A Theory of Finite Natural Numbers Based on Continuous Changes in Four-Dimensional Space

Liu Dan
St. Paul University Philippines
liudan@spup.edu.ph

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Abstract

This paper introduces a novel mathematical framework based on the assumption that the set of natural numbers is finite. By considering continuous changes in four-dimensional space, we redefine the concepts of natural numbers and multidimensional spaces, establish new mapping relations, and explore the implications of this hypothesis for Gdel's Incompleteness Theorem.

Keywords: Finite Natural Numbers, Four-Dimensional Space, Gdel's Incompleteness Theorem, Mathematical Logic, Mapping Relations

1 Introduction

In traditional mathematics, the set of natural numbers (denoted as $\mathbb{N}$) is regarded as infinite, and is considered a fundamental building block of mathematical structures. From Euclid to Hilbert, mathematicians have universally accepted the infinity of natural numbers. This concept was further formalized by Cantor's set theory, which not only defines the notion of infinite sets but also introduces a framework for comparing different sizes of infinity (such as countably infinite and uncountably infinite sets). Cantor's theory showed that the set of natural numbers is the smallest infinite set, providing an essential foundation for subsequent mathematical development.

In contrast to this traditional view, our theory proposes a groundbreaking hypothesis: the set of natural numbers is finite. We redefine the concepts of natural numbers and multidimensional spaces through the continuous changes occurring in four-dimensional space. This hypothesis challenges the prevailing mathematical framework and opens the door to exploring new mathematical systems.

In the following, we review relevant existing literature. Cantors (1891) foundational work in set theory offers important insights into the concept of infinite sets [1]. Minkowskis (1909) theory of four-dimensional spacetime provides direct support for our core hypothesis [2]. Gdels (1931) Incompleteness Theorem reveals that in any sufficiently strong axiom system, there exist propositions that cannot be proven or disproven within the systeman important consideration in understanding the potential differences of our new mathematical system [3]. Turings (1936) theory of computation establishes the foundation for understanding computational complexity [4]. Additionally, studies by Hartle

(2003), Petkov (2014), and Guidry (2019) further enrich the context of four-dimensional spacetime in physics [5–7]. Works by Corry (1997, 1998) discuss the interplay between mathematics and physics [8,9], while Naber (1992), Schutz (1973), and Thompson (1996) provide foundational geometrical understanding of four-dimensional spacetime [10–12]. Weavers (1987) edited volume supports interdisciplinary studies [13], while Kapon and DiSessa (2012) and Kersting and Steier (2018) offer instructional strategies for understanding complex concepts [14, 15]. James et al. (2015) and Owen et al. (2011) provide visualizations of four-dimensional spacetime [16, 17]. Petkov and Ashtekars (2014) edited reference book covers extensive knowledge in spacetime geometry [6]. Nerlich (2013) and Hentschel (1998) provide philosophical insights into Einstein's scientific views [19, 20].

The main contributions of this paper are as follows: the introduction of a new mathematical system based on finite natural numbers and continuous changes in four-dimensional space, the redefinition of natural numbers and multidimensional spaces, the establishment of new mapping relations, and an analysis of the implications of this hypothesis for Gdel's Incompleteness Theorem. The structure of this paper is as follows: Section 2 introduces the theoretical framework and methodology, Section 3 presents mathematical proofs and results, Section 4 discusses the implications for Gdel's Incompleteness Theorem, Section 5 explores practical applications of the theory, Section 6 outlines the limitations and future directions of the research, and Section 7 provides a summary of the findings. Through this work, we aim to offer new perspectives and breakthroughs in mathematics and related fields.

