]}} \colon P_1^{\bullet -}(I_T^-) &\to P_0(I_T^-) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}) =: \mathcal{W}(I_T^-; \mathbb{A}_{-T}) \\ \xi &\mapsto \Big(D_A\xi, \pi_+^{\mathbb{A}_{-T}} \xi_{-T}\Big) \end{aligned}$$

[5] *Bullet notation.* An interval has two boundary points, left and right. The bullet '•' symbolizes 'no boundary condition' at that boundary point whose position corresponds to the position of the bullet, left or right. The sign $+/-$ tells that the positive/negative part of the spectrum must vanish at the other boundary point.

and

$$\mathfrak{D}^{+\bullet}_{A|_{[0,T]}} : P_1^{+\bullet}(I_T^+) \to P_0(I_T^+) \times H^-_{\frac{1}{2}}(\mathbb{A}_T) =: \mathcal{W}(I_T^+;\mathbb{A}_T)$$
$$\eta \mapsto \left(D_A\eta, \pi_-^{\mathbb{A}_T}\eta_T\right)$$

are defined by the usual formula $\partial_s + A(s)$. Observe that one boundary condition is imposed on the domain and the other one on the co-domain as follows

$$P_1^{\bullet -}(I_T^-) := \{\xi \in P_1(I_T^-) \mid \xi_-(0) \overset{(4.52)}{=} 0\},$$
$$P_1^{+\bullet}(I_T^+) := \{\eta \in P_1(I_T^+) \mid \eta_+(0) \overset{(4.52)}{=} 0\}.$$

The proof proceeds in six steps 0-5 as enumerated in the nutshell.

Step 0. The operators appearing in the nutshell are all Fredholm.

Proof. The operators $\mathfrak{D}_A$, $\mathfrak{D}_{A|_{[-T,0]}}$, and $\mathfrak{D}_{A|_{[0,T]}}$ are Fredholm, by Corollary 4.14, and $\mathfrak{D}_{A,1}$ and $\mathfrak{D}_{A,0}$ are Fredholm, by Proposition 4.16. The operators $\mathfrak{D}^{\bullet +}_{A|_{[-T,0]}}$ and $\mathfrak{D}^{-\bullet}_{A|_{[0,T]}}$ are Fredholm by the arguments in the proof of Corollary 4.14. This concludes Step 0. □

Step 1. $\mathfrak{D}_A = \mathfrak{D}_{A,1}$

Proof. This follows immediately using the obvious identification of $P_1(I_T)$ with

$$\mathfrak{P}_1 = \{(\xi,\eta) \in P_1(I_T^-) \times P_1(I_T^+) \mid \xi_-(0) = \eta_-(0) \;\wedge\; \xi_+(0) = \eta_+(0)\}.$$

by cutting the elements of $P_1(I_T)$ at time 0 into two pieces. See also Appendix C on the evaluation map. □

Step 2. Homotopy of Fredholm operators between $\mathfrak{D}_{A,0}$ and $\mathfrak{D}_{A,1}$.

Proof. Proposition 4.16. □

Step 3. $\mathfrak{D}_{A,0}$ and $\mathfrak{D}^+_{A|_{[-T,0]}} \oplus \mathfrak{D}^-_{A|_{[0,T]}}$ correspond naturally.

Proof. This follows from equal domain

$$\mathfrak{P}_0 := \{(\xi,\eta) \in P_1(I_T^-) \times P_1(I_T^+) \mid \xi_-(0) = 0 = \eta_+(0)\} = P_1^{\bullet -}(I_T^-) \times P_1^{+\bullet}(I_T^+)$$

and, in the co-domain, the identification of $P_0(I_T)$ with $P_0(I_T^-) \times P_0(I_T^+)$. □

Step 4. Direct sum of Fredholm operators is Fredholm of index the index sum.

Proof. Well known. □

Step 5. $\operatorname{index} \mathfrak{D}^{\bullet -}_{A|_{[-T,0]}} = \operatorname{index} \mathfrak{D}_{A|_{[-T,0]}}$ and $\operatorname{index} \mathfrak{D}^{+\bullet}_{A|_{[0,T]}} = \operatorname{index} \mathfrak{D}_{A|_{[0,T]}}$.

Proof. This follows by the arguments in the proof of Corollary 4.14. □

This concludes the proof of Theorem 4.15. □

Proof of Proposition 4.16. We define, for each $r \in [0,1]$, a bounded linear map

$$F_r \colon X := P_1(I_T^-) \times P_1(I_T^+) \to H^-_{\frac{1}{2}}(\mathbb{A}_0) \times H^+_{\frac{1}{2}}(\mathbb{A}_0) =: Z$$
$$(\xi,\eta) \mapsto (\xi_-(0) - r\eta_-(0), r\xi_+(0) - \eta_+(0)) .$$

The kernel is the domain $\mathfrak{P}_r = \ker F_r$ of the restriction $\mathfrak{D}_{A,r} \colon \mathfrak{P}_r \to Y$ of

$$\mathfrak{D}_A^{\#} \colon X \to Y, \quad (\xi,\eta) \mapsto \Big((D_A\xi)\#(D_A\eta), \pi_+^{\mathbb{A}_{-T}}\xi_{-T}, \pi_-^{\mathbb{A}_T}\xi_T \Big) .$$

a) $\mathfrak{D}_{A,r} \colon \mathfrak{P}_r \to Y$ is Fredholm $\forall r \in [0,1]$.

By Proposition 4.17 below each operator $\mathfrak{D}_{A,r} \colon \mathfrak{P}_r \to Y$ is semi-Fredholm. Theorem E.2 for $D = \mathfrak{D}_A^{\#}$ and $D_r = \mathfrak{D}_{A,r}$ therefore implies that its semi-Fredholm index is independent of $r \in [0,1]$. By Step 1 in the proof of Theorem 4.15 the operator $\mathfrak{D}_{A,1}$ is Fredholm and therefore has finite index. Hence, by independence of r, every $\mathfrak{D}_{A,r}$ has finite index and is therefore Fredholm.

b) $\operatorname{index} \mathfrak{D}_{A,r}$ does not depend on r.

Use Theorem E.2 for $D = \mathfrak{D}_A^{\#}$ and $D_r = \mathfrak{D}_{A,r}$. This proves Proposition 4.16. □

Proposition 4.17. *Let $A \in \mathcal{A}^*_{I_T}$. Then there is a constant $c > 0$ such that*

$$\|\xi\|_{P_1(I_T^-)} + \|\eta\|_{P_1(I_T^+)} \le c\Big(\|\xi\|_{P_0(I_T^-)} + \|\eta\|_{P_0(I_T^+)} + \|D_{\mathbb{A}}\xi\|_{P_0(I_T^-)} + \|D_{\mathbb{A}}\eta\|_{P_0(I_T^+)} + \|\xi_+(-T)\|_{\frac{1}{2}} + \|\eta_-(T)\|_{\frac{1}{2}} . \Big)$$

for every $(\xi,\eta) \in \mathfrak{P}_r$ and $r \in [0,1]$.

Proof. The proof follows the same way as the proof of Theorem 4.7. The only step which needs to be adjusted is the estimate (4.34) in step 1. For this adjustment the assumption that $r \in [0,1]$ is crucial. Therefore it suffices to show for any constant path $A(s) \equiv \mathbb{A} = \mathbb{A}_T = \mathbb{A}_{-T}$ that consists of an invertible operator the existence of a constant $c > 0$ such that

$$\|\xi\|_{P_1(I_T^-)} + \|\eta\|_{P_1(I_T^+)} \le c\Big(\|D_{\mathbb{A}}\xi\|_{P_0(I_T^-)} + \|D_{\mathbb{A}}\eta\|_{P_0(I_T^+)} + \|\xi_+(-T)\|_{\frac{1}{2}} + \|\eta_-(T)\|_{\frac{1}{2}} . \Big) \tag{4.54}$$

for every $(\xi,\eta) \in \mathfrak{P}_r$.

To see this we proceed as follows. As in Step 1 in the proof of Theorem 4.7, by changing the constant C_1 if necessary, we can assume without loss of generality, as explained in Section 2.3, that $\mathbb{A} \colon H_1 \to H_0$ is a symmetric isometry. We abbreviate $\pi_\pm := \pi_\pm^{\mathbb{A}}$. Analogous to (4.37) it holds that

$$\|D_{\mathbb{A}}\xi\|^2_{P_0(I_T^-)} \stackrel{2}{=} \|\mathbb{A}\xi\|^2_{P_0(I_T^-)} + \|\partial_s\xi\|^2_{P_0(I_T^-)} + \langle \xi_0, \mathbb{A}\xi_0\rangle_0 - \langle \xi_{-T}, \mathbb{A}\xi_{-T}\rangle_0$$

for every $\xi \in P_1(I_T^-)$ and that

$$\|D_{\mathbb{A}}\eta\|^2_{P_0(I_T^+)} = \|\mathbb{A}\eta\|^2_{P_0(I_T^+)} + \|\partial_s\eta\|^2_{P_0(I_T^+)} + \langle\eta_T, \mathbb{A}\eta_T\rangle_0 - \langle\eta_0, \mathbb{A}\eta_0\rangle_0$$

for any $\eta \in P_1(I_T^+)$. The opposite signs are crucial. As $\mathbb{A}$ is an isometry we get

$$\begin{aligned}
\langle\xi(0), \mathbb{A}\xi(0)\rangle_0 &= \langle\xi_+(0), \mathbb{A}\xi_+(0)\rangle_0 + \langle\xi_-(0), \mathbb{A}\xi_-(0)\rangle_0 = \|\xi_+(0)\|^2_{\frac{1}{2}} - \|\xi_-(0)\|^2_{\frac{1}{2}},\\
\langle\eta(0), \mathbb{A}\eta(0)\rangle_0 &= \langle\eta_+(0), \mathbb{A}\eta_+(0)\rangle_0 + \langle\eta_-(0), \mathbb{A}\eta_-(0)\rangle_0 = \|\eta_+(0)\|^2_{\frac{1}{2}} - \|\eta_-(0)\|^2_{\frac{1}{2}}.
\end{aligned}$$

Taking the difference and using the relations in $\mathfrak{P}_r$ we obtain

$$\begin{aligned}
&\langle\xi(0), \mathbb{A}\xi(0)\rangle_0 - \langle\eta(0), \mathbb{A}\eta(0)\rangle_0\\
&= \|\xi_+(0)\|^2_{\frac{1}{2}} - \|\xi_-(0)\|^2_{\frac{1}{2}} - \|\eta_+(0)\|^2_{\frac{1}{2}} + \|\eta_-(0)\|^2_{\frac{1}{2}}\\
&= \|\xi_+(0)\|^2_{\frac{1}{2}} - r^2\|\eta_-(0)\|^2_{\frac{1}{2}} - r^2\|\xi_+(0)\|^2_{\frac{1}{2}} + \|\eta_-(0)\|^2_{\frac{1}{2}}\\
&= (1-r^2)\left(\|\xi_+(0)\|^2_{\frac{1}{2}} + \|\eta_-(0)\|^2_{\frac{1}{2}}\right)\\
&\geq 0
\end{aligned} \tag{4.55}$$

where we used that $r \in [0, 1]$. By definition (1.5) of the P_1 norm we have

$$\begin{aligned}
&\|\xi\|^2_{P_1(I_T^-)} + \|\eta\|^2_{P_1(I_T^+)}\\
&= \|\mathbb{A}\xi\|^2_{P_0(I_T^-)} + \|\partial_s\xi\|^2_{P_0(I_T^-)} + \|\mathbb{A}\eta\|^2_{P_0(I_T^+)} + \|\partial_s\eta\|^2_{P_0(I_T^+)}\\
&\overset{2}{=} \|D_{\mathbb{A}}\xi\|^2_{P_0(I_T^-)} + \underline{\langle\xi_{-T}, \mathbb{A}\xi_{-T}\rangle_0} - \langle\xi_0, \mathbb{A}\xi_0\rangle_0\\
&\quad + \|D_{\mathbb{A}}\eta\|^2_{P_0(I_T^+)} + \langle\eta_0, \mathbb{A}\eta_0\rangle_0 - \underline{\langle\eta_T, \mathbb{A}\eta_T\rangle_0}\\
&\overset{3}{\leq} \|D_{\mathbb{A}}\xi\|^2_{P_0(I_T^-)} + \|\xi_+(-T)\|^2_{H_{\frac{1}{2}}} + \|D_{\mathbb{A}}\eta\|^2_{P_0(I_T^+)} + \|\eta_+(-T)\|^2_{H_{\frac{1}{2}}}.
\end{aligned}$$

Equality 2 uses the two displayed identities after (4.54). In inequality 3 we used (4.55) to drop the terms at time $s = 0$ and we used the estimates (4.38) on the underlined terms. This proves (4.54) and Proposition 4.17. □

4.2.6 Index and spectral content

Definition 4.18. Given $A \in \mathcal{A}_{I_T}$, pick non-eigenvalues $\lambda_{\pm T} \in \mathcal{R}(A(\pm T))$, set

$$\boxed{\bar\lambda := (\lambda_{-T}, \lambda_T)}, \quad \mathbb{A}_{-T}^{\lambda_{-T}} := A(-T) - \lambda_{-T}\iota, \quad \mathbb{A}_T^{\lambda_T} := A(T) - \lambda_T\iota,$$

then $\mathbb{A}_{-T}^{\lambda_{-T}}$ and $\mathbb{A}_T^{\lambda_T}$ lie in $\mathcal{L}^*_{symo_0}(H_1, H_0)$. Define moreover $\mathfrak{D}_A^{\bar\lambda} = \mathfrak{D}_A^{\lambda_{-T}, \lambda_T}$ by

$$\begin{aligned}
\mathfrak{D}_A^{\bar\lambda} \colon P_1(I_T) &\to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}_{-T}^{\lambda_{-T}}) \times H^-_{\frac{1}{2}}(\mathbb{A}_T^{\lambda_T}) =: \mathcal{W}(I_T; \mathbb{A}_{-T}^{\lambda_{-T}}, \mathbb{A}_T^{\lambda_T})\\
\xi &\mapsto \left(D_A\xi\,,\, \pi_+^{\mathbb{A}_{-T}^{\lambda_{-T}}}\xi(-T)\,,\, \pi_-^{\mathbb{A}_T^{\lambda_T}}\xi(T)\right).
\end{aligned} \tag{4.56}$$

Lemma 4.19 (Difference of $\mathfrak{D}_A^{\bar\lambda}$ and Fredholm operator $\mathfrak{D}_{\bar A}$ is compact)**.**
a) If $A \in \mathcal{A}^*_{I_T}$, then $\mathfrak{D}_A^{0,0}$ is the Fredholm operator $\mathfrak{D}_A$ in Corollary 4.14.
b) If $A \in \mathcal{A}_{I_T}$ and $\lambda_{\mp T}$ are non-eigenvalues of $A_{\mp T}$, define a path $\bar A \in \mathcal{A}^*_{I_T}$ by

$$\bar A(s) := A(s) - h(s)\iota, \qquad h(s) := \left(1 - \tfrac{s+T}{2T}\right)\lambda_{-T} + \tfrac{s+T}{2T}\lambda_T, \quad s \in I_T.$$

Then $\mathfrak{D}_A^{\bar\lambda}$ and the Fredholm operator $\mathfrak{D}_{\bar A}$, see Corollary 4.14, differ

$$\mathfrak{D}_A^{\bar\lambda} - \mathfrak{D}_{\bar A} = h\iota \oplus 0 \oplus 0 \tag{4.57}$$

by a compact operator. Thus $\mathfrak{D}_A^{\bar\lambda}$ is Fredholm and of the same index.

Proof. a) Since $A \in \mathcal{A}^*_{I_T}$, zero is a non-eigenvalue of $A_{\mp T}$ and so $\mathfrak{D}_A^{0,0}$ is defined and, since there is no shift at the ends, it equals $\mathfrak{D}_A$.
b) Since the inclusion $\iota\colon H_1 \to H_0$ is compact, the operator denoted again by

$$\iota\colon P_1(I_T) \to P_0(I_T), \quad \xi \mapsto \iota\xi = [s \mapsto \iota\xi(s)]$$

is compact as well by [RS95, Le. 3.8]. Since $\|h\|_\infty = |\lambda_T - \lambda_{-T}|$ is finite, the product $h\iota$ is still a compact operator. The identity (4.57) is easy to see. □

Corollary 4.20 (Path and shift concatenation)**.** *Consider real numbers $a < b < c$ and a continuous operator path $A\colon [a,c] \to \mathcal{F}(H_1, H_0)$. Let λ_a, λ_b, λ_c be non-eigenvalues of the operators A_a, A_b, A_c, respectively. Then the Fredholm index is additive under concatenation*

$$\operatorname{index} \mathfrak{D}^{\lambda_a,\lambda_c}_{A|_{[a,c]}} = \operatorname{index} \mathfrak{D}^{\lambda_a,\lambda_b}_{A|_{[a,b]}} + \operatorname{index} \mathfrak{D}^{\lambda_b,\lambda_c}_{A|_{[b,c]}}.$$

Proof. Theorem 4.15 and Lemma 4.19. □

To compute the index difference of the Fredholm operator $\mathfrak{D}_A^{\bar\lambda}$ for different pairs $\bar\lambda$ we introduce the spectral content of an individual operator, not a path.

