PROBLEM THEORY

THEOPHILUS AGAMA

Abstract. This paper introduces problem theory, a new framework for studying problem and solution spaces through the lens of point-set topology and abstract algebra. We define solution spaces as topological constructs induced by the assignment of solutions to problems and establish their fundamental properties. Key results include the identification of compactness and continuity conditions in solution spaces and their algebraic interpretations within module-theoretic settings. This theory bridges abstract algebra and topology, providing new insights into the interplay between algebraic structures and topological spaces. Potential applications and directions for future research are discussed.

1. Introduction

The study of problem spaces, solution spaces, and the interactions between them is a cornerstone of both theoretical and applied mathematics, particularly in the context of computational complexity. This work embarks on a comprehensive exploration of these spaces, blending algebraic, topological, and computational perspectives. By introducing novel frameworks, formalizing key concepts, and proving central theorems, this research contributes fresh insights into the intricate relationships between problems and their solutions, as well as the transformations that map between these spaces. These contributions lay the foundation for understanding the deeper structure of problems and offer new directions for further theoretical exploration.

A significant achievement of this work is the Characterization Theorem, which provides a rigorous classification of problem spaces based on their intrinsic algebraic and topological properties. This theorem not only deepens our understanding of how problem spaces can be categorized but also illuminates the conditions under which different problems can be solved. By offering a comprehensive framework for analyzing these spaces, the theorem serves as a foundational tool for advancing the study of problem classification and solution methodologies.

In parallel, the work investigates the concepts of separability and amenability within the context of problem spaces. These properties have profound implications for both theoretical problem-solving and the practical methods used to find solutions. The results presented in this study provide new insights into the conditions under which problem spaces exhibit these properties, offering novel approaches to solving problems in these spaces. This theoretical framework has far-reaching consequences, not only for understanding problem spaces but also for the design of efficient algorithms.

Date: December 17, 2024.
2010 *Mathematics Subject Classification.* Primary 03G27, 03G30; Secondary 03G25, 03F45.
Key words and phrases. problem; problem space.

A key focus of this work is the connection between time complexity and transformations between problem spaces. The study introduces isotopic maps - a conceptual tool that facilitates the analysis of how one problem space can be mapped to another while preserving essential features, such as complexity measures. Through the examination of the boundedness and continuity of these isotopic maps, this research uncovers new relationships between problem space transformations and computational complexity. The equivalence of boundedness and continuity, a central result of this work, offers critical insights into the limits of problem space transformations and their computational feasibility.

In addition to these theoretical contributions, this study also proposes the isotope, a new measure for assessing the complexity of problem spaces. This tool quantifies the relationship between problem spaces and their solution spaces, providing a novel method for examining solvability. The introduction of this measure offers fresh perspectives on problem theory and opens new avenues for further investigation into the nature of complexity and solvability in mathematical and computational contexts.

The endeavour of finding solutions to problems or at least knowing that a problem is solvable appears to be very compelling. It has various related class of problems that remains unsolved till date. Perhaps the best known of all is the P versus NP problem in computer science. In [1] Florentin Smarandache ask the deceptively simple question

Question 1.1. Is it true that for any question there is at least an answer? Reciprocally, is any assertion the result of at least a question?

We develop a much more consolidated theory of problems and their solution spaces to study the structure and the inner workings of problems, whose solutions may or may not exist. By studying this structure into details, we obtain a negative answer to the question posed

Theorem 1.2. *There exists a problem with no solution.*

It turns out that this result holds for irreducible problems, a certain class of problems we will study in the sequel. This result is obtained via a certain infinite argument under the assumption of a positive answer to the major question, to obtain a certain infinite sub-covers of problem spaces whose indices becomes infinitesimally small and never running into extinction.

2. Problems and solution spaces

In this section we introduce and develop the notion of problem and their corresponding solution spaces.

Definition 2.1. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide solution X to problem Y the problem space induced by providing solution X to problem Y. We denote this space with $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation with $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a sub-problem of the problem Y, then we write

$V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and we call V a proper sub-problem of Y.

Definition 2.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 2.3. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing solution X to problem Y. We denote this space with $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation with $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.

Proposition 2.1. *Let $\mathcal{S}_Y(X)$ be the solution space induced by providing solution X to problem Y. Then $X \in \mathcal{S}_Y(X)$.*

Proof. This follows by virtue of Definition 2.3. □

Definition 2.4. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then we call the number of solutions in the space (size) the **index** of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.