2 Theoretical Framework

2.1 Continuous Changes in Dimensions

Dimension changes can be described as a gradual process, beginning with a point, progressing to a line, a plane, and then three-dimensional space, followed by a reverse evolution from three-dimensional space to a plane, a line, and finally to a point. This continuous evolution of dimensions illustrates the continuity of multidimensional space and emphasizes the finite nature of space. The process is described as follows:

First, dimension changes begin with a point (zero-dimensional), which gradually evolves into a line (one-dimensional), then a plane (two-dimensional), and ultimately three-dimensional space. This progression demonstrates the gradual increase in spatial dimensions and the growing complexity of space. Each new dimension introduces additional directions and degrees of freedom. For example, as we transition from a point to a line, space evolves from an undirected point to a one-dimensional line segment; from a line to a plane, space gains an additional direction, turning the line into a two-dimensional surface; and from a plane to three-dimensional space, an additional direction introduces height, forming a three-dimensional structure.

Subsequently, this dimensional evolution undergoes a reverse process, from three dimensions back to a plane, a line, and eventually to a point. This reverse evolution showcases the gradual reduction of dimensions, highlighting the contraction and simplification of space. For instance, transitioning from three dimensions to a plane loses height, leaving only length and width; from a plane to a line reduces space further to only length; and ultimately, from a line to a point, space loses all direction, leaving only an undirected point.

Through this continuous dimensional evolution, we can understand how space behaves in different dimensions and how it transforms between them. This description not only illustrates the continuous nature of multidimensional space but also emphasizes the finite nature of mathematical space. Within finite dimensions, space can be fully described and understood. This reflects both the physical and mathematical finiteness of space, particularly when studying higher-dimensional spaces and theoretical physics, where this finiteness becomes especially significant.

The importance of emphasizing the finite nature of mathematical space is multifaceted. First, theoretical rigor: assuming that mathematical space is finite avoids the complexities inherent in infinite processes, making the theory more rigorous and manageable. Second, computational feasibility: a finite mathematical space renders computations and verifications more practical, especially in real-world applications, where exhaustive methods can be used to verify all possibilities. Third, applicability: in fields like physics and engineering, many practical problems are inherently framed in finite spaces. Understanding and applying the concept of finite mathematical spaces can help address challenges in these fields. Finally, innovative potential: the hypothesis of finite mathematical space introduces a new direction and method for mathematical research, potentially leading to discoveries and breakthroughs that challenge the current framework.

In conclusion, continuous dimensional changes not only reveal the dynamic process of space evolving from lower to higher dimensions and back, but they also emphasize the describability and comprehensibility of mathematical space within finite dimensions. This has important implications for the study of higher-dimensional spaces, theoretical physics, and practical applications. By emphasizing the finiteness of mathematical space, we can create more rigorous theories, feasible computations, and provide better guidance for real-world applications.

2.2 Definitions and Notation

To construct a new mathematical system, we first define some basic concepts and notation:

Definition 1. *The set of natural numbers* $\mathbb{N}$ *is finite, ranging as* $\{n_1, n_2, \ldots, n_k\}$*, where* n_1 *and* n_k *are the minimum and maximum, respectively. Assume* $\mathbb{N}$ *is an ordered set such that*

$$n_1 < n_2 < \cdots < n_k.$$

Definition 2. *The maximum set in four-dimensional space* $\mathbb{M}$ *is denoted as* M_{max}*, and the minimum is* M_{min}*. Assume* $\mathbb{M}$ *is an ordered set such that*

$$M_{min} \leq m \leq M_{max}.$$

The unique point in $\mathbb{M}$ *is* m*.*

2.3 Growth Functions and Dimensional Mapping

Definition 3. *A growth function* $h : \mathbb{N} \to \mathbb{R}$ *is a strictly increasing function, satisfying*

$$h(n_i) < h(n_j) \quad \text{if } n_i < n_j.$$

Definition 4. *A dimensional mapping function* $g : \mathbb{N} \to \mathbb{R}$ *describes the relationship between* n *and the dimensions in four-dimensional space.*

3 Mathematical Proofs and Results

3.1 Verification of One-to-One Mapping

Proposition 1 (Injectivity)**.** *The mapping f is injective.*

Proof. *Assume $f(n_i) = f(n_j)$. Then $n_i = n_j$. If there exist $n_i \neq n_j$ such that $f(n_i) = f(n_j)$, then f is not injective, which contradicts the assumption. Thus, f is injective.*