Definition 4.21 (Spectral content)**.** Let $A \in \mathcal{L}(H_1, H_0)$ be a symmetrizable Fredholm operator of index zero, in symbols $A \in \mathcal{F}(H_1, H_0)$. For $a \in \operatorname{spec} A$ we denote by $E_a := \ker A - a\iota$ the eigenspace of A to the eigenvalue a. Pick non-eigenvalues $\lambda \le \mu$ of $\mathcal{R}(A)$. We define the **eigenspace interval**

$$E_{(\lambda,\mu)} := \bigoplus_{\substack{a \in \operatorname{spec} A \\ a \in (\lambda,\mu)}} E_a.$$

The resulting decomposition defines projections along $E_{(\lambda,\mu)}$, notation

$$\pi^A_{(\lambda,\mu)}\colon \underbrace{H^+_{\frac12}(\mathbb{A}^\lambda)}_{H^{>\lambda}_{\frac12}(A)} = E_{(\lambda,\mu)} \oplus \underbrace{H^+_{\frac12}(\mathbb{A}^\mu)}_{H^{>\mu}_{\frac12}(A)} \to H^+_{\frac12}(\mathbb{A}^\mu) \tag{4.58}$$

and

$$\pi^A_{(\mu,\lambda)}\colon \underbrace{H^-_{\frac12}(\mathbb{A}^\mu)}_{H^{<\mu}_{\frac12}(A)} = \underbrace{H^-_{\frac12}(\mathbb{A}^\lambda)}_{H^{<\lambda}_{\frac12}(A)} \oplus E_{(\lambda,\mu)} \to H^-_{\frac12}(\mathbb{A}^\lambda). \tag{4.59}$$

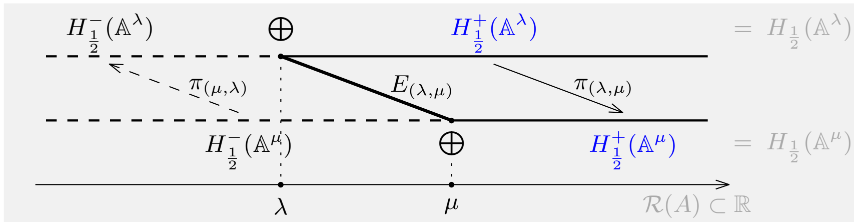

Figure 5: Projections associated to non-eigenvalues $\lambda \le \mu$

We define the **spectral content** of A between the two non-eigenvalues $\lambda \le \mu$ as the number of eigenvalues in between, with multiplicities, in symbols

$$\rho_A(\lambda,\mu) := \sum_{\substack{a\in \operatorname{spec} A\\ a\in(\lambda,\mu)}} \dim\ker(A - a\iota) = \dim E_{(\lambda,\mu)} \in \mathbb{N}_0. \tag{4.60}$$

Note that $\rho_A(\lambda,\lambda) = 0$. Moreover, we define

$$\rho_A(\mu,\lambda) := -\rho_A(\lambda,\mu) \in -\mathbb{N}_0. \tag{4.61}$$

This concludes Definition 4.21.

Due to the same summand $E_{(\lambda,\mu)}$ in (4.58) and (4.59), and as illustrated by Figure 5, we have equality of co-dimensions

$$\begin{aligned}\operatorname{codim}\left(H^+_{\frac12}(\mathbb{A}^\mu) \text{ in } H^+_{\frac12}(\mathbb{A}^\lambda)\right) &= \rho_A(\lambda,\mu)\\ &= \operatorname{codim}\left(H^-_{\frac12}(\mathbb{A}^\lambda) \text{ in } H^-_{\frac12}(\mathbb{A}^\mu)\right).\end{aligned} \tag{4.62}$$

Lemma 4.22 (Index difference is difference of spectral contents of overlaps)**.** *Given $A \in \mathcal{A}_{I_T}$, pick non-eigenvalues λ_{-T}, μ_{-T} of $A(-T)$ and non-eigenvalues λ_T, μ_T of $A(T)$. Set $\bar\lambda := (\lambda_{-T}, \lambda_T)$ and $\bar\mu := (\mu_{-T}, \mu_T)$. Then*

$$\operatorname{index} \mathfrak{D}^{\bar\mu}_A - \operatorname{index} \mathfrak{D}^{\bar\lambda}_A = \rho_{A_{-T}}(\lambda_{-T}, \mu_{-T}) - \rho_{A_T}(\lambda_T, \mu_T)$$

where ρ is the spectral content defined by (4.60).

Proof. In the proof we distinguish four cases.

Case 1. $\lambda_{-T} \le \mu_{-T}$ and $\lambda_T \ge \mu_T$

Proof. Recall the projections defined by (4.58–4.59). Since $\lambda_{-T} \le \mu_{-T}$ we have

$$\pi^{A_{-T}}_{(\lambda_{-T},\mu_{-T})}\colon H^+_{\frac12}(\mathbb{A}^{\lambda_{-T}}_{-T}) = E_{(\lambda_{-T},\mu_{-T})} \oplus H^+_{\frac12}(\mathbb{A}^{\mu_{-T}}_{-T}) \to H^+_{\frac12}(\mathbb{A}^{\mu_{-T}}_{-T}).$$

Since $\lambda_T \ge \mu_T$ we have

$$\pi^{A_T}_{(\lambda_T,\mu_T)}\colon H^-_{\frac12}(\mathbb{A}^{\lambda_T}_T) = H^-_{\frac12}(\mathbb{A}^{\mu_T}_T) \oplus E_{(\mu_T,\lambda_T)} \to H^-_{\frac12}(\mathbb{A}^{\mu_T}_T).$$

Consider the projection $p\colon Y \to Y$ onto its image $Z := \operatorname{im} p$ defined by

$$p = \left(1\!\!1\,,\, \pi^{A_{-T}}_{(\lambda_{-T},\mu_{-T})}\,,\, \pi^{A_T}_{(\lambda_T,\mu_T)}\right) :\ \underbrace{\mathcal{W}(I_T; \mathbb{A}^{\lambda_{-T}}_{-T}, \mathbb{A}^{\lambda_T}_{T})}_{=:Y} \to \underbrace{\mathcal{W}(I_T; \mathbb{A}^{\mu_{-T}}_{-T}, \mathbb{A}^{\mu_T}_{T})}_{=:Z\subset Y}.$$

Due to the inclusions of the second and third factors, there is the inclusion

$$\underbrace{P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}^{\mu_{-T}}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}^{\mu_T}_{T})}_{=Z} \subset \underbrace{P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}^{\lambda_{-T}}_{-T}) \times H^-_{\frac{1}{2}}(\mathbb{A}^{\lambda_T}_{T})}_{=Y}.$$

For the second factors the inclusion can be seen directly in Figure 5, for the third factors interchange λ and μ in the figure. From the inclusions of the second and third factors we also see that the image of p is Z and that the codimension of Z in Y is the sum of the co-dimensions of the second and third factors. Since

$$\mathfrak{D}^{\bar\mu}_A = p \circ \mathfrak{D}^{\bar\lambda}_A \colon P_1(I_T) \to Y \to Z \subset Y$$

we get, by Theorem E.4, identity 1 in

$$\begin{aligned}
\operatorname{index} \mathfrak{D}^{\bar\mu}_A - \operatorname{index} \mathfrak{D}^{\bar\lambda}_A &\overset{1}{=} \operatorname{codim}\, (Z \text{ in } Y) \\
&\overset{2}{=} \operatorname{codim}\left(H^+_{\frac{1}{2}}(\mathbb{A}^{\mu_{-T}}_{-T}) \text{ in } H^+_{\frac{1}{2}}(\mathbb{A}^{\lambda_{-T}}_{-T})\right) \\
&\quad + \operatorname{codim}\left(H^-_{\frac{1}{2}}(\mathbb{A}^{\mu_T}_{T}) \text{ in } H^-_{\frac{1}{2}}(\mathbb{A}^{\lambda_T}_{T})\right) \\
&\overset{3}{=} \rho_{A_{-T}}(\lambda_{-T}, \mu_{-T}) + \rho_{A_T}(\mu_T, \lambda_T) \\
&= \rho_{A_{-T}}(\lambda_{-T}, \mu_{-T}) - \rho_{A_T}(\lambda_T, \mu_T).
\end{aligned}$$

Identity 2 was explained above. Identity 3 is (4.62). Then we used (4.61). □

Case 2. $\lambda_{-T} \geq \mu_{-T}$ and $\lambda_T \leq \mu_T$

Proof. Interchanging the roles of $\bar\lambda$ and $\bar\mu$ in Case 1 we obtain

$$\operatorname{index} \mathfrak{D}^{\bar\lambda}_A - \operatorname{index} \mathfrak{D}^{\bar\mu}_A = \rho_{A_{-T}}(\mu_{-T}, \lambda_{-T}) - \rho_{A_T}(\mu_T, \lambda_T).$$

Taking the negative of both sides we obtain

$$\begin{aligned}
\operatorname{index} \mathfrak{D}^{\bar\mu}_A - \operatorname{index} \mathfrak{D}^{\bar\lambda}_A &= -\rho_{A_{-T}}(\mu_{-T}, \lambda_{-T}) + \rho_{A_T}(\mu_T, \lambda_T) \\
&= \rho_{A_{-T}}(\lambda_{-T}, \mu_{-T}) - \rho_{A_T}(\lambda_T, \mu_T)
\end{aligned}$$

where the second identity is by (4.61). □

Case 3. $\lambda_{-T} \leq \mu_{-T}$ and $\lambda_T \leq \mu_T$

Proof. Add zero to obtain

$$\begin{aligned}
&\operatorname{index}\mathfrak{D}_A^{\bar\mu}-\operatorname{index}\mathfrak{D}_A^{\bar\lambda}\\
&=\operatorname{index}\mathfrak{D}_A^{\mu_{-T},\mu_T}-\operatorname{index}\mathfrak{D}_A^{\lambda_{-T},\mu_T}+\operatorname{index}\mathfrak{D}_A^{\lambda_{-T},\mu_T}-\operatorname{index}\mathfrak{D}_A^{\lambda_{-T},\lambda_T}\\
&\stackrel{2}{=}\rho_{A_{-T}}(\lambda_{-T},\mu_{-T})-\rho_{A_T}(\mu_T,\mu_T)+\rho_{A_{-T}}(\lambda_{-T},\lambda_{-T})-\rho_{A_T}(\lambda_T,\mu_T)\\
&=\rho_{A_{-T}}(\lambda_{-T},\mu_{-T})-\rho_{A_T}(\lambda_T,\mu_T)
\end{aligned}$$

where in identity 2 we used Case 1 for the first difference and Case 2 for the second one. □

Case 4. $\lambda_{-T}\ge\mu_{-T}$ and $\lambda_T\ge\mu_T$

Proof. This follows by literally the same computation as in Case 3. What differs is the explanation: now in identity 2 we use Case 2 for the first difference and Case 1 for the second one. □

This proves Lemma 4.22. □

4.2.7 Theorem A – Index is spectral flow

In order to prove the assertion $\operatorname{index}\mathfrak{D}_A=\varsigma(A)$ of Theorem A, in (4.65) we use

Theorem 4.23 ([FW24, Thm. D])**.** *For a Hilbert space pair (H_0,H_1) the maps*

$$\pi_\pm\colon\mathcal{L}^*_{sym_0}(H_1,H_0)\to\mathcal{L}(H_{\frac12}),\quad \mathbb{A}\mapsto\pi_\pm^{\mathbb{A}}\tag{4.63}$$

are continuous.[6]

Proof of Theorem A – Index formula.
Let $A\in\mathcal{A}^*_{I_T}:=\{A\in\mathcal{A}_{I_T}\mid A(-T)\text{ and }A(T)\text{ are invertible}\}$. We write $\mathbb{A}(-T)$ and $\mathbb{A}(T)$ to indicate invertibility.

The proof takes three steps. Step 1 is the case that the whole operator path consists of invertible operators, in particular the spectral flow along the path is zero. Step 2 prepares for concatenation which is used in Step 3.

Step 1. Theorem A holds if every operator $\mathbb{A}(s)$ in the path $\mathbb{A}$ is invertible.

Proof. Homotop to constant invertible $\mathbb{A}(0)$. Consider the homotopy of paths $\mathbb{A}^r(s):=\mathbb{A}(rs)$ for $r\in[0,1]$. Then the initial path $\mathbb{A}^0\equiv\mathbb{A}(0)$ is constant and invertible and the end path $\mathbb{A}^1=\mathbb{A}$ is the given path. We claim the identity

$$\operatorname{index}\mathfrak{D}_{\mathbb{A}}=\operatorname{index}\mathfrak{D}_{\mathbb{A}(0)}.\tag{4.64}$$

The proof uses Theorem E.1. To homotopy member $\mathbb{A}^r$ we assign the operator

$$\begin{aligned}
\mathcal{D}^r\colon P_1(I_T)&\to P_0(I_T)\times H_{\frac12}\times H_{\frac12}\\
\xi&\mapsto(D_{\mathbb{A}^r}\xi,\xi(-T),\xi(T))
\end{aligned}$$

[6] $\mathcal{L}^*_{sym_0}(H_1,H_0)$ consists of the invertible $A\in\mathcal{L}(H_1,H_0)$ which are H_0-symmetric (1.1).

and the projection

$$p^r = \left(\mathbb{1}, \pi_+^{\mathbb{A}^r(-T)}, \pi_-^{\mathbb{A}^r(T)}\right) : P_0(I_T) \times H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to P_0(I_T) \times H_{\frac{1}{2}} \times H_{\frac{1}{2}}.$$

Observe that $\mathbb{A}(\pm rT) = \mathbb{A}^r(\pm T)$. Composing both operators we get

$$\mathfrak{D}_{A^r} := p^r \circ \mathcal{D}^r \colon P_1(I_T) \to P_0(I_T) \times H^+_{\frac{1}{2}}(\mathbb{A}(-rT)) \times H^-_{\frac{1}{2}}(\mathbb{A}(rT)).$$
$$\xi \mapsto \left(D_{\mathbb{A}^r}\xi, \pi_+^{\mathbb{A}(-rT)}\xi(-T), \pi_-^{\mathbb{A}(rT)}\xi(T)\right)$$

The projections p^r depend continuously on r in view of Theorem 4.23. Therefore Theorem E.1 implies that

$$\operatorname{index} \mathfrak{D}_{\mathbb{A}^0} = \operatorname{index} \mathfrak{D}_{\mathbb{A}^1}. \tag{4.65}$$

Since $\mathfrak{D}_{\mathbb{A}^0} = \mathfrak{D}_{\mathbb{A}(0)}$ and $\mathfrak{D}_{\mathbb{A}^1} = \mathfrak{D}_{\mathbb{A}}$ this proves the claimed identity (4.64).

By (4.35) in Step 1 of the proof of Theorem 4.7, the operator $\mathfrak{D}_{\mathbb{A}(0)}$ is an isomorphism and therefore a Fredholm operator of index zero. Hence, in view of (4.64), we have $\operatorname{index} \mathfrak{D}_{\mathbb{A}} = 0$. Since $\mathbb{A}(s)$ is invertible for every s, the spectral flow $\varsigma(\mathbb{A})$ is zero. This proves Theorem A in case of a family of invertible operators along a finite interval I_T. This proves Step 1. □

Step 2. Let $A \in \mathcal{A}^*_{I_T}$. There exists an integer $N \in \mathbb{N}_0$ and real numbers

$$-T = t_0 < t_1 < \dots < t_N < t_{N+1} = T, \qquad 0 = \lambda_0, \lambda_1, \dots, \lambda_{N-1}, \lambda_N$$

such that

$$\boxed{\mathbb{A}_j(s) := A(s) - \lambda_j \iota \colon H_1 \to H_0, \quad s \in [t_j, t_{j+1}],} \tag{4.66}$$

is invertible for any $s \in [t_j, t_{j+1}]$ whenever $j \in \{0, \dots, N\}$.

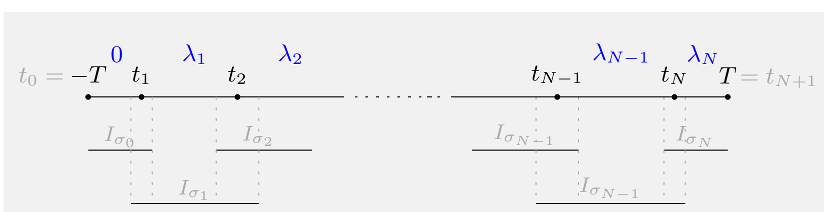

Figure 6: Step 2: Invertibility shifts λ_i and intervals $[t_j, t_{j+1}]$

Proof. For each $\sigma \in [-T, T]$ choose $\mu_\sigma \in \mathbb{R} \backslash \operatorname{spec} A(\sigma)$. So $A(\sigma) - \mu_\sigma \iota \colon H_1 \to H_0$ is invertible. Since invertibility is an open condition, there exists $\varepsilon_\sigma > 0$ such that $A(\tau) - \mu_\sigma \iota$ is invertible for every

$$\tau \in I_\sigma := (\sigma - \varepsilon_\sigma, \sigma + \varepsilon_\sigma) \cap [-T, T].$$

Since $\mathbb{A}(-T)$ is invertible we choose

$$\mu_{-T} = 0.$$

Since $[-T, T]$ is compact, there exists a finite subset $\mathfrak{S}$ of $[-T, T]$ such that the corresponding open intervals still cover $[-T, T]$, in symbols

$$\bigcup_{\sigma \in \mathfrak{S}} I_\sigma = [-T, T].], \qquad \mathcal{I} := \{I_\sigma \mid \sigma \in \mathfrak{S}\}.$$

We can assume without loss of generality that $-T, T \in \mathfrak{S}$, otherwise just add two intervals.