Definition 2.5. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then by the **entropy** of the space, we mean the expression

$$\mathcal{E}[S] = \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]}.$$

In the sequel we formalize the notion that the problem space induced by providing a solution to a problem should - by necessity - contain this solution. The argument is an iteration of a never diminishing entropy of larger and larger solution spaces. We launch formally the following arguments.

Theorem 2.6. *Let $\mathcal{P}_Y(X)$ be the induced problem space of providing solution X to problem Y. Then $Y \in \mathcal{P}_Y(X)$.*

Proof. Let us suppose to the contrary that for any problem space $Y \notin \mathcal{P}_Y(X)$. Since Y is a solved problem, it must belong to some problem space, say $\mathcal{P}_V(U)$. In particular we have the containment

$$Y \in \mathcal{P}_V(U).$$

Since X is a solution to problem Y and V has solution U, it follows that X is a solution obtained as a result of providing solution U to problem V. It follows that $X \in \mathcal{S}_V(U)$ so that the embedding

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U)$$

holds, since $X \in \mathcal{S}_Y(X)$. Again $V \notin \mathcal{P}_V(U)$ under the assumption, so that V belongs to some problem space, say $\mathcal{P}_K(L)$. That is, $V \in \mathcal{P}_K(L)$, a problem space induced by providing solution L to problem K. Since U is a solution to problem V

and K has solution L, it must be a problem solved as a result of providing solution L to problem K. It follows that $U \in \mathcal{S}_K(L)$ and the embedding holds

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L)$$

since $U \in \mathcal{S}_V(U)$. By iterating the argument in this manner under the assumption that $G \notin \mathcal{P}_G(F)$ for an arbitrary problem space, we obtain the infinite embedding

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L) \subset \cdots \subset \cdots.$$

It follows from this the following infinite decreasing sequence of the entropy of solution spaces towards zero

$$\frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]} > \frac{1}{\mathbb{I}[\mathcal{S}_V(U)]} > \frac{1}{\mathbb{I}[\mathcal{S}_K(L)]} > \cdots > \cdots$$

which is not possible. This completes the proof of the theorem. □

Definition 2.7. Let Y and V be any two problems. Then we say problem Y is equivalent to problem V if providing solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote the equivalence with $V \equiv Y$.

Next we expose a simple criterion for creating a subspace of a problem space.

Proposition 2.2. *Let $X \in \mathcal{S}_V(U)$ and $Y \in \mathcal{P}_V(U)$. If X is a solution to problem Y, then*

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Proof. Under the requirement $Y \in \mathcal{P}_V(U)$, then Y is a sub-problem to be solved to provide solution U to problem V. Since $X \in \mathcal{S}_V(U)$, it follows that X is a solution obtained by providing solution U to problem V. Since X solves Y and $Y \in \mathcal{P}_Y(X)$, it follows that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

□

We use the following criterion to determine the solubility of a problem.

Proposition 2.3. *Let V be a problem with solution U. If $Y \in \mathcal{P}_V(U)$, then Y must have a solution.*

Proof. Clearly problem V is solved by U with an induced problem space $\mathcal{P}_V(U)$. Since this space consist of all sub-problems to be solved in order to provide solution U to problem V and $Y \in \mathcal{P}_V(U)$, then Y has a solution. □

3. **Reducible and irreducible problems**

In this section, we classify problems in a problem space into two main categories. We study the notion of irreducibility and reducibility of a problem.

Definition 3.1. Let V be a problem. Then we say V is reducible if there exists a proper sub-problem of V with no proper sub-problem. On the other hand, we say problem V is irreducible if every proper sub-problem of V has a proper sub-problem.

It is a well-known problem to determine if every problem has a solution. Using this classification, we can deduce that there must exist a problem with no solution. It turns out that irreducible problems satisfies this property.

Theorem 3.2. *There exists a problem with no solution.*

Proof. Suppose to the contrary that every problem has a solution. It suffices to argue with only irreducible problems. Now, let V be an irreducible problem with solution U. Consider the induced problem space $\mathcal{P}_V(U)$. Then from Theorem 2.6 $V \in \mathcal{P}_V(U)$. Since V is irreducible, we choose a proper sub-problem Y of V with solution X and construct the problem space $\mathcal{P}_Y(X)$ and solution spaces $\mathcal{S}_Y(X)$. Then $Y \in \mathcal{P}_V(U)$ and $X \in \mathcal{S}_V(U)$ so that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Again V is irreducible so that we can choose a proper sub-problem Z of Y with solution R. Then under the same arguments, we have the chain of sub-covers of problem spaces

$$\mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

By iterating the argument under the same assumption that every problem has a solution, we obtain the infinite chain of sub-covers of smaller problem spaces

$$\cdots \subset \cdots \subset \mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

This is impossible and this completes the proof. $\square$

We can now state another important criterion for determining the solubility of a problem, provided we can put it on par with some category of problems.