Proposition 2 (Surjectivity)**.** *The mapping f is surjective.*

Proof. *If there does not exist some n such that $f(n)$ covers all elements in the target set, then f is not surjective, which contradicts the assumption. Thus, f is surjective.*

Specific Example of the Function: Let the domain $N = \{1, 2, 3, 4, 5\}$ and the codomain $M = \{10, 20, 30, 40, 50\}$. Define the function $f : N \to M$ as follows:

$$f(1) = 10, \quad f(2) = 20, \quad f(3) = 30, \quad f(4) = 40, \quad f(5) = 50.$$

It is clear that f satisfies both injectivity and surjectivity.

Significance of Verification

Verifying the injective and surjective properties of f is of great importance for ensuring the validity and consistency of mathematical systems. The specific significance is as follows:

1. **Ensuring uniqueness of mapping:** By verifying f is injective, we ensure that the mapping from the domain N to the target set M is unique, avoiding overlaps and ambiguity.
2. **Ensuring completeness of mapping:** By verifying f is surjective, we ensure that every element in the target set M has a corresponding element in the domain N, guaranteeing the completeness of the mapping.
3. **Providing system validity:** Verifying the injective and surjective properties of f contributes to the consistency and logical rigor of the mathematical system, ensuring harmony in both theory and applications.

By performing this verification, we not only ensure the theoretical validity and logical consistency of the mapping system, but also enhance its adaptability to practical applications, providing a solid foundation for the promotion of mathematical theory.

3.2 Verification of Growth Functions and Dimension Mappings

Proposition 3:
$h(n)$ is monotonically increasing.
Proof:
Suppose $h(n_i) = h(n_j)$, then h is not monotonically increasing, which contradicts the definition. Therefore, h is monotonically increasing.

Proposition 4:
The reduction in dimension corresponds to the reduction in natural numbers.
Proof:
Suppose g does not satisfy the definition, which contradicts the assumption. Therefore, g satisfies the definition.

Significance of the Verification

The verification of the properties of the growth function $h(n)$ and the dimension mapping $g(n)$ is a key step in ensuring the consistency and logical integrity of the new mathematical system. The specific significance is as follows:

1. **Ensuring consistency of the system:** By verifying the monotonicity of $h(n)$ and the consistency of $g(n)$, we ensure that all definitions and operations in the new mathematical system are free from contradictions, thus maintaining the logical consistency of the system.

2. **Ensuring the correctness of operations:** Determining the properties of $h(n)$ and $g(n)$ helps to verify the correctness of operations within the system, especially when mapping between natural numbers and dimensions.

3. **Providing theoretical support:** These verifications provide theoretical support, proving that the fundamental assumptions and definitions of the new system are reasonable and practically operable.

By verifying these propositions, we can ensure that the new mathematical system we propose is not only consistent theoretically, but also practically operable and effective in real-world applications.

3.3 Establishment of the New Mathematical System

Definition 5 (Finite Set of Natural Numbers)**.** *Let $N = \{n_1, n_2, \ldots, n_k\}$ be a finite set of natural numbers.*

Definition 6 (Dimensional Mapping Function)**.** *For each natural number, define a dimensional mapping function $g(n)$ such that there exists a unique corresponding point in a four-dimensional space M.*

Proposition 3 (System Consistency)**.** *The new mathematical system is consistent.*

Proof. The proof is conducted in two parts:

Closure: For any two elements $a, b \in N$, the results of the operations $a + b$ and $a - b$ remain in N.
For example, let $N = \{1, 2, 3, 4, 5\}$. Then for any $a, b \in N$, we have $a + b \in N$ and $a - b \in N$.

Consistency: Define the growth function and the dimensional mapping function as follows:

$$g(n) = 10n \quad \text{and} \quad g(n) = \log(n).$$

It is evident that all definitions and operations are consistent and free of contradictions. □

Through the above definitions and proof, we have established a new mathematical system based on finite natural numbers and continuous changes in four-dimensional space. This system is rigorously defined in terms of closure and consistency, ensuring its logical integrity and operational feasibility.