Out of this finite covering we construct recursively a further sub-covering $\mathcal{I} := \{I_{\sigma_0}, I_{\sigma_1}, \dots, I_{\sigma_N}\}$ beginning at $\sigma_0 := -T$ and such that exactly nearest neighbors overlap. If $T \in I_{\sigma_j}$, we set $N := j$ and we are done. If $T \notin I_{\sigma_j}$, then we choose $\sigma_{j+1} \in \mathfrak{S}$ satisfying the two conditions

1. $I_{\sigma_{j+1}} \cap I_{\sigma_j} \neq \emptyset$ — intersects predecessor j
2. $\sigma_{j+1} + \varepsilon_{\sigma_{j+1}} \geq \sigma + \varepsilon_\sigma$, $\forall \sigma \in \mathfrak{S} \colon I_\sigma \cap I_{\sigma_j} \neq \emptyset$ — farthest right among intersectors

Condition 1 means that the chosen interval $I_{\sigma_{j+1}}$ intersects its predecessor. Condition 2 means that the chosen interval $I_{\sigma_{j+1}}$ reaches farthest to the right among all intersectors. Furthermore, there are the following consequences

(i) $\sigma_{j+1} + \varepsilon_{\sigma_{j+1}} > \sigma_j + \varepsilon_{\sigma_j}$; — successor $j+1$ extends further right

(ii) If $I_{\sigma_i} \cap I_{\sigma_j} \neq \emptyset$ where $i, j \in \{1, \dots, N\}$, then $|i - j| \leq 1$. — only next neighbors can intersect

We prove (i) and (ii). (i) Since $T \notin I_{\sigma_j}$ it follows that $\sigma_j + \varepsilon_{\sigma_j} \leq T$. Therefore there exists $\sigma \in \mathfrak{S}$ such that $\sigma_j + \varepsilon_{\sigma_j} \in I_\sigma$. Since I_σ is open it follows that $I_\sigma \cap I_{\sigma_j} \neq \emptyset$ and $\sigma + \varepsilon_\sigma > \sigma_j + \varepsilon_{\sigma_j}$. Therefore, by condition 2, $\sigma_{i+1} + \varepsilon_{\sigma_{i+1}} \geq \sigma + \varepsilon_\sigma$ which is strictly larger than $\sigma_j + \varepsilon_{\sigma_j}$.

(ii) We assume by contradiction that there exists an interval I_{σ_i} intersecting I_{σ_j} where $0 \leq i < i + 2 \leq j \leq N$. Applying condition 2 for $j = i$ and using that $I_{\sigma_i} \cap I_{\sigma_j} \neq \emptyset$, we obtain that $\sigma_{i+1} + \varepsilon_{\sigma_{i+1}} \geq \sigma_j + \varepsilon_{\sigma_j}$. Now applying (i) we obtain that $\sigma_{i+2} + \varepsilon_{\sigma_{i+2}} > \sigma_{i+1} + \varepsilon_{\sigma_{i+1}}$ which as we saw is $\geq \sigma_j + \varepsilon_{\sigma_j}$. Using that $j \geq i + 2$ and using (i) again, we conclude that $\sigma_j + \varepsilon_{\sigma_j} \geq \sigma_{i+2} + \varepsilon_{\sigma_{i+2}}$ which as we saw is $> \sigma_j + \varepsilon_{\sigma_j}$. This contradiction proves (ii).

The family of intervals $\mathcal{I} := \{I_{\sigma_0}, I_{\sigma_1}, \dots, I_{\sigma_N}\}$ covers $[-T, T]$ and it has the property that exactly nearest neighbors overlap, see Figure 6. Set $t_0 := -T$ and $t_{N+1} := T$. For $i = 1, \dots, N$ choose $t_i \in I_{\sigma_{i-1}} \cap I_{\sigma_i}$ in the overlap interval. The finite set of real numbers is then defined by $\Lambda' := \{\lambda_i := \mu_{\sigma_i} \mid i = 0, \dots, N\}$. Note that $\lambda_0 := \mu_{-T} = 0$. This proves Step 2. $\square$

Step 3. We prove the index formula in Theorem A.

Proof. We continue the notation from Step 2. Choose a non-eigenvalue λ_{N+1} of $\mathbb{A}_T$. By Step 1, for $j = 0, \dots, N$, the index along each interval $[t_j, t_{j+1}]$ vanishes

$$\operatorname{index} \mathfrak{D}^{\lambda_j,\lambda_j}_{A|_{[t_j,t_{j+1}]}} \overset{1}{=} \operatorname{index} \mathfrak{D}_{A-\lambda_j\iota|_{[t_j,t_{j+1}]}} \overset{2}{=} \operatorname{index} \mathfrak{D}_{\mathbb{A}_j} \overset{3}{=} 0.$$

More precisely, identity 1 is by Lemma 4.19 b) which tells that both Fredholm indices are equal, identity 2 is (4.66), and identity 3 is by Step 1.

Furthermore, Lemma 4.22 and anti-symmetry (4.61) of ρ assert that

$$\begin{aligned}\operatorname{index} \mathfrak{D}^{\lambda_j,\lambda_{j+1}}_{A|_{[t_j,t_{j+1}]}} &= -\rho_{A(t_{j+1})}(\lambda_j, \lambda_{j+1}) + \rho_{A(t_j)}(\lambda_j, \lambda_j) \\ &= \rho_{A(t_{j+1})}(\lambda_{j+1}, \lambda_j).\end{aligned}$$

By path and shift concatenation, Corollary 4.20, and the previous identity we get

$$\operatorname{index} \mathfrak{D}^{\lambda_0,\lambda_{N+1}}_{A} = \sum_{j=0}^{N} \operatorname{index} \mathfrak{D}^{\lambda_j,\lambda_{j+1}}_{A|_{[t_j,t_{j+1}]}} = \sum_{j=0}^{N} \rho_{A(t_{j+1})}(\lambda_{j+1}, \lambda_j). \tag{4.67}$$

Given a time $s \in [-T, T]$ and a non-eigenvalue $\mu \in \mathcal{R}(A(s))$, we define

$$\nu_{\uparrow}(s; \mu) := \max\{\ell \in \Lambda \cup \{0\} \mid a_\ell(s) \le \mu\}$$

to be the largest eigenvalue number among all eigenvalues of $A(s)$ below or equal to μ; for the enumeration see (i) after (3.18). Then

$$\begin{aligned}\nu_{\uparrow}(T; 0) &= -\varsigma(A) \quad \text{, cf. (3.19),} \\ \rho_{A(t_{j+1})}(\lambda_{j+1}, \lambda_j) &= \nu_{\uparrow}(t_{j+1}; \lambda_j) - \nu_{\uparrow}(t_{j+1}; \lambda_{j+1}).\end{aligned} \tag{4.68}$$

Since $\mathbb{A}_j(s) := A(s) - \lambda_j\iota$ is invertible for every $s \in [t_j, t_{j+1}]$, no eigenvalue of $A(s)$ crosses λ_j along $[t_j, t_{j+1}]$, and therefore

$$\nu_{\uparrow}(t_j; \lambda_j) = \nu_{\uparrow}(t_{j+1}; \lambda_j). \tag{4.69}$$

Identity 1 in the following calculation has been shown above

$$\begin{aligned}\operatorname{index} \mathfrak{D}^{\lambda_0,\lambda_{N+1}}_{A} &\overset{1}{=} \sum_{j=0}^{N} \rho_{A(t_{j+1})}(\lambda_{j+1}, \lambda_j) \\ &\overset{2}{=} \sum_{j=0}^{N} \big(\nu_{\uparrow}(t_{j+1}; \lambda_j) - \nu_{\uparrow}(t_{j+1}; \lambda_{j+1})\big) \\ &\overset{3}{=} \sum_{j=0}^{N} \big(\nu_{\uparrow}(t_j; \lambda_j) - \nu_{\uparrow}(t_{j+1}; \lambda_{j+1})\big) \\ &\overset{4}{=} \nu_{\uparrow}(t_0; \lambda_0) - \nu_{\uparrow}(t_{N+1}; \lambda_{N+1}) \\ &\overset{5}{=} \nu_{\uparrow}(-T; 0) - \nu_{\uparrow}(T; \lambda_{N+1}) \\ &\overset{6}{=} -\nu_{\uparrow}(T; \lambda_{N+1})\end{aligned} \tag{4.70}$$

Identity 2 is by (4.68) and identity 3 by (4.69). In identity 4 all terms cancel pairwise except the first and the last one. Identity 5 holds by the choices in Step 2 and identity 6 since $\nu_\uparrow(s;0)=0$.

CASE 1. $\lambda_{N+1}=0$

In this case $\mathfrak{D}_A^{0,0}=\mathfrak{D}_A$ and $-\nu_\uparrow(T;\lambda_{N+1})=-\nu_\uparrow(T;0)=\varsigma(A)$. Hence (4.70) tells that $\operatorname{index}\mathfrak{D}_A=\varsigma(A)$ and we are done.

CASE 2. $\lambda_{N+1}\neq 0$

By Lemma 4.22 we obtain identity 2 in the following calculation

$$\begin{aligned}
\operatorname{index}\mathfrak{D}_A^{\lambda_0,\lambda_{N+1}}-\operatorname{index}\mathfrak{D}_A &= \operatorname{index}\mathfrak{D}_A^{0,\lambda_{N+1}}-\operatorname{index}\mathfrak{D}_A^{0,0}\\
&\stackrel{2}{=}\rho_{\mathbb{A}(-T)}(0,0)-\rho_{\mathbb{A}(T)}(0,\lambda_{N+1})\\
&\stackrel{3}{=}0-\nu_\uparrow(T;\lambda_{N+1})+\nu_\uparrow(T;0)\\
&\stackrel{4}{=}-\nu_\uparrow(T;\lambda_{N+1})-\varsigma(A)\\
&\stackrel{5}{=}\operatorname{index}\mathfrak{D}_A^{\lambda_0,\lambda_{N+1}}-\varsigma(A).
\end{aligned} \tag{4.71}$$

Identities 3 and 4 hold by (4.68), identity 5 by (4.70). This proves Step 3, hence Theorem A. □

The proof of Theorem A is complete. □

4.3 Half infinite forward interval

Pick a Hessian path $A\in\mathcal{A}^*_{I_+}$ along the half infinite forward interval $I_+=[0,\infty)$; see Definition 1.5. Then $A\colon[0,\infty)\to\mathcal{F}=\mathcal{F}(H_1,H_0)$ takes values in the symmetrizable Fredholm operators of index zero; cf. Remarks 1.4 and 2.3. In order to eventually get to Fredholm operators, it is not enough that the Hessian at zero and the limit at infinity are invertible, notation

$$\mathbb{A}_0:=A(0),\qquad \mathbb{A}^+:=\lim_{s\to\infty}A(s).$$

In addition, one must impose a boundary condition at zero formulated in terms of the spectral projection $\pi_+^{\mathbb{A}_0}$ sitting at time zero; see (2.14).

4.3.1 Estimate for D_A

Let $A\in\mathcal{A}^*_{I_+}$. The Hilbert spaces $P_0(\mathbb{R}_+)$ and $P_1(\mathbb{R}_+)$ are defined by (1.4) for $I=\mathbb{R}_+$. In this section we study the linear operator ∂_s+A as a map

$$D_A\colon P_1(\mathbb{R}_+)\to P_0(\mathbb{R}_+),\quad \xi\mapsto\partial_s\xi+A(s)\xi. \tag{4.72}$$

As in the case of the finite interval, Section 4.2.1, this operator is *not* Fredholm: although it has closed image and finite dimensional co-kernel, the kernel is infinite dimensional in the Floer and Morse case; see Figure 7.

Theorem 4.24. *Given $A \in \mathcal{A}^*_{I_+}$, there exist constants $T, c > 0$ such that*

$$\|\xi\|_{P_1(\mathbb{R}_+)} \leq c\left(\|\xi\|_{P_0([0,T])} + \|D_A\xi\|_{P_0(\mathbb{R}_+)} + \|\pi_+^{\mathbb{A}_0}\xi(0)\|_{\frac{1}{2}}\right)$$

for every $\xi \in P_1(\mathbb{R}_+)$.

This estimate becomes a semi-Fredholm estimate for D_A restricted to those $\xi \in P_1(\mathbb{R}_+)$ with $\pi_+^{\mathbb{A}_0}\xi(0) = 0$ or even $\xi(0) = 0$. We study this in Section 4.3.3.

Proof of Theorem 4.24. We prove the theorem in four steps. It is sometimes convenient to abbreviate $A_s := A(s)$. We enumerate the constants by the step where they appear, e.g. constant C_1 arises in Step 1.

Step 1 (Asymptotic estimate)**.** There exist constants $T_1, C_1 > 0$ such that the following is true. Suppose $\beta \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfies $\operatorname{supp}\beta \subset (T_1, \infty)$. Then

$$\|\beta\xi\|_{P_1(\mathbb{R}_+)} \leq C_1\left(\|\beta D_A\xi\|_{P_0(\mathbb{R}_+)} + \|\beta'\xi\|_{P_0(\mathbb{R}_+)}\right)$$

for every $\xi \in P_1(\mathbb{R}_+)$.

Proof. Step 3 in the proof of the Rabier Theorem 4.2. □

Step 2 (Small interval at left boundary)**.** There are constants $\varepsilon_2 > 0$ and $C_2 > 0$ such that for every compactly supported $\beta \in C^\infty(\mathbb{R}_+, \mathbb{R})$ with the property

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \leq \varepsilon_2$$

it holds that

$$\begin{aligned}&\|\beta\xi\|_{P_1(\mathbb{R}_+)}\\ &\leq C_2\left(\|\beta D_A\xi\|_{P_0(\mathbb{R}_+)} + \|\beta'\xi\|_{P_0(\mathbb{R}_+)} + \|\beta\xi\|_{P_0(\mathbb{R}_+)} + \|\pi_+^{\mathbb{A}_0}\beta(0)\xi(0)\|_{\frac{1}{2}}\right)\end{aligned}$$

for every $\xi \in P_1(\mathbb{R}_+)$.

Proof. Step 4 in the proof of the finite interval Theorem 4.7. □

Step 3 (Small interior interval)**.** There is a finite subset $\Lambda' \subset \mathbb{R}$ and constants $\varepsilon_3, C_3 > 0$ such that for every $\beta \in C^\infty(\mathbb{R}_+, \mathbb{R})$ which has compact support in $(0, \infty)$ and has the property

$$\sup_{\sigma,\tau \in \operatorname{supp}\beta} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \leq \varepsilon_3$$

it holds that

$$\|\beta\xi\|_{P_1(\mathbb{R}_+)} \leq C_3\left(\|\beta D_A\xi\|_{P_0(\mathbb{R}_+)} + \|\beta'\xi\|_{P_0(\mathbb{R}_+)} + \|\beta\xi\|_{P_0(\mathbb{R}_+)}\right)$$

for every $\xi \in P_1(\mathbb{R}_+)$.

Proof. This is Step 6 in the proof of the Rabier Theorem 4.2 with $\frac{1}{C_3} = \varepsilon_3$. $\square$

Step 4 (Partition of unity)**.** We prove Theorem 4.24.

Proof. Set $\varepsilon := \min\{\varepsilon_2, \varepsilon_3\}$ and $C := \max\{C_1, C_2, C_3\}$. Choose $T > T_1$ and a finite partition of unity $\{\beta_j\}_{j=0}^{M+1}$ for $[0, \infty)$ with the properties that β_0 is compactly supported in $[0, T)$ and

$$\beta_0(0) = 1, \qquad \sup_{\sigma,\tau\in\operatorname{supp}\beta_0} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \varepsilon, \quad \operatorname{supp}\beta_{M+1} \subset (T_1, \infty),$$

and

$$\sup_{\sigma,\tau\in\operatorname{supp}\beta_j} \|A_\sigma - A_\tau\|_{\mathcal{L}(H_1,H_0)} \le \varepsilon, \qquad \operatorname{supp}\beta_j \subset (0, T),$$

for $j = 1, \dots, M$. That such a partition exists follows from the continuity of $s \mapsto A(s)$ and the fact that on the compact set $[0, T_1]$ continuity becomes uniform continuity. Let $\xi \in P_1(\mathbb{R}_+)$. Then by Steps 2,3,1 we have the estimates

$$\|\beta_0\xi\|_{P_1(\mathbb{R}_+)} \overset{2}{\le} C\left(\|\beta_0 D_A\xi\|_{P_0(\mathbb{R}_+)} + \|\beta_0'\xi\|_{P_0(\mathbb{R}_+)} + \|\beta_0\xi\|_{P_0(\mathbb{R}_+)} + \|\pi_+^{\mathbb{A}_0}\xi(0)\|_{\frac{1}{2}}\right)$$

and

$$\begin{aligned}
\|\beta_j\xi\|_{P_1(\mathbb{R}_+)} &\overset{3}{\le} C\left(\|\beta_j D_A\xi\|_{P_0(\mathbb{R}_+)} + \left\|\beta_j'\xi\right\|_{P_0(\mathbb{R}_+)} + \|\beta_j\xi\|_{P_0(\mathbb{R}_+)}\right) \\
\|\beta_{M+1}\xi\|_{P_1(\mathbb{R}_+)} &\overset{1}{\le} C\left(\|\beta_{M+1} D_A\xi\|_{P_0(\mathbb{R}_+)} + \left\|\beta_{M+1}'\xi\right\|_{P_0(\mathbb{R}_+)}\right)
\end{aligned}$$

for $j = 1, \dots, M$. We abbreviate $B := \max\{\|\beta_0'\|_\infty, \|\beta_1'\|_\infty, \dots, \|\beta_{M+1}'\|_\infty\}$. Putting these estimates together we obtain

$$\begin{aligned}
\|\xi\|_{P_1(\mathbb{R}_+)} &\le \sum_{j=0}^{M+1} \|\beta_j\xi\|_{P_1(\mathbb{R}_+)} \\
&\le C\sum_{j=0}^{M+1}\left(\|\beta_j D_A\xi\|_{P_0(\mathbb{R}_+)} + \left\|\beta_j'\xi\right\|_{P_0([0,T])}\right) \\
&\quad + C\sum_{j=0}^{M} \|\beta_j\xi\|_{P_0([0,T])} + C\|\pi_+^{\mathbb{A}_0}\xi(0)\|_{\frac{1}{2}} \\
&\le C(M+2)\,\|D_A\xi\|_{P_0(\mathbb{R}_+)} + C\left(B(M+2) + M + 1\right)\|\xi\|_{P_0([0,T])} \\
&\quad + C\|\pi_+^{\mathbb{A}_0}\xi(0)\|_{\frac{1}{2}}
\end{aligned}$$

where in the second inequality we replaced the $P_0(\mathbb{R}_+)$ norm by the $P_0([0, T])$ norm due to the supports of the β_j's and their derivatives.[7] Setting

$$c := \max\{C(M+2), C\left(B(M+2) + M + 1\right)\}$$

proves Step 4. $\square$

The proof of Theorem 4.24 is complete. $\square$

[7] Along $[T, \infty)$ we have $\beta_{M+1} \equiv 1$, so $\beta_{M+1}' \equiv 0$.