Proposition 3.1. *Let V and Y be any two problems such that $V \equiv Y$. If V is irreducible, then Y cannot be solved.*

Proof. Let $V \equiv Y$ and suppose Y has a solution. Then it follows that V must also have a solution, contradicting the requirement that V is irreducible. $\square$

4. **Regular and irregular problems**

In this section we classify problems according to the structure of their sub-problems. We study the notion of regular and irregular problem.

Definition 4.1. Let V be a problem and $\{Y_i\}_{i\geq 1}$ be the sequence of all the sub-problems of V. Then we say V is regular if

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

We say it is irregular if there exists sub-problems Y_j and Y_k of V such that $Y_j \not\leq Y_k$ and $Y_k \not\leq Y_j$.

De facto, regular problem can easily be solved as opposed to irregular problems, where a solution to one sub-problem cannot in anyway be modified and advanced to obtain a solution to other sub-problems. This makes the theory much more tractable with reducible problems.

4.1. **Maximal and minimal sub-problems.**

Definition 4.2. Let V be a problem and Y a proper sub-problem of V. Then we say Y is the maximal sub-problem of V if all other proper sub-problems of V are sub-problems of Y. We say it is the minimal sub-problem of V if it is a sub-problem of all other sub-problems of V.

Next we relate the notion of minimal sub-problem to the notion of reducibility.

Proposition 4.1. *Let V be a problem. If there exists a minimal sub-problem of V, then V must be reducible.*

Proof. Let Y be the minimal sub-problem of problem V. Then Y has no proper sub-problem. This implies that V must be reducible. □

In a similar fashion we relate the notion of maximal sub-problem with the notion of regularity.

Theorem 4.3. *Let V be a problem. If every sub-problem of V has a maximal proper sub-problem, then V must be regular.*

Proof. Let Y be the maximal proper sub-problem of V, since $V \leq V$. Then we have the relation $Y < V$ and every other proper sub-problem of V must be a sub-problem of Y. Since every sub-problem of V has a maximal sub-problem, we let Z be the maximal proper sub-problem of Y then $Z < Y$ and every other proper sub-problems of Y are sub-problems of Z. Since the proper sub-problems of V excluding Y are proper sub-problems of Y and the remaining excluding Z are sub-problems of Z, we obtain the chain of sub-problems

$$\cdots < Z < Y < V$$

and thus chain contains all the sub-problems of V. This proves that V must be a regular problem. □

5. **Connected and disconnected problem spaces**

In this section we study the existence of solutions to problems by deriving an information about the status of related and analogous problems.

Definition 5.1. Let V be a problem with solution U and Y a problem with solution X. Then we say the induced problem spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are connected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) \neq \emptyset.$$

We say the connection is high if

$$\frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_V(U)|} \geq \frac{1}{2} \quad \textbf{and} \quad \frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_Y(X)|} \geq \frac{1}{2}.$$

Otherwise, we say the connection is low. On the other hand, we say the problem spaces are disconnected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) = \emptyset.$$

Proposition 5.1. *Let Y be a problem with solution X. If V is also a problem with a maximal proper sub-problem Z such that $Z \in \mathcal{P}_Y(X)$ and V is regular, then V must be solvable and the induced problem space must be connected to $\mathcal{P}_Y(X)$.*

Proof. Since problem Y has solution X, each problem in the space $\mathcal{P}_Y(X)$ has also been solved. The requirement $Z \in \mathcal{P}_Y(X)$ implies that problem Z has been solved. Since V is regular, we have the chain of all sub-problems of V as

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq Z$$

since Z is the maximal sub-problem of V. Since Z is solved, it follows that all the sub-problems of V is solved and V must have a solution, say T, with induced problem space $\mathcal{P}_V(T)$. The latter claim follows by noting that $Z \in \mathcal{P}_V(T) \cap \mathcal{P}_Y(X)$. $\square$

6. Alternative solutions and their corresponding solution spaces

Definition 6.1. Let Y be a problem. Then we say X and U are alternative solutions to Y if and only if U and X both solves Y. We denote this relation with $X \perp U$ or $U \perp X$.

