Rainbow, Great Pyramid, Icosahedron: Mathematics in an Entertaining Way

Hans Hermann Otto

Materials Science and Crystallography, Clausthal University of Technology, Clausthal-Zellerfeld, Lower Saxony, Germany

E-mail: hhermann.otto@web.de

Abstract

The rainbow angle of about 42° is comparable with the golden mean based angle between edge and base of the Great Pyramid. It allows bringing together different areas of knowledge in an amusing way using simple geometry besides laws of optics. The mathematical exercise may encourage students to understand spectacles of nature in a simple and didactical manner.

Keywords: Rainbow, Great Pyramid, Icosahedron, Golden Mean, Comparative Geometry

1. Introduction

The rainbow is a spectacle of nature, which plays a dominant role in all ancient civilizations. It is caused by refraction and internal reflection of sun light by water drops forming a circularly arranged sequence of spectral colors. The sun must be behind the viewer. The exploring of this phenomenon is associated with the names of great researchers: *Alhazen* [1], Al Quaräfi [2], *von Freiberg* [3], *Snell* [4], *Descartes* [5], *Huygens* [6], *Newton* [7], *Young* [8], *Airy* [9], and many others. The first correct interpretation of the rainbow was already given in the 11th century by *Alkazen* (*Abu Ali al-Hasan Ibn Al-Haithan*), a contemporary of *Avicenna* in Persia.

Figure 1. Picture of a main rainbow

2. Rainbow, Great Pyramid, Icosahedron: a Formal Geometric Comparison

Curiously, a light deflection angle around 42° is known as the main rainbow angle between the incident sunlight and the refracted and internally reflected light rays within a water drop (**Figure 2**). Because the light incidence condition is given on a circle on the water drop sphere, the dispersed light appears on a bow. This deflection depends on the refractivity $n(\lambda)$ of water. According to *Snell*'s law of refraction the following relation exists between the angle of incident *i* and the angle of refraction *r* [4]

$$n(\lambda) \ = \frac{\sin(i)}{sn(r)} \tag{1}$$

For the complete deflection of the light beam by refraction and single or multiple internal reflection within a water drop one obtains the angle according to [10] (**Figure 2**)

$$\Theta_m = 2(i - r) + m(\pi - 2r) \, modulo \, 2\pi \tag{2}$$

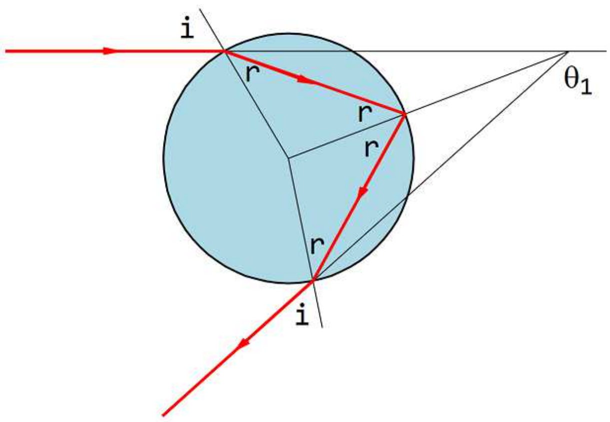

Figure 2. Deflection of a ray of light (red) within a water drop explaining the rainbow. Angle of incident *i*, angle of refraction *r*, and Θ_1 see relation (4) below. Support for the construction of the light beam path is given in the **Appendix**.

We now use for *m* = 1 (case of main rainbow)

$$\sin(i) = x_1 \ \text{and} \ \sin(r) = \frac{x_1}{n} \tag{3}$$

and get

$$\Theta_1 = 180 + 2 \cdot \arcsin x_1 - 4 \cdot \arcsin(\frac{x_1}{n}) \tag{4}$$

and

$$180 - \Theta_1 = 4 \cdot \arcsin\left(\frac{x_1}{n}\right) - 2 \cdot \arcsin x_1 \tag{5}$$

The extremum (minimum) of relation (4) can be calculated by setting the first derivative to zero [10] [11]

$$\frac{d\Theta_1}{dx_1} = \frac{2}{\sqrt{1+x^2}} - \frac{4}{\sqrt{n^2-x^2}} = 0 \tag{6}$$