4.3.2 Estimate for the adjoint D_A^*

Let $A \in \mathcal{A}^*_{\mathbb{R}_+}$. We call the following operator the **adjoint of D_A**, namely

$$D_A^* := D_{-A^*}\colon P_1(\mathbb{R}_+; H_0^*, H_1^*) \to P_0(\mathbb{R}_+; H_1^*), \quad \eta \mapsto \partial_s \eta - A(s)^* \eta.$$

Corollary 4.25. *For $A \in \mathcal{A}^*_{\mathbb{R}_+}$ there exists a constant $c > 0$ such that*

$$\|\eta\|_{P_1(\mathbb{R}_+; H_0^*, H_1^*)} \le c\Big(\|\eta\|_{P_0(\mathbb{R}_+; H_1^*)} + \|D_A^*\eta\|_{P_0(\mathbb{R}_+; H_1^*)} + \|\pi_+^{-\mathbb{A}_0^*}\eta(0)\|_{\frac{1}{2}}\Big)$$

for every $\eta \in P_1(\mathbb{R}_+; H_0^*, H_1^*)$.

Proof. Theorem 4.24 and Lemma 2.7; see also Remark 1.4. □

4.3.3 Fredholm under boundary conditions: D_A^+

Given $A \in \mathcal{A}^*_{\mathbb{R}_+}$, let $\pi_\pm := \pi_\pm^{\mathbb{A}_0}$ be defined by (2.14). To get from Theorem 4.24 to semi-Fredholm we restrict the domain of the operator $D_A\colon P_1(\mathbb{R}_+) \to P_0(\mathbb{R}_+)$ by the boundary condition $\pi_+^{\mathbb{A}_0}\xi(0) = 0$ which cuts the operator kernel down to finite dimension and but still leads to a finite dimensional co-kernel. Hence $\operatorname{coker} D_A$ is finite dimensional, too.

To this end define a subspace of the Hilbert space $P_1(\mathbb{R}_+) = P_1(\mathbb{R}_+; H_1, H_0)$ from (1.4) as follows

$$P_1^+(\mathbb{R}_+, \mathbb{A}_0) = P_1^+(\mathbb{R}_+, \mathbb{A}_0; H_1, H_0) := \{\xi \in P_1(\mathbb{R}_+) \mid \pi_+^{\mathbb{A}_0}\xi(0) = 0\}. \tag{4.73}$$

The restriction of the operator $D_A\colon P_1(\mathbb{R}_+) \to P_0(\mathbb{R}_+)$ in (4.72) we denote by

$$D_A^+\colon P_1^+(\mathbb{R}_+, \mathbb{A}_0) \to P_0(\mathbb{R}_+), \quad \xi \mapsto \partial_s \xi + A(s)\xi.$$

Remark 4.26 (Goal and idea of proof)**.** Our goal is to show that D_A has finite dimensional cokernel.

To achieve this goal we show that D_A^+ is a Fredholm operator whose co-kernel is isomorphic to $\ker D_{-A^*}^+$. The fact that D_A^+ has closed image is crucial to show that D_A itself has closed image (since it contains $\operatorname{im} D_A^+$).

For the proof that D_A^+ is a semi-Fredholm operator we need the full strength of the estimate in Theorem 4.24, in particular, that the third term on the right is just $\|\pi_+^{\mathbb{A}_0}\xi(0)\|_{H_{1/2}}$ and not $\|\xi(0)\|_{H_{1/2}}$.

Theorem 4.27 (Fredholm)**.** $D_A^+\colon P_1^+(\mathbb{R}_+, \mathbb{A}_0; H_1, H_0) \to P_0(\mathbb{R}_+; H_0)$ *is a Fredholm operator for any Hessian path $A \in \mathcal{A}^*_{\mathbb{R}_+}$.*

Corollary 4.28. *The operator $D_A\colon P_1(\mathbb{R}_+; H_1, H_0) \to P_0(\mathbb{R}_+; H_0)$ in (4.72) has closed image of finite co-dimension for any Hessian path $A \in \mathcal{A}^*_{\mathbb{R}_+}$.*

Proof. By Theorem 4.27 the image of D_A^+ is closed and of finite co-dimension. Since D_A^+ is a restriction of D_A we have inclusion $\operatorname{im} D_A^+ \subset \operatorname{im} D_A \subset P_0(\mathbb{R}_+)$. So $\operatorname{im} D_A$ is of finite co-dimension. Thus $\operatorname{im} D_A$ is closed by [Bre11, Prop. 11.5]. □

	$D_A\colon P_1 \to P_0$		$D_A^+\colon P_1^+ \to P_0$
dim ker	∞		$k < \infty$
dim coker	$\leq \ell$	$\Leftarrow$	$\ell < \infty$
	co-semi-Fredholm		Fredholm
image	closed	$\Leftarrow$	closed
coker			coker $D_A^+ \simeq$ ker $D_{-A^*}^+$
ker	huge		ker $D_A^+ \simeq$ coker $D_{-A^*}^+$

Figure 7: $D_A = \partial_s + A(s)$ on P_1 and its restriction D_A^+ to P_1^+

Proof of Theorem 4.27. Pick $A \in \mathcal{A}^*_{\mathbb{R}_+}$, then $\mathbb{A}_0 := A(0)$ is invertible. By Corollary 4.25 the operator D_A^+ (and also $D_{-A^*}^+$) has finite dimensional kernel and closed image. By the same reasoning as in the proof of Theorem 4.11 one shows that the co-kernel of D_A^+ can be identified with the kernel of $D_{-A^*}^+$, in symbols

$$\operatorname{coker} D_A^+ \simeq \ker D_{-A^*}^+. \tag{4.74}$$

This proves Theorem 4.27. □

4.3.4 Theorem A – Fredholm property

Corollary 4.29 (to Theorem 4.24, Fredholm)**.** *For any $A \in \mathcal{A}^*_{\mathbb{R}_+}$ the operator*

$$\begin{aligned} \mathfrak{D}_A = \mathfrak{D}_A^{\mathbb{R}_+}\colon P_1(\mathbb{R}_+) &\to P_0(\mathbb{R}_+) \times H^+_{\frac{1}{2}}(\mathbb{A}_0) =: \mathcal{W}(\mathbb{R}_+; \mathbb{A}_0) \\ \xi &\mapsto \left(D_A\xi, \pi_+^{\mathbb{A}_0}\xi_0\right) \end{aligned}$$

is Fredholm, where $\mathbb{A}_0 := A(0)$, and of the same index as D_A^+. More precisely, the kernels coincide and the co-kernels are of equal dimension.

Proof. By Theorem 4.24 the operator $\mathfrak{D}_A$ is semi-Fredholm. So it has finite dimensional kernel and closed image. That kernel and image of $\mathfrak{D}_A$ are equal, respectively isomorphic, to those of the Fredholm operator D_A^+ from Theorem 4.27 follows by the arguments in the proof of Corollary 4.14. □

4.3.5 Paths of invertibles

Also in the half infinite forward interval case, the proof of the index formula index $\mathfrak{D}_A = \varsigma(A)$ in Theorem A is based on Theorem 4.23 (main result in [FW24, Thm. D]) which enters the following preparation for the index formula.

Proposition 4.30 (Constant path)**.** *Let $\mathbb{A} \in \mathcal{A}^*_{\mathbb{R}_+}$ be a constant path, then the Fredholm operator $D_{\mathbb{A}}^+\colon P_1^+(\mathbb{R}_+, \mathbb{A}_0) \to P_0(\mathbb{R}_+)$ is an isomorphism and therefore its Fredholm index vanishes.*

Proof. The proof is in three steps. After replacing the inner products by $\mathbb{A}$-adaptable inner products, see Definition 2.8, we can assume without loss of generality that $\mathbb{A}\colon H_1 \to H_0$ is a symmetric isometry.

Step 1: $\ker D_{\mathbb{A}}^+ = \{0\}$**.** We first show that the kernel of $D_{\mathbb{A}}^+$ vanishes. For this purpose suppose that $\xi \in P_1^+(\mathbb{R}_+, \mathbb{A}_0)$ lies in the kernel of $D_{\mathbb{A}}^+$. Then ξ is a solution of the problem $\partial_s \xi(s) = -\mathbb{A}\xi(s)$ and $\pi_+\xi(0) = 0$.
Pick an orthonormal basis $\mathcal{V}(\mathbb{A}) = \{v_\ell\}_{\ell\in\Lambda}$ of H_0 consisting of eigenvectors $\mathbb{A}v_\ell = a_\ell v_\ell$. We write $\xi = \sum_{\ell\in\mathbb{Z}^*} \xi_\ell v_\ell$. Then each coefficient ξ_ℓ satisfies the ODE in one variable $\partial_s\xi_\ell(s) = -a_\ell\xi_\ell(s)$ whose solution is $\xi_\ell(s) = e^{-a_\ell s}\xi_\ell(0)$. Since $\pi_+\xi(0) = 0$ we have $\xi_\nu(0) = 0$ for every $\nu \in \mathbb{N}$. Therefore $\xi_\nu = 0$ for every $\nu \in \mathbb{N}$. Since $a_{-\nu} < 0$ is negative for $\nu \in \mathbb{N}$ we have that $\xi_{-\nu}(s) = e^{-a_{-\nu}s}\xi_{-\nu}(0)$ grows exponentially unless $\xi_{-\nu}(0) = 0$. Since $\xi \in P_1^+(\mathbb{R}_+, \mathbb{A}_0) \subset L^2(\mathbb{R}_+, H_1)$ negative modes $\xi_{-\nu}$ cannot grow exponentially which implies that $\xi_{-\nu} \equiv 0$ for every $\nu \in \mathbb{N}$. This shows that $\xi = 0$. So $\ker D_{\mathbb{A}}^+ = \{0\}$ is trivial.

Step 2: $\operatorname{coker} D_{\mathbb{A}}^+ = \{0\}$**.** But $\operatorname{coker} D_{\mathbb{A}}^+ \simeq \ker D_{-\mathbb{A}^*}^+$, by (4.74), and the latter is zero by Step 1.

Step 1 and Step 2 show that $D_{\mathbb{A}}^+$ is bijective and hence, by the open mapping theorem, an isomorphism. This proves Proposition 4.30. □

Corollary 4.31. *Assume that $\mathbb{A} \in \mathcal{A}_{\mathbb{R}_+}^*$ has the property that $\mathbb{A}(s)$ is invertible for every $s \in \mathbb{R}_+$. Then the Fredholm index* $\operatorname{index} \mathfrak{D}_{\mathbb{A}} = 0$ *vanishes.*

Proof. For constant paths this is true by Proposition 4.30. The family of paths $\{\mathbb{A}_r\}_{r\in[0,1]} \subset \mathcal{A}_{\mathbb{R}_+}^*$ defined by

$$\mathbb{A}_r(s) := \mathbb{A}(s + \varphi(r)), \qquad \varphi\colon [0,1] \to \mathbb{R}_+ \cup \{\infty\}, \quad r \mapsto \tfrac{r^2}{1-r^2}, \tag{4.75}$$

provides a homotopy between $\mathbb{A}$ and the constant path $\mathbb{A}^+$ at infinity. Therefore, since by Theorem E.1 the Fredholm index is invariant under homotopies

$$r \mapsto \mathfrak{D}_{\mathbb{A}_r} = \mathcal{D}_r \circ p_r \colon P_1 \to P_0 \times H_{\frac{1}{2}}^+(\mathbb{A}_r(0))$$

through Fredholm operators (true by Corollary 4.29), the index of $\mathfrak{D}_{\mathbb{A}_r}$ is constant. In the case at hand the operators are the following

$$\mathcal{D}_r \colon P_1 \to P_0 \times H_{\frac{1}{2}}, \quad \xi \mapsto (D_{A_r}\xi, \xi(0))$$

and

$$p_r = \left(\mathrm{Id}, \pi_+^{\mathbb{A}_r(0)}\right) \colon P_0 \times H_{\frac{1}{2}} \to P_0 \times H_{\frac{1}{2}}.$$

The map $r \mapsto p_r$ is continuous by [FW24, Thm. D], see Theorem 4.23. It remains to show continuity of the homotopy $[0,1] \ni r \mapsto \mathcal{D}_r$, hence of $r \mapsto D_{A_r}$. Next we show this at $r = 1$. By continuity of the path $\sigma \mapsto A(\sigma)$, given $\varepsilon > 0$, there exists $\sigma_0 = \sigma_0(\varepsilon) > 0$ such that $\|\mathbb{A}^+ - A(\sigma)\|_{\mathcal{L}(H_1,H_0)} \le \varepsilon$ for every $\sigma \ge \sigma_0$. Let r_0 be such that $r_0^2/(1 - r_0^2) = \sigma_0$. Since the function φ is monotone increasing, for every $r \in [r_0, 1]$ we have $r^2/(1-r^2) \geq \sigma_0$. Therefore for every $s \in \mathbb{R}_+$ we have $\|\mathbb{A}^+ - A_r(s)\|^2_{\mathcal{L}(H_1,H_0)} \leq \varepsilon$. Hence there is the estimate

$$\begin{aligned}
\|(D_{\mathbb{A}^+} - D_{A_r})\xi\|^2_{P_0(\mathbb{R}_+)} &= \int_0^1 \|(\mathbb{A}^+ - A_r(s))\xi(s)\|_0^2 ds \\
&\leq \int_0^1 \|\mathbb{A}^+ - A_r(s)\|^2_{\mathcal{L}(H_1,H_0)} \|\xi(s)\|_1^2 ds \\
&\leq \varepsilon^2 \|\xi\|^2_{P_1(\mathbb{R}_+)}.
\end{aligned}$$

This proves that $\|D_{\mathbb{A}^+} - D_{A_r}\|_{\mathcal{L}(P_1^+, P_0)} \leq \varepsilon$. This shows continuity at $r = 1$. For $r \in [0,1]$ one compares D_{A_r} and $D_{A_{\tilde{r}}}$ by a similar argument where, in addition, uniform continuity of $\sigma \mapsto A(\sigma)$ enters. Now it follows from Theorem E.1 that $\operatorname{index} \mathfrak{D}_{\mathbb{A}_0} = \operatorname{index} \mathfrak{D}_{\mathbb{A}_1}$. Since $\mathbb{A}_0 = \mathbb{A}$ and $\mathbb{A}_1 \equiv \mathbb{A}^+$, we get identity 1 in

$$\operatorname{index} \mathfrak{D}_{\mathbb{A}} \stackrel{1}{=} \operatorname{index} \mathfrak{D}_{\mathbb{A}^+} \stackrel{2}{=} \operatorname{index} D^+_{\mathbb{A}^+} \stackrel{3}{=} 0$$

where identity 2 holds by the same arguments as in the proof of Corollary 4.14 and identity 3 is by Proposition 4.30. This proves Corollary 4.31. □

4.3.6 Theorem A – Index is spectral flow

Pick $A \in \mathcal{A}^*_{\mathbb{R}_+}$. Choose $T > 0$ sufficiently large such that $A(s)$ is invertible for every $s \geq T$. Analogous to Theorem 4.15 there is concatenation formula 1

$$\begin{aligned}
\operatorname{index} \mathfrak{D}_A &\stackrel{1}{=} \operatorname{index} \mathfrak{D}_A|_{[0,T]} + \operatorname{index} \mathfrak{D}_A|_{[T,\infty)} \\
&\stackrel{2}{=} \operatorname{index} \mathfrak{D}_A|_{[0,T]} \\
&\stackrel{3}{=} \varsigma(\mathfrak{D}_A|_{[0,T]}) \\
&\stackrel{4}{=} \varsigma(\mathfrak{D}_A).
\end{aligned}$$

Identity 2 is Corollary 4.31 and identity 3 is the spectral flow formula of Theorem A in the already proved finite interval case. Identity 4 holds since $A(s)$ is invertible for every $s \in [T, \infty]$; no eigenvalues cross zero.

4.4 Half infinite backward interval – Theorem A

Let $\mathbb{R}_- = (-\infty, 0]$. For $k = 0, 1$ we define the Hilbert space isomorphism

$$\mathcal{R}_k \colon P_k(\mathbb{R}_-) \to P_k(\mathbb{R}_+), \quad \xi \mapsto \mathcal{R}_k \xi := [s \mapsto \xi(-s)]$$

which reverses time from negative to positive. We define the reversal map

$$\mathcal{R} \colon \mathcal{A}^*_{\mathbb{R}_-} \to \mathcal{A}^*_{I_+}, \quad A \mapsto \mathcal{R}A := [s \mapsto -A(-s)]\,.$$

Let $A \in \mathcal{A}^*_{\mathbb{R}_-}$. Consider

$$\begin{aligned}
\mathfrak{D}_A = \mathfrak{D}_A^{\mathbb{R}_-} \colon P_1(\mathbb{R}_-) &\to P_0(\mathbb{R}_-) \times H^-_{\frac{1}{2}}(\mathbb{A}_0) =: \mathcal{W}(\mathbb{R}_-; \mathbb{A}_0) \\
\xi &\mapsto \left(D_A \xi, \pi_-^{\mathbb{A}_0} \xi_0\right)
\end{aligned}$$

where $\mathbb{A}_0 := A(0)$. Note that

$$(\mathcal{R}_0, \mathbb{1}) \circ \mathfrak{D}_A \circ \mathcal{R}_1 = \mathfrak{D}_{\mathcal{R}A} \colon P_1(\mathbb{R}_+) \to P_0(\mathbb{R}_+) \times \underbrace{H^-_{\frac{1}{2}}(\mathbb{A}_0)}_{H^+_{\frac{1}{2}}((\mathcal{R}A)_0)} .$$

Hence $\mathfrak{D}_A$ is Fredholm since $\mathfrak{D}_{\mathcal{R}A}$ is, by Section 4.3. The index is given by

$$\begin{aligned} \operatorname{index} \mathfrak{D}_A &\stackrel{1}{=} \operatorname{index} \mathfrak{D}_{\mathcal{R}A} \\ &\stackrel{2}{=} \varsigma(\mathcal{R}A) \\ &\stackrel{3}{=} \varsigma(A). \end{aligned}$$

Here identity 1 is by the previous displayed conjugation, identity 2 is by the already proven Theorem A for $\mathbb{R}_+$, and identity 3 holds since the path $\mathcal{R}A$ is the negative of the path A traversed backwards, the two minus signs cancel.