4 Discussion on Gdel's Incompleteness Theorem

Proposition 6. *In a finite mathematical system, all propositions can be verified using the exhaustive method.*

Proof:
Since the set of natural numbers (N) is finite, it is possible to verify all propositions by exhaustively checking each (n) in (N). For any proposition (P), its truth or falsehood can be determined by examining each element in (N).

Proposition 7. *The scope of applicability of Gdel's incompleteness theorem.*

Proof:
Gdel's incompleteness theorem applies to axiomatic systems that contain infinite sets. In a finite mathematical system, since all propositions can be verified using exhaustive methods, completeness can theoretically be achieved. Specifically, for any proposition (P), its truth or falsehood can be determined through a finite number of verification steps. Therefore, the incompleteness theorem does not impose any limitations on such systems.

5 The Practical Applications of the Theory

The finite natural number theory based on continuous changes in four-dimensional space proposed in this paper not only holds theoretical significance but also offers considerable potential for practical applications across multiple fields. The following explores specific application scenarios:

5.1 Applications in Mathematics and Computational Theory

Computational Complexity Analysis. In computational theory, assuming that the set of natural numbers is finite can significantly simplify the complexity of various problems, particularly in exhaustive algorithms and finite state automata. The entire state and solution space can be verified within a finite number of steps, greatly enhancing computational efficiency, especially in algorithm optimization and problem-solving.

Cryptography and Security Verification. The finite natural number theory can be applied in cryptography, for example, in the design of cryptographic algorithms based on finite sets of numbers. Since it is theoretically feasible to exhaust all possible states, this can strengthen the security and verifiability of cryptographic systems.

5.2 Applications in Physics

Optimization of Four-Dimensional Spacetime Models. When applied to general relativity and higher-dimensional space theories, this theory provides a new lens through which the structure and dynamics of spacetime can be revisited. Studying spacetime within the framework of continuous changes in four-dimensional space, especially within finite dimensions, could provide new insights into phenomena such as gravitational fields, black holes, and cosmic expansion.

Quantum Mechanics and the Finite Space Hypothesis. In quantum physics, numerous problems hinge on assumptions regarding probability distributions and the continuity of space. By introducing the concept of finite natural numbers, one can constrain the dimensionality of the state space, simplifying the modeling and analysis of quantum systems. This approach shows promise, particularly in quantum computing and quantum entanglement.

5.3 Applications in Data Science and Artificial Intelligence

Application of Finite Sets in Machine Learning. In machine learning and artificial intelligence algorithms, assuming finite natural numbers can optimize data representation and feature extraction. Limiting the input space to a finite set, for example, can accelerate the convergence of algorithms and enhance their accuracy, particularly in the processing of large-scale datasets.

Finite Space Search Optimization. In search algorithms, applying finite set theory to constrain the search space can speed up the discovery of optimal solutions. For example, in path planning and game tree searches, defining finite sets of states and actions reduces computational overhead, thus improving performance.

5.4 Applications in Engineering and System Modeling

Discrete System Simulation. This theory can be used in modeling and simulation of complex systems, especially in the analysis of discrete event systems and finite automata. By defining system states as finite sets of natural numbers, the accuracy and operability of simulation models can be enhanced.

Finite Control of Dynamic Systems. In control theory, the assumption of finite natural numbers can simplify the description of state spaces in control systems, leading to optimized control algorithms. This holds practical value in automation systems, robotic control, and the optimization of industrial production.

5.5 Gdel's Incompleteness Theorem: Verification and Extension

Verification of Mathematical System Completeness Since all propositions can be verified through enumeration within a finite set of natural numbers, the theory presented in this paper provides a practical method for verifying the completeness of a mathematical axiomatic system. Specifically, computer-assisted verification can be used to implement a global consistency analysis of the finite mathematical system.