Then we simply get for the angle of incidence

$$\sin(i) = x_1 = \sqrt{\frac{4-n^2}{3}} \tag{7}$$

If we deliberately choose n = 1.3337448, we calculate the angle of incidence respectively refraction as i = 59.367194°, r = 40.176077°, and

$$\Theta_1 = 138.03008° \qquad 180 - \Theta_1 = 41.969922° \tag{8}$$

Interestingly, the secondary rainbow (m = 2) shows an inverted sequence of colours. The reader may study the contribution of *Jackson* [10] and the elaborated treatise in German given in [11] to learn more. The deliberately chosen refractivity index n can be realized for water at 11.5°C for yellow light of the Na-D excitation doublet with the mean wavelength of $\lambda \approx 589.295\ nm$.

In a way, this particular deflection angle $180 - \Theta_1 = 41.969922°$ is a golden one, suggested by the following relation

$$cos^2(180 - \theta_1) = 0.5527864 = \frac{2}{\varphi+3} \tag{9}$$

where $\varphi = \frac{5-1}{2} = 0.6180339887 \ldots$ is he golden ratio. Curiously, this angle equals almost exactly the angle α_E between the edge and the base of the Great Pyramid [12] [13]. Furthermore, also an approximation holds that connects the rainbow angle with the angle α_P between faces and height of the Great Pyramid

$$\frac{\arccos(\varphi)}{2\varphi} = 41.92916° \approx 180 - \theta_1 \tag{10}$$

We use the *Pythagorean* theorem to determine the angle α_E between the edge and the base of the Great Pyramid (**Figure 3**).

When the base length is set to 2, then the high is $\sqrt{\Phi}$. With big $\Phi = \varphi + 1$ we can determine the angle with the following relations

$$\left(\sqrt{\Phi}\right)^2 + \left(\sqrt{2}\right)^2 = \Phi + 2 = \varphi + 3 = 3.6180339887 \tag{11}$$

$$\cos(\alpha_E) = \frac{\sqrt{2}}{\sqrt{\Phi+2}} = \sqrt{\frac{2}{\varphi+3}} \tag{12}$$

$$\alpha_E = 41.969915° \tag{13}$$

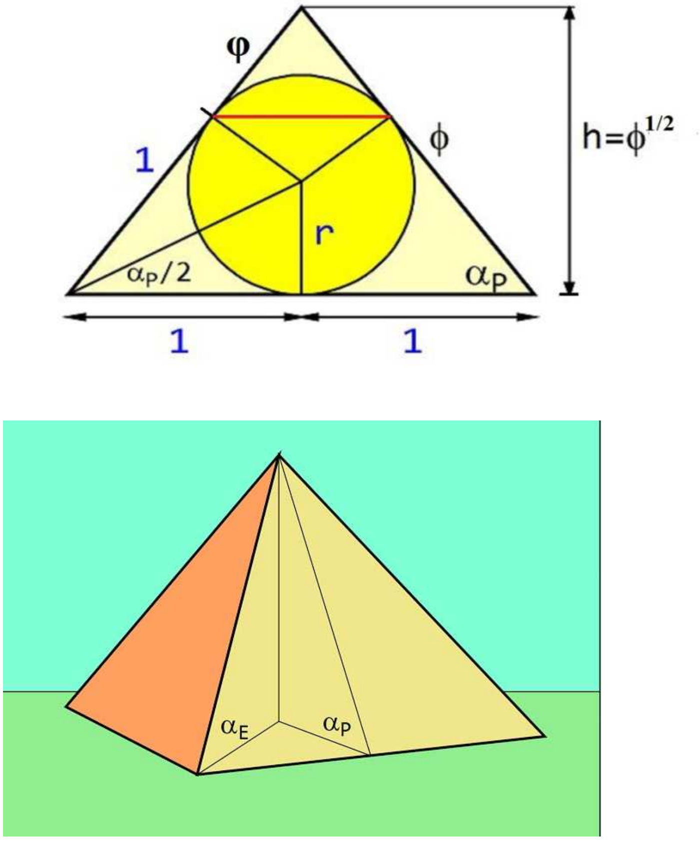

Figure 3. Geometry of the Great Pyramid with outlined dimensioning. Above**:** Cut through the middle of a Great Pyramid's face down the apex with yellow displayed in-sphere projection. Big ϕ denotes the inverse of φ: $\phi = \varphi^{-1} = 1 + \varphi$. The length of the red secant yields $2 \cdot \varphi^2$ [12] [13]. Below: Pyramid sketch with outlined angles.