4.5 Real line – Theorem A

Let $A \in \mathcal{A}^*_{\mathbb{R}}$. Corollary 4.6 shows that $\mathfrak{D}_A = D_A \colon P_1(\mathbb{R}) \to P_0(\mathbb{R})$ is Fredholm.

To show the spectral flow formula pick $T > 0$ such that $A(s)$ invertible whenever $s \geq |T|$. In the following calculation identity 1 is by concatenation

$$\begin{aligned} \operatorname{index} \mathfrak{D}_A &\stackrel{1}{=} \operatorname{index} \mathfrak{D}_A|_{(-\infty,-T]} + \operatorname{index} \mathfrak{D}_A|_{[-T,T]} + \operatorname{index} \mathfrak{D}_A|_{[T,\infty)} \\ &\stackrel{2}{=} \operatorname{index} \mathfrak{D}_A|_{[-T,T]} \\ &\stackrel{3}{=} \varsigma(\mathfrak{D}_A|_{[-T,T]}) \\ &\stackrel{2}{=} \varsigma(\mathfrak{D}_A). \end{aligned}$$

Identity 2 is Corollary 4.31 and identity 3 is the spectral flow formula of Theorem A in the already proved finite interval case. Identity 4 holds since $A(s)$ is invertible whenever $|s| \geq T$; no eigenvalues cross zero.

A Hilbert space pairs

A.1 Interpolation and extrapolation: Hilbert $\mathbb{R}$-scales

Let $H = (H_0, H_1)$ be a Hilbert space pair. Then both Hilbert spaces H_0 and H_1 are separable by [FW24, Cor. A.5]. By Riesz' theorem there is a unique bounded linear map $T \in \mathcal{L}(H_1)$, called the **growth operator** of the pair, with

$$\langle \xi, \eta \rangle_0 = \langle \xi, T\eta \rangle_1 \tag{A.76}$$

for all $\xi, \eta \in H_1$. Since $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ are inner products, the operator T is positive definite and symmetric. Moreover, in [FW24, Le. A.7] we showed that compactness of the inclusion $\iota \colon H_1 \to H_0$ implies that the operator T is compact. In particular, the spectrum of T consists of positive eigenvalues κ, of finite multiplicity m_κ, whose only accumulation point is zero. Define

$$\forall \kappa \in \operatorname{spec} T, \quad V_\kappa := \operatorname{Eig}_\kappa T := \{v \in H_1 \mid Tv = \kappa v\}, \quad m_\kappa := \dim V_\kappa < \infty,$$

then the **eigenspace core** of the pair (H_0, H_1) is the direct sum of eigenspaces

$$V := \bigoplus_{\kappa \in \operatorname{spec} T} V_\kappa, \qquad V \subset H_1 \subset H_0.$$

For later use, the direct sum is in decreasing eigenvalue order $\kappa_1 > \kappa_2 > \dots > 0$. As a consequence of the spectral theorem for compact symmetric operators

$$H_1 = \overline{V}^{\|\cdot\|_1}.$$

Since H_1 is a dense subset of H_0 we further have

$$H_0 = \overline{V}^{\|\cdot\|_0}.$$

Lemma A.1. *Let $\kappa_1 \neq \kappa_2$ be different eigenvalues of T. Then the eigenspaces V_{κ_1} and V_{κ_2} are orthogonal with respect to both inner products $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$. Two vectors of V are 0-orthogonal iff they are 1-orthogonal, in symbols $\perp_1 \Leftrightarrow \perp_0$.*

Proof. Pick $\xi_1 \in V_{\kappa_1}$ and $\xi_2 \in V_{\kappa_2}$. This means that $T\xi_1 = \kappa_1 \xi_1$ and $T\xi_2 = \kappa_2 \xi_2$. Using (A.76) we compute

$$\begin{aligned} \kappa_2 \langle \xi_1, \xi_2 \rangle_1 &= \langle \xi_1, T\xi_2 \rangle_1 \\ &\stackrel{2}{=} \langle \xi_1, \xi_2 \rangle_0 \\ &= \langle \xi_2, \xi_1 \rangle_0 \\ &= \langle \xi_2, T\xi_1 \rangle_1 \\ &= \kappa_1 \langle \xi_1, \xi_2 \rangle_1 . \end{aligned}$$

The hypothesis $\kappa_1 \neq \kappa_2$ implies 1-orthogonality $\langle \xi_1, \xi_2 \rangle_1 = 0$. So $\langle \xi_1, \xi_2 \rangle_0 = 0$, by equality 2. For $\xi_1, \xi_2 \in V_{\kappa_2}$ equality 2 proves assertion two of the lemma. □

Another immediate consequence of (A.76) is the **length relation** in V_κ, namely

$$\xi \in V_\kappa \qquad \Rightarrow \qquad \|\xi\|_1 = \frac{1}{\sqrt{\kappa}} \|\xi\|_0. \tag{A.77}$$

We write $\xi \in V$ uniquely in the form $\xi = \sum_{\kappa \in \operatorname{spec} T} \xi_\kappa$ where $\xi_\kappa \in V_\kappa$. Then

$$\|\xi\|_0^2 = \sum_{\kappa \in \operatorname{spec} T} \|\xi_\kappa\|_0^2, , \qquad \|\xi\|_1^2 = \sum_{\kappa \in \operatorname{spec} T} \frac{1}{\kappa} \|\xi_\kappa\|_0^2. \tag{A.78}$$

The first formula is by 0-orthogonality in Lemma A.1 and the second formula by 1-orthogonality in Lemma A.1 combined with (A.77).

For any real $r \in \mathbb{R}$ we define an r-norm for $\xi \in V$ by

$$\|\xi\|_{H_r} := \Bigg(\sum_{\kappa \in \operatorname{spec} T} \frac{1}{\kappa^r} \|\xi_\kappa\|_0^2 \Bigg)^{\frac{1}{2}}.$$

Since V is a direct product, for any element ξ only finitely many components ξ_κ are non-zero, hence the number of non-zero summands, also in (A.78), is finite. By (A.78), the definition of the r-norm coincides for $r = 0, 1$ with the original norms in H_0 and H_1, respectively. Now we take the completion

$$H_r := \overline{V}^{\|\cdot\|_{H_r}}. \tag{A.79}$$

We endow H_r with the **pair r-inner product** and the **pair r-norm** defined by

$$\langle \xi, \eta \rangle_{H_r} := \sum_{\kappa \in \operatorname{spec} T} \frac{1}{\kappa^r} \langle \xi_\kappa, \eta_\kappa \rangle_0 , \quad \|\xi\|_{H_r} := \Bigg(\sum_{\kappa \in \operatorname{spec} T} \frac{1}{\kappa^r} \|\xi_\kappa\|_0^2 \Bigg)^{\frac{1}{2}}, \tag{A.80}$$

whenever $\xi, \eta \in H_r$. Here the number of non-zero summands could be infinite, but the sum is still finite due to the completion property.

To summarize, a Hilbert space pair $H = (H_0, H_1)$ canonically induces a **Hilbert $\mathbb{R}$-scale**, roughly speaking a real family of Hilbert spaces H_r, notation

$$H_\mathbb{R} := (H_r)_{r \in \mathbb{R}}. \tag{A.81}$$

The **dual Hilbert $\mathbb{R}$-scale** is defined by $H_\mathbb{R}^* = (H_r^*)_{r \in \mathbb{R}}$ where $H_r^* := \mathcal{L}(H_r, \mathbb{R})$.

A.1.1 The model Hilbert $\mathbb{R}$-scale

Let $f \colon \mathbb{N} \to (0, \infty)$ be a **growth function**, i.e. a monotone unbounded function. Let $\ell^2_f = \ell^2_f(\mathbb{N})$ be the space of all real sequences $x = (x_\nu)_{\nu \in \mathbb{N}}$ with

$$\sum_{\nu=1}^\infty f(\nu) x_\nu^2 < \infty.$$

The space ℓ^2_f is a Hilbert space with respect to the inner product

$$\langle x, y\rangle_f := \sum_{\nu\in\mathbb{N}} f(\nu) x_\nu y_\nu, \qquad \|x\|_f := \Big(\sum_{\nu\in\mathbb{N}} f(\nu) x_\nu^2\Big)^{\frac{1}{2}}, \tag{A.82}$$

where $\|x\|_f = \sqrt{\langle x, x\rangle_f}$ is the induced norm. Note that $\ell^2_{f^0} = \ell^2$.

Hilbert space pair. The pair (ℓ^2, ℓ^2_f) is a Hilbert space pair by [Fra09, Le. 2.1]; see also [FW21, Thm. 8.1]. For $\nu \in \mathbb{N}$ let $e_\nu = (0, \dots, 0, 1, 0, \dots)$ be the sequence whose members are all 0 except for member ν which is 1. The set of all e_ν's

$$\mathcal{E} = \{e_\nu\}_{\nu\in\mathbb{N}} \tag{A.83}$$

is called the **standard basis** of $\ell^2 = \ell^2(\mathbb{N})$. While $\mathcal{E}$ is an orthonormal basis of ℓ^2, it is still an orthogonal basis of ℓ^2_f.

Growth operator. The growth operator $T \in \mathcal{L}(\ell^2_f)$ is characterized by the identity $\langle y, x\rangle_{\ell^2} = \langle y, Tx\rangle_{\ell^2_f}$ for all $x, y \in \ell^2_f$. Thus the growth operator $T\colon \ell^2_f \to \ell^2_f$ of the pair (ℓ^2, ℓ^2_f) is given by

$$T(x_\nu) = (\tfrac{x_\nu}{f(\nu)}), \qquad (x_\nu) := (x_\nu)_{\nu\in\mathbb{N}}. \tag{A.84}$$

By monotonicity and unboundedness of f there exists ν_0 such that for any $\nu \le \nu_0$ it holds $\frac{1}{f(\nu)^2} \le \frac{1}{f(1)^2}$ and for any $\nu \ge \nu_0 + 1$ it holds $\frac{1}{f(\nu)} \le f(\nu)$. Thus

$$\langle Tx, Tx\rangle_f = \sum_{\nu=1}^\infty \tfrac{x_\nu^2}{f(\nu)^2} f(\nu) = \sum_{\nu=1}^{\nu_0} \tfrac{x_\nu^2}{f(\nu)} \tfrac{f(\nu)}{f(\nu)} + \sum_{\nu=\nu_0+1}^{\infty} \tfrac{x_\nu^2}{f(\nu)} \le \max\{\tfrac{1}{f(1)^2}, 1\} \langle x, x\rangle_f$$

which shows that T indeed maps ℓ^2_f to ℓ^2_f. The elements of the standard basis $\mathcal{E}$ are the eigenvectors e_ν of T with eigenvalues $\kappa(\nu) = \frac{1}{f(\nu)}$, in symbols

$$Te_\nu = \tfrac{1}{f(\nu)} e_\nu, \quad \kappa(\nu) = \tfrac{1}{f(\nu)}, \qquad \kappa(\nu) \ge \kappa(\nu+1) > 0, \quad \kappa(\nu) \searrow 0.$$

Eigenspace core. We write the eigenvalues $\kappa(\nu)$ to the eigenvectors e_ν in the form of a list $\mathcal{S}(T) = (\frac{1}{f(\nu)})_{\nu\in\mathbb{N}}$ in which can occur finite repetitions. The eigenspace core of T is then equal to

$$V = \bigoplus_{\nu\in\mathbb{N}} \mathbb{R} e_\nu = \mathbb{R}^\infty_0, \qquad \mathbb{R}^\infty_0 \subset \ell^2_f \subset \ell^2.$$

Scale levels. Let $\ell^2_{f^r}$, for $r \in \mathbb{R}$, consist of all sequences $x = (x_\nu)$ such that

$$\|\xi\|_{f^r} = \Big(\sum_{\nu\in\mathbb{N}} f(\nu)^r x_\nu^2\Big)^{\frac{1}{2}} < \infty$$

is finite. The r-inner product is given by $\langle x, y\rangle_{f^r} = \sum_{\nu\in\mathbb{N}} f(\nu)^r x_\nu y_\nu$.

Real scale. The real Hilbert scale associated to (ℓ^2, ℓ^2_f) is the family

$$\ell^{2,f}_{\mathbb{R}} := (\ell^2_{f^r})_{r\in\mathbb{R}}.$$

Because the function f is monotone increasing, it follows that whenever $s \le r$ there is an inclusion $\ell^2_{f^r} \hookrightarrow \ell^2_{f^s}$ of Hilbert spaces and the corresponding linear inclusion operator is bounded. For strict inequality $s < r$, by unboundedness of f, the inclusion operator is compact. Moreover, its image is dense. For details we refer to [FW21, Thm. 8.1] and [FW24, Sec. 2].

A.2 Scale bases

Let (H_0, H_1) be a Hilbert space pair. Then both Hilbert spaces H_0 and H_1 are infinite dimensional, by definition, and separable, by [FW24, Cor. A.5].

Definition A.2. A Hilbert space is called **separable** if it contains a countable dense subset. An **orthonormal basis** (**ONB**) of a separable Hilbert space H is a countable orthonormal subset of H whose linear span is dense in H. Weakening the condition from norm 1 to positive norm we speak of an **orthogonal basis**.

Each separable Hilbert space admits an ONB. To see this pick a dense sequence $(v_k)_{k\in\mathbb{N}}$, throw out any member v_k if it is a linear combination of $v_1, \dots, v_{k-1}$, then apply Gram-Schmidt orthogonalization to what remains.

Definition A.3. A **scale basis** for a Hilbert space pair (H_0, H_1) is an orthonormal basis $E = \{E_\nu\}_{\nu\in\mathbb{N}}$ of H_0 that is simultaneously an orthogonal basis of H_1, and which is ordered such that the function

$$h\colon \mathbb{N} \to (0, \infty), \quad \nu \mapsto \|E_\nu\|_1^2 \tag{A.85}$$

is monotone increasing. Following [FW24, Thm. A.4] we refer to h as the **pair growth function** of H. It is automatically unbounded.

Existence of scale bases. We can construct a scale basis as follows. We associated to (H_0, H_1) an operator $T\colon H_1 \to H_1$ by (A.76) . For every eigenvalue $\kappa \in \operatorname{spec} T$ we choose an ordered H_0-orthonormal basis of $V_\kappa := \operatorname{Eig}_\kappa T$, notation

$$E^\kappa = \{E^\kappa_1, \dots, E^\kappa_{m_\kappa}\}. \tag{A.86}$$

By Lemma A.1 the basis E^κ of V_κ is H_1-orthogonal as well and, furthermore, all vectors have the same H_1-length, namely in (A.77) we obtained

$$\|E^\kappa_i\|_1 = \frac{1}{\sqrt{\kappa}}.$$

We order the eigenvalues of T decreasingly

$$\kappa_1 > \kappa_2 > \dots > 0.$$

Now we define a function $\kappa\colon \nu \mapsto \kappa_{j(\nu)}$ that enlists the eigenvalues accounting for multiplicities.[8] More precisely, for $\nu \in \mathbb{N}$ we define

$$j(\nu) := \min\Big\{ j \in \mathbb{N} \mid \sum_{i=1}^{j} m_{\kappa_i} \geq \nu \Big\}, \qquad \kappa(\nu) := \kappa_{j(\nu)},$$

and set

$$E_\nu := E^{\kappa(\nu)}_{\nu - \sum_{i=1}^{j(\nu)-1} m_{\kappa_i}}, \qquad E := \{E_\nu\}_{\nu\in\mathbb{N}}.$$

Note that the ordered orthonormal basis E of H_0 starts at $E_1 = E^1_1$. The pair growth function is related to the growth operator eigenvalues $\kappa(\nu)$ by

$$h(\nu) = \frac{1}{\kappa(\nu)}. \tag{A.87}$$

Next we address the question of moduli of scale bases. For this we show the following lemma.

Lemma A.4. *All elements of a scale basis $E = \{E_\nu\}_{\nu\in\mathbb{N}}$ are T-eigenvectors*

$$TE_\nu = \tfrac{1}{\|E_\nu\|_1^2} E_\nu, \qquad \forall \nu \in \mathbb{N}.$$

Proof. For $\nu \in \mathbb{N}$ write $TE_\nu = \sum_{\mu\in\mathbb{N}} t_{\mu\nu} E_\mu$. Then

$$\delta_{\rho\nu} \stackrel{\perp_0}{=} \langle E_\rho, E_\nu\rangle_0 \stackrel{\text{(A.76)}}{=} \langle E_\rho, TE_\nu\rangle_1 = \sum_{\mu\in\mathbb{N}} t_{\mu\nu} \langle E_\rho, E_\mu\rangle_1 \stackrel{\perp_1}{=} t_{\rho\nu}\|E_\rho\|_1^2$$

where the last step uses that $\langle E_\rho, E_\mu\rangle_1$ is 0 for $\mu \neq \rho$ and $\|E_\rho\|_1^2$ otherwise. □

Moduli of scale bases.