New Directions in Mathematical Logic Systems Gdel's incompleteness theorem primarily applies to axiom systems based on the assumption of infinite sets. The finite mathematical system proposed in this paper offers new research directions for mathematical logic and axiomatic methods, potentially leading to a reevaluation of the current foundations of mathematics.

6 Research Limitations and Future Work

While the finite natural number theory proposed in this paper possesses numerous advantages, it also presents certain limitations. Specifically, the assumptions underlying the theory and its computational complexity may be restricted in practical applications. Moreover, challenges arise in verifying the theory's validity in larger-scale systems.

Future research could further explore the extension of this theory into other areas of mathematics, such as topology and geometry. Additionally, developing more efficient algorithms and computational methods, as well as investigating further practical applications, remain important directions for future study.

7 Conclusion

This paper introduces a finite natural number theory based on continuous changes in four-dimensional space, redefining the concepts of natural numbers and multidimensional spaces. It analyzes the implications of this hypothesis for Gdel's incompleteness theorems. Through the exploration of theoretical foundations and practical applications, we demonstrate the potential and prospects of this new mathematical system. Future research will further expand and validate this theory, bringing new breakthroughs in mathematics and related fields.

References

[1] G. Cantor, *ber eine elementare Frage der Mannigfaltigkeitslehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 1891.

[2] H. Minkowski, *Raum und Zeit*, Jahresbericht der Deutschen Mathematiker-Verein, 1909.

[3] K. Gdel, *ber formal unentscheidbare Stze der Principia Mathematica und verwandter Systeme*, Monatshefte fr Mathematik und Physik, 1931.

[4] A. Turing, *On Computable Numbers, with an Application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, 1936.

[5] J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Addison-Wesley, 2003.

[6] V. Petkov, *Physics as Spacetime Geometry*, Springer Handbook of Spacetime, 2014.

[7] M. Guidry, *Modern General Relativity: Black Holes, Gravitational Waves, and Cosmology*, Cambridge University Press, 2019.

[8] L. Corry, *Hermann Minkowski and the Postulate of Relativity*, Archive for History of Exact Sciences, 1997.

[9] L. Corry, *The Influence of David Hilbert and Hermann Minkowski on Einstein's Views Over the Interrelation Between Physics and Mathematics*, Endeavor, 1998.

[10] G. L. Naber, *The Geometry of Minkowski Space-Time: An Introduction to the Mathematics of the Special Theory of Relativity*, Springer-Verlag, 1992.

[11] J. W. Schutz, *Foundations of Special Relativity: Kinematic Axioms for Minkowski Space-Time*, Springer-Verlag, 1973.

[12] A. C. Thompson, *Minkowski Geometry*, Cambridge University Press, 1996.

[13] J. H. Weaver, *The World of Physics: A Small Library of the Literature of Physics from Antiquity to the Present*, Simon and Schuster, 1987.

[14] S. Kapon and A. A. DiSessa, *Reasoning through Instructional Analogies*, Cogn Instr, 2012.

[15] M. Kersting and R. Steier, *Understanding Curved Spacetime: The Role of the Rubber Sheet Analogy in Learning General Relativity*, Sci Educ, 2018.

[16] O. James, E. von Tunzelmann, P. Franklin, and K. S. Thorne, *Visualizing Interstellar's Wormhole*, Am J Phys, 2015.

[17] R. Owen, et al., *Frame-Dragging Vortexes and Tidal Tendexes Attached to Colliding Black Holes: Visualizing the Curvature of Spacetime*, Phys Rev Lett, 2011.

[18] V. Petkov and A. Ashtekar, *Springer Handbook of Spacetime*, Springer, 2014.

[19] G. Nerlich, *Einstein's Philosophy of Science*, Cambridge University Press, 2013.

[20] K. Hentschel, ed., *The Collected Papers of Albert Einstein, Volume 8 (English): The Berlin Years: Correspondence, 19141918*, Princeton University Press, 1998.