Between the angles α_P and α_E exists the approximate relationship (see relation 10)

$$\alpha_P \approx 2\varphi \cdot \alpha_E \quad (14)$$

Interestingly, $2r_c = \sqrt{\varphi + 3}$ is the circumsphere diameter of an icosahedron with unit triangle edge length and the mid-sphere radius is exactly $r_m = \frac{1}{2\varphi}$ (see **Figure 4** and **Appendix**) [12] [13] [14] [15].

We can assume that the knowledge about the mathematics of the golden ratio, but also of the icosahedron and the rainbow were already present in the antiquity, may be without any known documentation. The icosahedron is one of *Platon*'s five solids, described in his *Theaetetus* in the fourth century B.C. However, well before *Platon*'s time such solids were obviously known and worked with as documented by the Scottish carved stone balls that are stone 'spheres' of ancient origin (Ashmolean museum of the university of Oxford) [16].

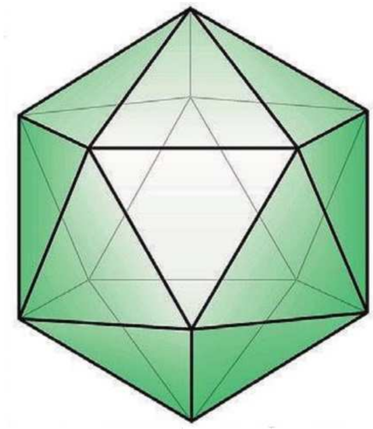

Figure 4. Projection of the Regular Icosahedron Solids Down the Threefold Axis. It is composed of 20 equilateral triangles, 12 vertices and 30 edges.

Conclusion

We really constantly underestimate the skills and knowledge of ancient times. The golden geometry of the Great Pyramid, the early explanation of the rainbow geometry and the beautiful fivefold geometry of the icosahedron, all represent knowledge of times long past, but a mutual influence may be assumed. They have in common purely formal geometric aspects that we tried to compare in this contribution to stimulate the imagination of young researchers.

Conflicts of Interest

The author declares no conflict of interests regarding the publication of this paper.

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Appendix

Table 1. Coordinates of points $P(x,y)$ constructing the light beam path in a spherical water drop of unit radius at the origin by using n = 1.3337448, i = 59.367194°, r = 40.176077°

***P*(*x*,*y*)**	***x***		***y***	
P_1	-cos(i)	-0.50953	sin(i)	0.86045
P_2	cos($2r$-i)	0.93367	sin($2r$-i)	0.35812
P_3	cos($4r$-i)	0.19658	sin($4r$-i)	-0.98049
P_4	sin(i)/tan($2r$-i)	2.24331	sin(i)	0.86045
P_5		-1		-2.5677

Formulas for the Icosahedron (triangle edge length = a, $\varphi = \frac{\sqrt{5}-1}{2} = 0.61803398 \ldots$) [13]

$$V_I = \frac{5}{6}\varphi^{-2}a^3 \tag{15}$$

$$r_i = \frac{\varphi^{-2}}{2\sqrt{3}}a \tag{16}$$

$$r_c = \frac{\sqrt{\varphi+3}}{2}a \tag{17}$$

$$r_m = \frac{a}{2\varphi} \tag{18}$$

$$V_{sph} = \pi \cdot \frac{\varphi^{-6}}{18\sqrt{3}}a^3 \tag{19}$$

$$\frac{V_{sph}}{V_I} = \pi \cdot \frac{\varphi^{-4}}{15\sqrt{3}} = \pi \cdot 0.263814507 = 0.8287977 \approx 2(\sqrt{2}-1) \tag{20}$$

$$A_I = 5 \cdot \sqrt{3}\,a^2 \tag{21}$$

$$A_{sph} = \pi\frac{\varphi^{-4}}{3}a^2 \tag{22}$$

$$\frac{A_{sph}}{A_I} = \pi \cdot \frac{\varphi^{-4}}{15\sqrt{3}} \tag{23}$$