In view of Lemma A.4 all scale bases are constructed as in (A.86). In particular, a scale basis is unique up to an action by the group $\oplus_{\kappa\in\operatorname{spec} T} \mathrm{O}(E^\kappa)$.

A.2.1 Isometry to model Hilbert $\mathbb{R}$-scale.

Consider a Hilbert space pair $H = (H_0, H_1)$. Let h be a pair growth function and let $H_\mathbb{R} = (H_r)_{r\in\mathbb{R}}$ be the Hilbert $\mathbb{R}$-scale associated to the pair. Any scale basis $E = \{E_\nu\}_{\nu\in\mathbb{N}}$ of H determines, for each $r \in \mathbb{R}$, a Hilbert space isometry

$$\Psi^E_r\colon H_r \to \ell^2_{h^r}, \quad \xi = \sum_{\nu\in\mathbb{N}} \xi_\nu E_\nu \mapsto (\xi_\nu)_{\nu\in\mathbb{N}}$$

[8] E.g. if the eigenvalues κ_i and their respective multiplicities m_{κ_i} are

$$\sqrt{5} > \tfrac{4}{7} > \tfrac{1}{2} > \cdots > 0, \qquad 2, 4, m_{\kappa_3 = \frac{1}{2}}, \dots$$

the functions $\nu \mapsto j(\nu)$ and $\nu \mapsto \kappa_{j(\nu)}$ return, respectively, the values

$$\underbrace{1,1}_{m_{\kappa_1}}, \underbrace{2,2,2,2}_{m_{\kappa_2}}, 3, \dots, \qquad \sqrt{5}, \sqrt{5}, \tfrac{4}{7}, \tfrac{4}{7}, \tfrac{4}{7}, \tfrac{4}{7}, \tfrac{1}{2}, \dots$$

by assigning to ξ its coordinate sequence; see [FW24, proof of Thm. A.4]. So

$$\langle \xi, \eta \rangle_{H_r} = \sum_{\nu\in\mathbb{N}} h(\nu)\xi_\nu\eta_\nu, \qquad \|\xi\|_{H_r} = \Bigl(\sum_{\nu\in\mathbb{N}} h(\nu)x_\nu^2\Bigr)^{\frac{1}{2}}, \tag{A.88}$$

for all $\xi, \eta \in H_r$ and where h relates to the growth operator eigenvalues $\kappa(\nu)$ by

$$\frac{1}{\kappa(\nu)} \overset{\text{(A.87)}}{=} h(\nu) \overset{\text{(A.85)}}{=} \|E_\nu\|_1^2.$$

A.3 Musical $\mathbb{R}$-scale isometry $\flat$ and shift isometries

Let $H = (H_0, H_1)$ be a Hilbert space pair and $E = \{E_\nu\}_{\nu\in\mathbb{N}}$ a scale basis. With H comes the growth function $h\colon \mathbb{N} \to [0,\infty)$ and the Hilbert $\mathbb{R}$-scale $H_\mathbb{R}$.

Definition A.5 (Canonical $\mathbb{R}$-scale isometry $\flat = \sharp^{-1}\colon H_{-r} \to H_r^*$)**.** For $r \in \mathbb{R}$ insertion into the 0-inner product

$$\flat\colon H_{-r} \to H_r^*, \quad \xi \mapsto \xi^\flat := \langle \xi, \cdot \rangle_0 \tag{A.89}$$

is for $\xi = \sum_{\nu\in\mathbb{N}} \xi_\nu E_\nu \in H_{-r}$ and $\eta = \sum_{\nu\in\mathbb{N}} \eta_\nu E_\nu \in H_r$ given by the sum

$$(\flat\xi)\eta = \sum_{\nu\in\mathbb{N}} \xi_\nu \eta_\nu.$$

We show that $\flat\colon H_{-r} \to H_r^*$ is an isometry.

The special case $H_0 \to H_0^*$, $\xi \mapsto \langle \xi, \cdot \rangle_0$, is the usual insertion isometry. For their common notation $\flat$ and $\sharp := \flat^{-1}$ these are called **musical isometries**.[9]

To see that $\flat$ is well defined, note that $\xi \in H_{-r}$ and $\eta \in H_r$ implies finiteness

$$\|\xi\|_{-r}^2 = \sum_{\nu\in\mathbb{N}} \xi_\nu^2 h(\nu)^{-r} < \infty, \qquad \|\eta\|_r^2 = \sum_{\nu\in\mathbb{N}} \eta_\nu^2 h(\nu)^r < \infty.$$

Thus by Cauchy-Schwarz the sum

$$\sum_{\nu\in\mathbb{N}} \xi_\nu\eta_\nu = \sum_{\nu\in\mathbb{N}} \xi_\nu h(\nu)^{\frac{-r}{2}} \eta_\nu h(\nu)^{\frac{r}{2}} \le \Bigl(\sum_{\nu\in\mathbb{N}} \xi_\nu^2 h(\nu)^{-r}\Bigr)^{\frac{1}{2}} \Bigl(\sum_{\nu\in\mathbb{N}} \eta_\nu^2 h(\nu)^{r}\Bigr)^{\frac{1}{2}} < \infty$$

is finite, so $\flat$ is well defined. The fact that $\sum_{\nu\in\mathbb{N}} \xi_\nu\eta_\nu = 0$ for every η implies $\xi = 0$ and this proves injectivity of $\flat\colon H_{-r} \to H_r^*$. Since H_0 is a Hilbert space, in particular complete, $\flat\colon H_0 \to H_0^*$ is an isomorphism, as is well known.[10] To see that $\flat\colon H_{-r} \to H_r^*$ is an isomorphism, in fact an isometry, whenever $r \in \mathbb{R}$ consider the shift isometries introduced next, then apply Lemma A.7.

[9] $\flat E_\nu = E_\nu^*$ since $(\flat E_\nu)E_\mu = \langle E_\nu, E_\mu \rangle_0 = \delta_{\nu\mu}$. Exactly isometries take ONB's to ONB's.

[10] Surjective: pick $\eta \in H_0^*$ non-zero, then $\ker\eta \subset H_0$ is a closed subspace of co-dimension 1. Hence $(\ker\eta)^\perp = \mathbb{R}\hat{v}$ for a unit vector $\hat{v} \in H_0$. Now $\eta = \flat_0((\eta\hat{v})\hat{v}) = \langle (\eta\hat{v})\hat{v}, \cdot \rangle_0$ since both sides are equal on $\ker\eta = (\mathbb{R}\hat{v})^\perp$, namely zero, and on $\hat{v}$, namely $\eta\hat{v}$ since $\langle \hat{v}, \hat{v} \rangle_0 = 1$.

Definition A.6 (Shift isometries)**.** Given reals $r, s \in \mathbb{R}$, we define

$$\phi_r^s\colon H_r \to H_s, \quad \xi \mapsto \sum_{\nu\in\mathbb{N}} \xi_\nu h(\nu)^{\frac{r-s}{2}} E_\nu$$

where $\xi = \sum_{\nu\in\mathbb{N}} \xi_\nu E_\nu$.

The maps ϕ_r^s are norm preserving with inverse $(\phi_r^s)^{-1} = \phi_s^r$. For $\xi \in H_r$ we compute

$$\|\phi_r^s \xi\|_s^2 = \sum_{\nu\in\mathbb{N}} h(\nu)^s \left(\xi_\nu h(\nu)^{\frac{r-s}{2}}\right)^2 = \sum_{\nu\in\mathbb{N}} \xi_\nu^2 h(\nu)^r = \|\xi\|_r^2$$

which proves norm preservation. But this implies inner product preservation by the polarization identity. Thus adjoint is inverse: $(\phi_r^s)^* = (\phi_r^s)^{-1} = \phi_s^r$. In particular, the adjoint is an isometry as well.

Lemma A.7. *$\flat = (\phi_r^0)^* \flat \phi_{-r}^0\colon H_{-r} \to H_r^*$ is composed of isometries $\forall r \in \mathbb{R}$.*

Proof. The maps on the right hand side are isometries, as was shown above. Given $\xi \in H_{-r}$ and $\eta \in H_r$, use the characterization of the adjoint to get

$$\begin{aligned}
({\phi_r^0}^* \flat \phi_{-r}^0 \xi)\eta &= (\flat \phi_{-r}^0 \xi)\phi_r^0 \eta \\
&\stackrel{2}{=} \left\langle \textstyle\sum_{\nu\in\mathbb{N}} \xi_\nu h(\nu)^{-\frac{r}{2}} E_\nu, \sum_{\mu\in\mathbb{N}} \eta_\mu h(\mu)^{\frac{r}{2}} E_\mu \right\rangle_0 \\
&= \sum_{\nu,\mu\in\mathbb{N}} \xi_\nu h(\nu)^{-\frac{r}{2}} \eta_\mu h(\mu)^{\frac{r}{2}} \underbrace{\langle E_\nu, E_\mu\rangle_0}_{\delta_{\nu\mu}} \\
&= \sum_{\nu\in\mathbb{N}} \xi_\nu h(\nu)^{-\frac{r}{2}} \eta_\nu h(\nu)^{\frac{r}{2}} \\
&= \sum_{\nu\in\mathbb{N}} \xi_\nu \eta_\nu \\
&= (\flat \xi)\eta
\end{aligned}$$

where in equality two we used the definition of $\phi_{-r}^0 \xi$ and of $\phi_r^0 \eta$. □

Lemma A.8 ($\mathbb{A}^* \simeq \mathbb{A}$)**.** *Assume that $\mathbb{A}\colon H_1 \to H_0$ is a symmetric isometry. Then the composition of $\mathbb{A}^*\colon H_0^* \to H_1^*$ with the four isometries*

$$\mathbb{A}\colon H_1 \xrightarrow{\phi_1^0} H_0 \xrightarrow{\flat} H_0^* \xrightarrow{\mathbb{A}^*} H_1^* \xrightarrow{\flat^{-1}} H_{-1} \xrightarrow{\phi_{-1}^0} H_0$$

is equal to $\mathbb{A}$.

Proof. The matrix of $\mathbb{A}$ for an eigenvector orthonormal basis $\mathcal{V}(\mathbb{A})$ of H_0, (2.11), is diagonal. Let $\xi \in H_1$. To show equality $\flat {\phi_0}^{-1} \mathbb{A}\xi = \mathbb{A}^* \flat \phi_1^0 \xi \in H_1^*$, apply both sides to $\eta \in H_1$. By linearity, basis elements $\xi = v_\nu$ and $\eta = v_\mu$ suffice. We get

$$\begin{aligned}
(\flat {\phi_0}^{-1} A v_\nu) v_\mu &\stackrel{\flat}{=} \langle {\phi_0}^{-1} a_\nu v_\nu, v_\mu\rangle_0 \\
&= \langle a_\nu |a_\nu| v_\nu, v_\mu\rangle_0 \\
&= a_\nu |a_\nu| \delta_{\nu\mu}
\end{aligned}$$

and

$$\begin{aligned}(A^* \flat \phi_1^0 v_\nu) v_\mu &= (\flat \phi_1^0 v_\nu) A v_\mu \\ &\overset{\flat}{=} \left\langle \phi_1^0 v_\nu, A v_\mu \right\rangle_0 \\ &= \langle |a_\nu| v_\nu, a_\mu v_\mu \rangle_0 \\ &= |a_\nu| a_\mu \delta_{\nu\mu}.\end{aligned}$$

This proves Lemma A.8. □

B Quantitative invertibility

In the proof of Theorem 4.2 we will use the following well-known lemma which shows, also quantitatively, that invertibility is an open condition.

Lemma B.1 (Quantitative invertibility)**.** *Given Banach spaces X and Y, suppose the operator $T \in \mathcal{L}(X,Y)$ is invertible and $P \in \mathcal{L}(X,Y)$ is small in the sense that $\|P\| < 1/\|T^{-1}\|$. Then $T+P$ is invertible as well with bound*

$$\|(T+P)^{-1}\| \le \frac{\|T^{-1}\|}{1 - \|T^{-1}\| \cdot \|P\|}$$

where all norms are operator norms.

Proof. We define $S := \mathrm{Id} - T^{-1}(T+P)$ and we estimate

$$\|S\| = \|\mathrm{Id} - T^{-1}(T+P)\| = \|T^{-1}P\| \le \|T^{-1}\|\|P\| < 1. \tag{B.90}$$

Hence $T^{-1}(T+P) = \mathrm{Id} - S$ is invertible with the help of the Neumann series

$$(\mathrm{Id} - S)^{-1} = \sum_{n=0}^{\infty} S^n$$

whose norm we can estimate via the geometric series

$$\|(\mathrm{Id} - S)^{-1}\| \le \sum_{n=0}^{\infty} \|S\|^n = \frac{1}{1 - \|S\|}.$$

An inverse of $T^{-1}(T+P)$ is $(T^{-1}(T+P))^{-1} = (\mathrm{Id} - S)^{-1}$ and bounded by

$$\|(T^{-1}(T+P))^{-1}\| = \|(\mathrm{Id} - S)^{-1}\| \le \frac{1}{1 - \|S\|} \overset{\text{(B.90)}}{\le} \frac{1}{1 - \|T^{-1}\|\|P\|}.$$

Therefore $T+P = T(T^{-1}(T+P))$ is invertible and the inverse $(T+P)^{-1} = (T^{-1}(T+P))^{-1}T^{-1}$ is bounded by $\|(T+P)^{-1}\| \le \|T^{-1}\|/(1 - \|T^{-1}\|\|P\|)$. □

C Evaluation map $P_1 \to H_{1/2}$

Let $H = (H_0, H_1)$ be a Hilbert space pair. Let $h \geq 1$ be a growth function representing the pair growth type. For the time interval $I = [0,1]$ we define the path space $P_1 = P_1(I)$ by (1.4). Let $E = \{E_\nu\}_{\nu\in\mathbb{N}}$ be a scale basis of H; see Appendix A.2. Corollary C.4 is used in Section 4.2.4.

Proposition C.1. *Let $x \in W^{1,2}([0,1], H_0) \cap L^2([0,1], H_1)$, then $x(0) \in H_{1/2}$.*

Proof. Writing $x = \sum_\nu x_\nu E_\nu$ we estimate for $s \in \left[0, \frac{1}{\sqrt{h(\nu)}}\right]$ the initial point

$$
\begin{aligned}
|x_\nu(0)| &\leq |x_\nu(s)| + \int_0^s |\partial_t x_\nu(t)| dt \\
&\leq |x_\nu(s)| + \int_0^{\frac{1}{\sqrt{h(\nu)}}} |\partial_t x_\nu(t)| dt \\
&\leq |x_\nu(s)| + \frac{1}{h(\nu)^{1/4}} ||\partial_t x_\nu||_{L^2}
\end{aligned}
$$

where the last step is by Hölder's inequality. Therefore

$$
\begin{aligned}
x_\nu(0)^2 &\leq \left(|x_\nu(s)| + \frac{1}{h(\nu)^{1/4}} ||\partial_t x_\nu||_{L^2}\right)^2 \\
&\leq 2x_\nu(s)^2 + \frac{2}{\sqrt{h(\nu)}} ||\partial_t x_\nu||^2_{L^2}.
\end{aligned}
$$

Taking advantage of this estimate in step four we obtain that

$$
\begin{aligned}
||x||^2_{L^2(H_1)} &= \int_0^1 ||x(s)||^2_h ds \\
&= \sum_{\nu=1}^\infty \int_0^1 h(\nu) x_\nu(s)^2 ds \\
&\geq \sum_{\nu=1}^\infty \int_0^{\frac{1}{\sqrt{h(\nu)}}} h(\nu) x_\nu(s)^2 ds \\
&\geq \frac{1}{2} \sum_{\nu=1}^\infty \int_0^{h(\nu)^{-1/2}} h(\nu)^1 x_\nu(0)^2 ds - \sum_{\nu=1}^\infty \int_0^{\frac{1}{\sqrt{h(\nu)}}} \sqrt{h(\nu)} ||\partial_t x_\nu||^2_{L^2} ds \\
&= \frac{1}{2} \sum_{\nu=1}^\infty h(\nu)^{1-1/2} x_\nu(0)^2 - \sum_{\nu=1}^\infty ||\partial_t x_\nu||^2_{L^2} \\
&\geq \frac{1}{2} ||x(0)||^2_{H_{1/2}} - ||x||^2_{W^{1,2}(H_0)}.
\end{aligned}
$$

Hence

$$
||x(0)||_{H_{1/2}} \leq \sqrt{2} ||x||_{L^2(H_1) \cap W^{1,2}(H_0)}.
$$

This completes the proof of Proposition C.1. ∎

Definition C.2. By Proposition C.1 we obtain well defined evaluation maps

$$\mathrm{ev}\colon P_1 = W^{1,2}([0,1], H_0) \cap L^2([0,1], H_1) \to H_{1/2}, \quad x \mapsto x(0)$$

and

$$\mathrm{Ev}\colon P_1 \to H_{1/2} \times H_{1/2}, \quad x \mapsto (x(0), x(1)).$$

The evaluation maps are linear continuous maps between Hilbert spaces.

Proposition C.3. *The evaluation map* $\mathrm{ev}\colon P_1 \to H_{1/2}$ *is surjective.*

Proof. Suppose that $x^0 = (x^0_\nu)_{\nu\in\mathbb{N}} \in H_{1/2}$. Define $x_\nu \in C^\infty([0,1], \mathbb{R})$ by

$$x_\nu(s) = e^{-\sqrt{h(\nu)}s} x^0_\nu, \quad s \in [0,1].$$

Note that

$$x_\nu(0) = x^0_\nu$$

so that if we set $x = (x_\nu)_{\nu\in\mathbb{N}}$ we have

$$\mathrm{ev}(x) = x^0.$$

Therefore in order to prove the proposition it suffices to show that

$$x \in W^{1,2}([0,1], H_0) \cap L^2([0,1], H_1).$$

In order to achieve this we estimate

$$\begin{aligned}
||x||^2_{W^{1,2}(H_0)\cap L^2(H_1)} &= ||x||^2_{L^2(H_1)} + ||x||^2_{W^{1,2}(H_0)} \\
&= \sum_{\nu=1}^\infty \int_0^1 \left(h(\nu)x_\nu^2(s) + \partial_s x_\nu(s)^2 + x_\nu(s)^2\right) ds \\
&= \sum_{\nu=1}^\infty \int_0^1 (2h(\nu)+1)e^{-2\sqrt{h(\nu)}s}\left(x^0_\nu\right)^2 ds \\
&= -\sum_{\nu=1}^\infty \frac{2h(\nu)+1}{2\sqrt{h(\nu)}}\left(x^0_\nu\right)^2 e^{-2\sqrt{h(\nu)}s}\Big|_0^1 \\
&= \sum_{\nu=1}^\infty \frac{2h(\nu)+1}{2\sqrt{h(\nu)}}\left(x^0_\nu\right)^2 \left(1 - e^{-2\sqrt{h(\nu)}}\right) \\
&\leq \sum_{\nu=1}^\infty 2\sqrt{h(\nu)}\left(x^0_\nu\right)^2 \\
&= 2||x||^2_{H_{1/2}}.
\end{aligned}$$

This finishes the proof of the proposition. □

Corollary C.4. *The evaluation map* $\mathrm{Ev}\colon P_1 \to H_{1/2} \times H_{1/2}$ *is surjective.*

Proof. Given $x^0, x^1 \in H_{1/2} \times H_{1/2}$, there exist, by Proposition C.3, paths $y^0, y^1 \in P_1$ such that $y^0(0) = x^0$ and $y^1(1) = x^1$. Pick cutoff functions $\beta_0, \beta_1 \in C^\infty([0,1],[0,1])$ such that $\beta_0(0) = 1$ and $\beta_0 \equiv 0$ on $[1/2, 1]$ and $\beta_1(1) = 1$ and $\beta_1 \equiv 0$ on $[0, 1/2]$. Then the combination $y := \beta_0 y^0 + \beta_1 y^1$ still lies in P_1 and $y(0) = y^0(0) = x^0$ and $y(1) = y^1(1) = x^1$. □

D Self-adjoint Hilbert space pair operators

Theorem D.1. *Let (H_0, H_1) be a Hilbert space pair. Suppose the bounded linear map $A\colon H_1 \to H_0$ is H-self-adjoint.*[11] *Then the following is true. As unbounded operator on H_0 with dense domain H_1 the operator $A = A^*$ is selfadjoint. The spectrum of A consists of infinitely many discrete real eigenvalues a_ℓ, of finite multiplicity each,*[12] *which accumulate either at $+\infty$, or at $-\infty$, or at both. Moreover, there exists a countable orthonormal basis $\mathcal{V}(A) = \{v_\ell\}$ of H_0 composed of eigenvectors $v_\ell \in H_1$ of A.*

In a Hilbert space pair both Hilbert spaces are separable by [FW24, Cor. A.5].

Lemma D.2. *For any H-self-adjoint operator $A\colon H_1 \to H_0$ the resolvent set*

$$\mathcal{R}(A) := \{\lambda \in \mathbb{R} \mid A - \lambda\iota\colon H_1 \to H_0 \text{ is bijective}\} \neq \emptyset$$

is non-empty and the complement, the spectrum of A, contains only eigenvalues.

Proof. Since H_1 is separable the operator A, as a map $H_1 \to H_0$, can only have countably many eigenvalues. Hence there exists $\lambda \in \mathbb{R}$ which is not an eigenvalue, i.e. $A - \lambda\iota\colon H_1 \to H_0$ is injective where $\iota\colon H_1 \hookrightarrow H_0$ is inclusion.

We prove that injective $\Rightarrow$ bijective: Since λ is not an eigenvalue of A there is no eigenvector, in symbols $\ker(A - \lambda\iota) = \{0\}$. Since $\iota\colon H_1 \to H_0$ is compact and $A\colon H_1 \to H_0$ is Fredholm of index zero, so is $A - \lambda\iota\colon H_1 \to H_0$, by stability of the Fredholm index under compact perturbation. By Fredholm index zero we conclude $\dim \operatorname{coker}(A - \lambda\iota) = \dim\ker(A - \lambda\iota) = 0$ which proves surjectivity. □

Proof of Theorem D.1. There are two cases for A, injective and not injective.

Case 1: A is injective.

By the Fredholm property the image of A is closed, hence $(\operatorname{im} A)^\perp = \operatorname{coker} A$. Since the Fredholm index is zero and A is injective we conclude $\dim \operatorname{coker} A = \dim\ker A = 0$. Thus the operator $A\colon H_1 \to H_0$ is surjective, hence bijective. Since A is also bounded the inverse $A^{-1}\colon H_0 \to H_1$ is bounded, too, by the open mapping theorem. Composed with the compact inclusion $\iota\colon H_1 \to H_0$, the inverse as an operator on H_0 is not only bounded, but even a compact operator with dense image

$$A^{-1}\colon H_0 \xrightarrow{\text{cp.}} H_0, \qquad \operatorname{im} A^{-1} = H_1 \overset{\text{compact}}{\underset{\text{dense}}{\hookrightarrow}} H_0.$$

Now, by H_0-symmetry of A, the inverse $A^{-1} \in \mathcal{L}(H_0)$ is symmetric

$$\langle A^{-1}x, y\rangle = \langle A^{-1}x, AA^{-1}y\rangle = \langle AA^{-1}x, A^{-1}y\rangle = \langle x, A^{-1}y\rangle, \quad \forall x, y \in H_0,$$

which, by boundedness, is equivalent to self-adjointness $(A^{-1})^* = A^{-1} \in \mathcal{L}(H_0)$.

To summarize, the inverse is a self-adjoint compact operator $A^{-1}\colon H_0 \to H_0$. These are exactly the hypotheses of the Hilbert-Schmidt theorem, see e.g. [RS80,

[11] Fredholm of index 0 and symmetric as unbounded operator on H_0 with dense domain H_1.

[12] The **multiplicity** of an eigenvalue a is the dimension of its eigenspace $\ker(A - a\,\mathrm{Id})$.

thm. VI.16], which asserts that there is an orthonormal basis $\{v_k\}_{k\in\mathbb{N}}$ of H_0 such that $A^{-1}v_k = b_k v_k$ for non-zero real numbers $b_k \to 0$, as $k \to \infty$. Moreover, the multiplicity of each eigenvalue b_k, namely the dimension of its eigenspace $\mathrm{Eig}_{b_k}(A^{-1}) := \ker(A^{-1} - b_k \,\mathrm{Id})$, is finite.

Note that, while the list $(b_k)_{k\in\mathbb{N}}$ may contain finite repetitions, there are still infinitely many different members. Note further that, since $\mathrm{im}\, A^{-1} = H_1$, the eigenvectors $v_k \in H_0$ lie simultaneously in H_1: indeed $b_k v_k = A^{-1}v_k \in H_1$. Hence we may apply A to $A^{-1}v_k = b_k v_k$ and divide by b_k to obtain

$$Av_k = a_k v_k, \quad a_k := \tfrac{1}{b_k} \in \mathbb{R}\setminus\{0\}, \quad k \in \mathbb{N}, \qquad |a_k| \overset{k\to\infty}{\longrightarrow} \infty.$$

Set $\mathcal{V}(A) := \{v_k\}_{k\in\mathbb{N}} \subset H_1$ to get an ONB of H_0 consisting of A-eigenvectors.

Self-adjointness $A = A^*$: The operator $A^{-1} \in \mathcal{L}(H_0)$ satisfies the hypothesis of [Rud91, Thm. 13.11 part (b)], namely to be self-adjoint and injective. The conclusion is that the operator inverse $(A^{-1})^{-1}\colon H_0 \supset \mathrm{im}\, A^{-1} \to H_0$ is self-adjoint. This proves Theorem D.1 for injective A (Case 1).

Case 2: A is not injective.

The linear map $A\colon H_0 \supset H_1 \to H_0$ decomposes as follows

$$\begin{array}{ccc} H_0 = \ker A \overset{\perp_0}{\oplus} X_0 & & (\mathrm{im}\, A)^{\perp_0} \overset{\perp_0}{\oplus} X_0 = H_0 \\ {\scriptstyle \text{dense}\atop \text{compact}} \big\uparrow \iota & \nearrow_{A = 0\oplus A|} & \\ H_1 = \ker A \oplus X_1 & & \end{array} \tag{D.91}$$

where

$$X_0 := (\ker A)^{\perp_0} \subset H_0, \quad X_1 := \iota^{-1}(X_0) = X_0 \cap H_1 \subset H_1, \quad X_0 = \mathrm{im}\, A.$$

We used that, by the Fredholm property, the kernel of A is finite dimensional, so a closed subspace of H_0, as well as of H_1. Let X_0 be the orthogonal complement of $\ker A$ in H_0. Orthogonal complements are closed subspaces. Since X_0 is closed and ι is continuous, the pre-image $X_0 \cap H_1$ is a closed subspace of H_1.

Again by the Fredholm property, the image of A is closed, hence it too admits an orthogonal complement which, by Fredholm index zero, is of the same finite dimension as $\ker A$. We show that $\mathrm{im}\, A = X_0$. '$\subset$' Given $y = Ax \in \mathrm{im}\, A$ and $z \in \ker A$, by symmetry of A we get $\langle y, z\rangle_0 = \langle Ax, z\rangle_0 = \langle x, Az\rangle_0 = \langle x, 0\rangle_0 = 0$. '$=$' Since the orthogonal complements $\ker A$ of X_0 and $(\mathrm{im}\, A)^{\perp_0}$ of $\mathrm{im}\, A$ are of the same finite dimension, inclusion $\mathrm{im}\, A \subset X_0$ can only be true in case of equality (otherwise the co-dimensions would be different).

We show that H_1 is the direct sum $\ker A \oplus X_1$. Note that $\ker A \cap X_1 = \ker A \cap X_0 \cap H_1 = \{0\} \cap H_1 = \{0\}$ and $\ker A + X_1 = H_1$: '$\subset$' obvious. '$\supset$' Pick $x \in H_1$. Since $H_1 \subset H_0 = \ker A \oplus X_0$ write $x = x_* + x_0$ for unique elements $x_* \in \ker A$ and $x_0 \in X_0$. Then $x_0 = x - x_* \in H_1 \cap X_0 = X_1$.

Step 1: The restriction $A|\colon X_0 \supset X_1 \to X_0$ meets the hypothesis of Case 1:

(a) inclusion $\iota|\colon X_1 \hookrightarrow X_0$ is compact and X_1 is a dense subset of X_0;

(b) $A|$ is X_0-symmetric;

(c) $A|\colon X_1 \to X_0$ is a bounded bijection (hence Fredholm of index zero).

Proof of Step 1. (a) Compactness: Let B be a bounded subset of X_1. Then B is also subset of H_1, H_0, and X_0. The closure of B in H_0 is compact since $X_1 \to H_1 \to H_0$ is the composition of a bounded and a compact inclusion map, hence itself compact. But X_0 is a closed subspace of H_0 which contains B. Thus the closure of B is contained in X_0 as well.
Density: The proof relies on $\ker A$ serving as finite dimensional complement in *both* H_0 and H_1. Fix $x \in X_0 \subset H_0$. Since H_1 is dense in H_0, there exists a H_0-convergent sequence $H_1 \ni x_\nu \to x$. We use the orthogonal sum $H_0 = \ker A \oplus X_0$ to write $x_\nu = c_\nu + z_\nu$ for unique $c_\nu \in \ker A \subset H_1$ and $z_\nu \in X_0$. Now $z_\nu - x + c_\nu = x_\nu - x \to 0$ in H_0 and $z_\nu = x_\nu - c_\nu \in H_1$. Thus $z_\nu \in X_0 \cap H_1 = X_1$. Since $x_\nu - x = c_\nu + (z_\nu - x)$ with $c_\nu \in \ker A$ and $z_\nu - x \in X_0$ being H_0-orthogonal Pythagoras provides the equality

$$\|c_\nu\|_0^2 + \|z_\nu - x\|_0^2 = \|x_\nu - x\|_0^2 \overset{\nu\to\infty}{\longrightarrow} 0.$$

This proves H_0-convergence $H_1 \ni z_\nu \to x \in X_0$ and concludes the proof of (a).

(b) Since $X_1 \subset H_1$ and $X_0 \subset H_0$, part (b) is true by H_0-symmetry of A.

(c) Injective and surjective are obvious. The restriction of a bounded linear map to a closed subspace is bounded. □

Step 2: We prove Theorem D.1.

Proof of Step 2. We decompose $A = 0 \oplus A|$ into two summands as in (D.91).
Summand $A|\colon X_0 \supset X_1 \to X_0$. By Step 1 the restriction $A|$ meets the hypothesis of Case 1. Thus $A|$ is self-adjoint as an unbounded operator and its spectrum $\operatorname{spec} A|$ consists of infinitely many discrete real eigenvalues $a \neq 0$ of finite multiplicity each, which accumulate either at $+\infty$, or at $-\infty$, or at both. Moreover, there is an ONB $\mathcal{V}(A|) = \{v_k\}_{k\in\mathbb{N}} \subset X_1$ of X_0 consisting of eigenvectors of $A|$.
Summand 0: $\ker A \to (\operatorname{im} A)^{\perp_0}$. The spectrum consists of the eigenvalue 0. The dimension of the eigenspace $\ker A$ is at least 1 (Case 2) and finite (Fredholm assumption). Choose an H_0-ONB of $\ker A$, notation $\mathcal{V}(\ker A)$.

To see that $A\colon H_0 \supset H_1 \to H_0$ is self-adjoint, unpack the definition of the domain of an adjoint operator to get the first identity

$$\operatorname{dom} A^* = \ker A \oplus D(A|^*) = \ker A \oplus D(A|) = \ker A \oplus X_1 = \operatorname{dom} A$$

whereas the second identity holds since $A|$ is self-adjoint by Case 1.

The spectrum of A is the union of the spectrum of $A|$ and $\{0\}$. The union

$$\mathcal{V}(A) := \mathcal{V}(\ker A) \cup \mathcal{V}(A|)$$

consists of eigenvectors of A. It is an ONB of H_0 (eigenvectors to different eigenvalues are orthogonal since $A = A^*$). This proves Step 2 and Case 2. □

This concludes the proof of Theorem D.1. □

E Invariance of the Fredholm index

E.1 Varying target space

Assume that X and Y are Hilbert spaces and $\mathcal{D}_r\colon X \to Y$ for $r \in [0,1]$ is a continuous family of bounded linear maps. Assume further that $p_r \in \mathcal{L}(Y)$ is a family of projections, orthogonal or not, depending continuously on $r \in [0,1]$. Since p_r is a projection ($p_r p_r = p_r$) its image is equal to its fixed point set which is closed by continuity. Hence $\operatorname{im} p_r$ is a closed subspace of Y. We abbreviate the composition by

$$\mathfrak{D}_r\colon X \xrightarrow{\mathcal{D}_r} Y \xrightarrow{p_r} \operatorname{im} p_r \subset Y.$$

Theorem E.1. *Assume that $\mathfrak{D}_r\colon X \to \operatorname{im} p_r$ is Fredholm for any $r \in [0,1]$, then its Fredholm index is independent of r.*

Proof. The case $p_r = \mathrm{Id}_Y$ is well known. We first discuss that case as warmup.

Case 1: $p_r \equiv \mathrm{Id}_Y$. The Fredholm index of $\mathcal{D}_r\colon X \to Y$ is independent of r.

Proof of Case 1. In this case $\mathfrak{D}_r = \mathcal{D}_r\colon X \to Y$ is a Fredholm operator between fixed Hilbert spaces. For fixed $r, s \in [0,1]$ we abbreviate

$$D := \mathcal{D}_r\colon X \to Y, \qquad Q := \mathcal{D}_s - \mathcal{D}_r\colon X \to Y.$$

Abbreviate $X_0 := \ker D$ and $Y_1 := \operatorname{im} D$ and decompose orthogonally

$$X = \underbrace{X_0}_{\ker D} \overset{\perp}{\oplus} X_1, \qquad Y = Y_0 \overset{\perp}{\oplus} \underbrace{Y_1}_{\operatorname{im} D}.$$

Let $D_{ij}\colon X_i \to Y_j$ denote the restriction of D to X_i followed by projection onto Y_j, and similarly for Q. Note that D is of the form

$$D = \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D_{11} \end{pmatrix} : X_0 \oplus X_1 \to Y_0 \oplus Y_1$$

where $D_{11}\colon X_1 \to Y_1$ is bijective, hence an isomorphism by the open mapping theorem. The operator Q is of the form

$$Q = \begin{pmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix} : X_0 \oplus X_1 \to Y_0 \oplus Y_1$$

If s is close to r, then Q is close to the zero operator, and so is Q_{11}. So by openness of invertibility $D_{11} + Q_{11}\colon X_1 \to Y_1$ is still an isomorphism. The linear map between finite dimensional vector spaces

$$F := Q_{00} - Q_{01}(D_{11} + Q_{11})^{-1} Q_{10}\colon X_0 = \ker D \to Y_0 = \operatorname{coker} D$$

is Fredholm and its index is the dimension difference of domain and target

$$\operatorname{index} F = \dim X_0 - \dim Y_0 = \operatorname{index} D.$$

Claim 1. $\dim\ker(D+Q)=\dim\ker F$.

Write $x\in\ker(D+Q)\subset X_0\oplus X_1$ uniquely in the form $x=x_0+x_1$ where $x_i\in X_i$. Then we get two equations in the form

$$\begin{pmatrix}0\\0\end{pmatrix}=(D+Q)x=\begin{pmatrix}Q_{00}&Q_{01}\\Q_{10}&D_{11}+Q_{11}\end{pmatrix}\begin{pmatrix}x_0\\x_1\end{pmatrix}=\begin{pmatrix}Q_{00}x_0+Q_{01}x_1\\Q_{10}x_0+(D_{11}+Q_{11})x_1\end{pmatrix}.$$

The second equation tells that

$$x_1=-(D_{11}+Q_{11})^{-1}Q_{10}x_0. \tag{E.92}$$

Insert this into equation one to get $0=Q_{00}x_0-Q_{01}(D_{11}+Q_{11})^{-1}Q_{10}x_0=Fx_0$. Consequently projection to the X_0-component is well defined as a map

$$\pi_0\colon X_0\oplus X_1\subset\ker(D+Q)\to\ker F\subset X_0,\quad x=x_0+x_1\mapsto x_0.$$

We show that π_0 is an isomorphism by constructing an inverse, the candidate is

$$\tau\colon\ \ker F\to\ker(D+Q),\quad x_0\mapsto(x_0,-(D_{11}+Q_{11})^{-1}Q_{10}x_0).$$

The image of τ lies in the kernel of $D+Q$, indeed

$$\begin{pmatrix}Q_{00}&Q_{01}\\Q_{10}&D_{11}+Q_{11}\end{pmatrix}\begin{pmatrix}x_0\\-(D_{11}+Q_{11})^{-1}Q_{10}x_0\end{pmatrix}=\begin{pmatrix}Fx_0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Clearly $\pi_0\tau=\mathrm{Id}$. Vice versa $\tau\pi_0=\mathrm{Id}$ holds by (E.92). This proves Claim 1.

Claim 2. $\dim\operatorname{coker}(D+Q)=\dim\operatorname{coker}F$.

This amounts to prove that the dimensions of the orthogonal complements $(\operatorname{im}D+Q)^\perp$ and $(\operatorname{im}F)^\perp$ coincide.

Suppose that $y=y_0+y_1\in Y_0\oplus Y_1$ is element of $(\operatorname{im}(D+Q))^\perp$, equivalently

$$0=\langle Q_{00}x_0+Q_{01}x_1,y_0\rangle+\langle Q_{10}x_0+(D_{11}+Q_{11})x_1,y_1\rangle \tag{E.93}$$

for every $x=x_0+x_1\in X_0\oplus X_1$. We take two particular choices.

Firstly, for the choice $x_0=0$ condition (E.93) reduces to

$$0=\langle Q_{01}x_1,y_0\rangle+\langle(D_{11}+Q_{11})x_1,y_1\rangle=\langle x_1,Q_{01}^*y_0+(D_{11}+Q_{11})^*y_1\rangle$$

for every $x_1\in X_1$. By non-degeneracy of the inner product this means that

$$y_1=-{(D_{11}+Q_{11})^*}^{-1}Q_{01}^*y_0 \tag{E.94}$$

whenever $y_0+y_1\in Y_0\oplus Y_1$ is element of $(\operatorname{im}(D+Q))^\perp$.

Secondly, in (E.93) choose x_1 according to (E.92). Then the first factor in the first inner product is Fx_0 and in the second inner product the first factor is 0, thus what remains is $0=\langle Fx_0,y_0\rangle_Y$ for every $x_0\in X_0$. Hence $y_0\perp\operatorname{im}F$ and therefore projection to the Y_0-component is well defined as a map

$$\Pi_0\colon Y_0\oplus Y_1\supset(\operatorname{im}(D+Q))^\perp\to(\operatorname{im}F)^\perp\subset Y_0,\quad y_0+y_1\mapsto y_0.$$

We show that Π_0 is an isomorphism by constructing an inverse, the candidate is

$$\mathcal{T}\colon (\mathrm{im}\, F)^\perp \to (\mathrm{im}\,(D+Q))^\perp, \quad y_0 \mapsto y_0 + y_1$$

where y_1 is given by (E.94). To see that the image of $\mathcal{T}$ lies in $(\mathrm{im}\,(D+Q))^\perp$, insert $\mathcal{T}y_0 = y_0 + y_1$ into the right hand side of condition (E.93) and note that

$$\begin{aligned}
&\langle Q_{00}x_0, y_0\rangle_Y + \underline{\langle Q_{01}x_1, y_0\rangle_Y} + \langle Q_{10}x_0, -(D_{11}+Q_{11})^{*^{-1}} Q_{01}^* y_0\rangle_Y \\
&\quad + \underline{\langle (D_{11}+Q_{11})x_1, -(D_{11}+Q_{11})^{*^{-1}} Q_{01}^* y_0\rangle_Y} \\
&= \langle Q_{00}x_0 - Q_{01}(D_{11}+Q_{11})^{-1}Q_{10}x_0, y_0\rangle_Y \\
&= \langle Fx_0, y_0\rangle_Y \\
&= 0
\end{aligned}$$

indeed vanishes for every $x = x_0 + x_1 \in X_0 \oplus X_1$. This proves that $\mathcal{T}y_0 \in (\mathrm{im}\,(D+Q))^\perp$. In the calculation the two underlined terms canceled each other and the last equality is due to the domain of $\mathcal{T}$, namely $y_0 \in (\mathrm{im}\, F)^\perp$.
Clearly $\Pi_0\mathcal{T} = \mathrm{Id}$. Vice versa $\mathcal{T}\Pi_0 = \mathrm{Id}$ holds by (E.94). This proves Claim 2.

We prove Claim 1. By definition of D and Q the above discussion shows that

$$\begin{aligned}
\mathrm{index}\, \mathcal{D}_s &= \mathrm{index}(D+Q) \\
&= \dim\ker(D+Q) - \dim\mathrm{coker}\,(D+Q) \\
&= \dim\ker F - \dim\mathrm{coker}\, F \\
&= \mathrm{index}\, F \\
&= \mathrm{index}\, D \\
&= \mathrm{index}\, \mathcal{D}_r
\end{aligned}$$

for all $s, r \in [0,1]$ sufficiently close. This proves the well known Case 1. □

Case 2: General**.** The Fredholm index of the composed operator

$$\mathfrak{D}_r := p_r \circ \mathcal{D}_r \colon X \to Y \to \mathrm{im}\, p_r$$

is independent of $r \in [0,1]$.

Proof of Case 2. We reduce the proof of Case 2 to Case 1 via Step 1:
Step 1. For any $r \in [0,1]$ there is $\varepsilon > 0$ such that $p_r|_{\mathrm{im}\, p_s} \colon \mathrm{im}\, p_s \to \mathrm{im}\, p_r$ is an isomorphism for every $s \in (r-\varepsilon, r+\varepsilon) \cap [0,1]$.
To see this, given $r \in [0,1]$, by continuity of projections we choose $\varepsilon > 0$ sufficiently small such that $\|p_r - p_s\|_{\mathcal{L}(Y)} \le \min\{1/4\|p_r\|_{\mathcal{L}(Y)}, \frac{1}{2}\}$ for every $s \in (r-\varepsilon, r+\varepsilon) \cap [0,1]$. Now, for any such s, we estimate

$$\begin{aligned}
\|p_r \circ p_s|_{\mathrm{im}\, p_r} - \mathbb{1}_{\mathrm{im}\, p_r}\|_{\mathcal{L}(\mathrm{im}\, p_r)} &= \|p_r \circ p_s|_{\mathrm{im}\, p_r} - p_r \circ p_r|_{\mathrm{im}\, p_r}\|_{\mathcal{L}(\mathrm{im}\, p_r)} \\
&= \|p_r\, (p_s|_{\mathrm{im}\, p_r} - p_r|_{\mathrm{im}\, p_r})\|_{\mathcal{L}(\mathrm{im}\, p_r)} \\
&\le \|p_r\|_{\mathcal{L}(Y)} \cdot \|p_s - p_r\|_{\mathcal{L}(Y)} \\
&\le \tfrac{1}{4}.
\end{aligned}$$

Analogously we get the estimate

$$\begin{aligned}\|p_s \circ p_r|_{\operatorname{im} p_s} - \mathbb{1}|_{\operatorname{im} p_s}\|_{\mathcal{L}(\operatorname{im} p_s)} &= \|p_s \circ p_r|_{\operatorname{im} p_s} - p_s \circ p_s|_{\operatorname{im} p_s}\|_{\mathcal{L}(\operatorname{im} p_s)} \\ &= \|p_s\,(p_r|_{\operatorname{im} p_s} - p_s|_{\operatorname{im} p_s})\|_{\mathcal{L}(\operatorname{im} p_s)} \\ &\le \|p_s - p_r + p_r\|_{\mathcal{L}(Y)} \cdot \|p_r - p_s\|_{\mathcal{L}(Y)} \\ &\le \|p_s - p_r\|^2_{\mathcal{L}(Y)} + \|p_r\|_{\mathcal{L}(Y)} \cdot \|p_r - p_s\|_{\mathcal{L}(Y)} \\ &\le \tfrac{1}{4} + \tfrac{1}{4}.\end{aligned}$$

This proves that both compositions

$$p_r \circ p_s|_{\operatorname{im} p_r} \in \mathcal{L}(\operatorname{im} p_r), \qquad p_s \circ p_r|_{\operatorname{im} p_s} \in \mathcal{L}(\operatorname{im} p_s),$$

are invertible. Hence $p_r|_{\operatorname{im} p_s} \colon \operatorname{im} p_s \to \operatorname{im} p_r$ is surjective by the first composition and injective by the second, thus an isomorphism by the open mapping theorem. This proves Step 1.

STEP 2. We prove Case 2.

Fix $r \in [0,1]$. We consider the family of operators, continuous in $s \in [0,1]$, between fixed Hilbert spaces

$$p_r \circ \mathfrak{D}_s \colon X \to \operatorname{im} p_s \to \operatorname{im} p_r.$$

Let $\varepsilon > 0$ be as in Step 1. Because for $s \in (s-\varepsilon, s+\varepsilon) \cap [0,1]$ the projection $p_r|_{\operatorname{im} p_s} \colon \operatorname{im} p_s \to \operatorname{im} p_r$ is an isomorphism, we conclude that $p_r \circ \mathfrak{D}_s$ is a Fredholm operator[13] satisfying

$$\operatorname{index}(p_r \circ \mathfrak{D}_s) = \operatorname{index} \mathfrak{D}_s.$$

By Case 1 we further have

$$\operatorname{index}(p_r \circ \mathfrak{D}_s) = \operatorname{index}(p_r \circ \mathfrak{D}_r)$$

for every $s \in (r-\varepsilon, r+\varepsilon) \cap [0,1]$. Since $p_r \circ \mathfrak{D}_r = \mathfrak{D}_r$, we combine the two index equalities to obtain $\operatorname{index} \mathfrak{D}_s = \operatorname{index} \mathfrak{D}_r$ for every $s \in (r-\varepsilon, r+\varepsilon) \cap [0,1]$. This proves that the index is locally constant and, since $[0,1]$ is connected, we obtain the the index is globally constant on $[0,1]$. This proves the Case 2. □

This concludes the proof of Theorem E.1. □

E.2 Varying domain

Theorem E.2. *Let X, Y, Z be Hilbert spaces and $D \in \mathcal{L}(X,Y)$. Suppose that*

$$[0,1] \ni r \mapsto F_r \in \mathcal{L}(X,Z)$$

is a continuous family of linear surjections. Then the following is true. If, for each $r \in [0,1]$, the operator given by restricting D to $\ker F_r$, notation

$$D_r := D| \colon X \supset V_r \to Y, \qquad V_r := \ker F_r,$$

is semi-Fredholm, then the semi-Fredholm index[14] *of D_r does not depend on r.*

[13] same kernel, isomorphism preserves closedness of image and dimension of cokernel

[14] The semi-Fredholm index $D_r := \dim \ker D_r - \dim \operatorname{coker} D_r$ takes values in $\{-\infty\} \cup \mathbb{Z}$.

Proof. The proof is in two Steps.

Step 1. (Kernel of F_r as a graph)**.** For r near zero $V_r := \ker F_r$ is the graph of

$$T_r := (F_r|_{V_0^\perp})^{-1}(F_0 - F_r)\colon V_0 \to V_0^\perp$$

and $T_r \to 0$ in $\mathcal{L}(V_0, V_0^\perp)$, as $r \to 0$.

Proof. Given $x \in V_0$, we shall determine $y = y(x,r)$ such that a) $y \in V_0^\perp$ and b) $F_r(x+y) = 0$.
By b) and since $x \in \ker F_0$ we get $0 = F_r(x+y) = F_r x + F_r y = (F_r - F_0)x + F_r y$. Hence $F_r y = (F_0 - F_r)x$. Since F_0 is onto, it holds that the restriction to a complement of the kernel $F_0|_{V_0^\perp}\colon V_0^\perp \to Z$ is an isomorphism. Since the map $r \mapsto F_r \in \mathcal{L}(X,Z)$ is continuous, so is in particular $r \mapsto F_r|_{V_0^\perp} \in \mathcal{L}(V_0^\perp, Z)$, Since the condition to be an isomorphism is an open property, each

$$F_r|_{V_0^\perp}\colon V_0^\perp \xrightarrow{\simeq} Z, \quad r \ge 0 \text{ small},$$

is still an isomorphism.
Consequently y is given in the form $y = (F_r|_{V_0^\perp})^{-1}(F_0 - F_r)x$. We abbreviate

$$T_r := (F_r|_{V_0^\perp})^{-1}(F_0 - F_r)\colon V_0 \to V_0^\perp.$$

Then $V_r = \operatorname{graph} T_r$. The linear map $(F_r|_{V_0^\perp})^{-1}\colon Z \to V_0^\perp$ is bounded, uniformly in $r \ge 0$ small. Hence, since $r \mapsto F_r$ is continuous, it holds that T_r converges to the zero operator in $\mathcal{L}(V_0, V_0^\perp)$, as $r \to 0$. □

Step 2. We prove the theorem.

Proof. We show that the index is locally constant. Since the interval $[0,1]$ is connected this implies that the index is constant on the whole interval. To simplify notation we discuss local constancy at $r = 0$.

By Step 1 we can write for small $r \ge 0$ the subspace V_r of X as the graph of T_r. The graph map is the isomorphism $\Gamma_r\colon V_0 \to V_r$ defined by $x \mapsto (x, T_r x)$. We further set $D_r^0 := D_r \circ \Gamma_r\colon V_0 \to V_r \to Y$. Since D_r is a semi-Fredholm operator by hypothesis and Γ_r is an isomorphism it follows that D_r^0 is a semi-Fredholm operator of the same index, namely $\operatorname{index} D_r^0 = \operatorname{index} D_r$.
Note that $\Gamma_0 = \mathrm{Id}_{V_0}$, hence $D_0^0 = D_0$. Since $T_r \to 0$ in $\mathcal{L}(V_0, V_0^\perp)$, as $r \searrow 0$, The map $r \mapsto D_r^0$ is continuous: indeed $D_r^0 x = D(x + T_r x)$ and T_r depends continuously on r by Step 1. Hence $r \mapsto D_r^0 \in \mathcal{L}(V_0, Y)$ is a continuous family of semi-Fredholm operators between fixed Hilbert spaces and hence its semi-Fredholm index is constant as explained in Case 1 in the proof of Theorem E.1 for Fredholm operators; for semi-Fredholm operators we refer to [Mül07, §18 Thm. 4]. □

The proof of Theorem E.2 is complete. □

E.3 Composition

Theorem E.3 (Composition). *Let X, Y, Z be Banach spaces.*
i) Let $S\colon X \to Y$ and $T\colon Y \to Z$ be Fredholm operators between Banach spaces, then the composition $R \circ S$ is Fredholm and

$$\operatorname{index} R \circ S = \operatorname{index} R + \operatorname{index} S.$$

ii) If both $S\colon X \to Y$ and $T\colon Y \to Z$ are bounded linear maps with finite dimensional kernel and closed range, then the above index formula is still valid, although with values in $\mathbb{Z} \cup \{-\infty\}$.

Proof. See e.g. [Mül07, §16 Thm. 5 and Thm. 12]. □

Theorem E.4. *Let $D\colon X \to Y$ be a bounded linear operator between Hilbert spaces. Let $p\colon Y \to Y$ be a projection whose image $Z := \operatorname{im} p$ is of finite co-dimension. Then the following is true. The operator $D\colon X \to Y$ is Fredholm iff*

$$D_p := p \circ D\colon X \to Z$$

is Fredholm as a map to Z and in this case the indices are related by

$$\operatorname{index} D = \operatorname{index} D_p - \operatorname{codim} Z.$$

Proof. As a map $p\colon Y \to Z$ is Fredholm and $\operatorname{index} p = \dim\ker p = \operatorname{codim} Z$.

Case 1. $D\colon X \to Y$ is Fredholm.

Proof. The composition of Fredholm operators $D_p = p \circ D\colon X \to Y \to Z$ is Fredholm, by Theorem E.3, and $\operatorname{index} D_p = \operatorname{codim} Z + \operatorname{index} D$. □

Case 2. $p \circ D\colon X \to Y \to Z$ is Fredholm.

Proof. a) The kernel of D is finite dimensional: True since $\ker D \subset \ker (p \circ D)$.
b) The image of D is closed: It is the pre-image under the continuous map p of the, by assumption closed, image of $p \circ D$, in symbols $\operatorname{im} D = p^{-1}(\operatorname{im}(p \circ D))$.
c) The co-kernel of D is finite dimensional: By a) and b) part ii) of Theorem E.3 applies and its index formula yields that

$$\dim\operatorname{coker} D = \operatorname{codim} Z + \dim\ker D + \dim\operatorname{coker}(p \circ D) - \dim\ker(p \circ D).$$

But the right hand side is finite by a) and assumption. □

This concludes the proof of Theorem E.4. □

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