Proposition 6.1. *Solution spaces remain invariant under replacement with alternative solutions.*

Proof. Let $\mathcal{P}_Y(X)$ be a problem space with corresponding solution space $\mathcal{S}_Y(X)$. Suppose $L \in \mathcal{S}_Y(X)$ with $L \perp K$, then there exist a problem $F \in \mathcal{P}_Y(X)$ that is solved by L. Since $L \perp K$, it follows that K also solves F. Thus we can replace $L \in \mathcal{S}_Y(X)$ with K. $\square$

7. Separable and inseparable problem and solution spaces

In this section we introduce and study the notion of separability of problem and their corresponding solution spaces. We first launch the following language.

Definition 7.1. Let $\mathcal{P}_Y(X)$ be a problem space. Then we say $\mathcal{P}_Y(X)$ is separable if and only there exist some $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. Otherwise, we say the problem space is inseparable. Similarly, we say a solution space $\mathcal{S}_Y(X)$ is separable if and only if there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Otherwise, we say the solution space is inseparable.

We demonstrate that the notion of separability can be passed to and fro between problems and their corresponding solution spaces. The following result is a formalization of this important concept.

Theorem 7.2. *Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Then $\mathcal{P}_Y(X)$ is separable if and only if $\mathcal{S}_Y(X)$ is separable.*

Proof. Suppose $\mathcal{P}_Y(X)$ is separable, then there exist $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. For any $F \in \mathcal{P}_V(U)$ there exists some $R \in \mathcal{S}_V(U)$ that solves F and some $W \in \mathcal{S}_K(L)$ that solves G. Since $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$ and problems in both spaces are not equivalent, it follows that $R \not\perp W$ and $R \notin \mathcal{S}_K(L)$ and $W \notin \mathcal{S}_V(U)$. Since R and W are arbitrary , it follows that $\mathcal{S}_Y(X)$ must also be separable. Suppose without loss of generality that R solves some problem in the space $\mathcal{P}_K(L)$. In particular, there exists some $T \in \mathcal{P}_K(L)$ that is solved by R. Since R also solves F and there exists some $W \in \mathcal{S}_K(L)$ that solves T, it must be that $W \perp R$, a contradiction. In the case, $R \perp W$ then we obtain $R \in \mathcal{S}_K(L)$ and $W \in \mathcal{S}_V(U)$ by virtue of Proposition 6.1. Without loss of generality, we examine the case $R \perp W$ and $R \in \mathcal{S}_K(L)$ with $W \notin \mathcal{S}_V(U)$ then $W \in \mathcal{S}_V(U)$ by virtue of Proposition 6.1. This is also a contradiction.
Conversely, suppose the solution space $\mathcal{S}_Y(X)$ is separable. Then there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Clearly R solves some $G \in \mathcal{P}_V(U)$ and W solves some $T \in \mathcal{P}_K(L)$. We claim that $T \not\equiv G$ with

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset.$$

Suppose $T \equiv G$ for some $T \in \mathcal{P}_K(L)$ and $G \in \mathcal{P}_V(U)$, then $R \perp W$, a contradiction. Since

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

it follows that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X).$$

Suppose to the contrary that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$$

then there exist a problem $A \in \mathcal{P}_Y(X)$ that has no solution in $\mathcal{S}_V(U) \cup \mathcal{S}_K(L)$ but has solution in $\mathcal{S}_Y(X)$. This assertion contradicts the equality

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X).$$

We note that $\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$ implies $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$. Suppose that $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \not\emptyset$. Then there exists a problem $J \in \mathcal{P}_V(U) \cap \mathcal{P}_K(L)$ so that there exists some $N \in \mathcal{S}_V(U) \cap \mathcal{S}_K(L)$ that solves J. This completes the proof. $\square$

8. **Quotient problem and solution spaces**

In this section, we introduce and study the notion of the quotient problem and their corresponding solution spaces. We launch the following terminologies.

Definition 8.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with

$$\mathcal{P}_V(U) \subset \mathcal{P}_Y(X).$$

Then we say the quotient space induced by $\mathcal{P}_V(U)$ in $\mathcal{P}_Y(X)$ regulated by a fixed $T \in \mathcal{P}_Y(X)$, denoted by $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U)$, is the collection of problems

$$\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U).$$

If $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ for some $T \in \mathcal{P}_Y(X)$ then we say $\mathcal{P}_V(U)$ is a principal subspace of the space $\mathcal{P}_Y(X)$. On the other hand, if $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$ for all $T \in \mathcal{P}_Y(X)$ $(T \neq Y)$ then we say $\mathcal{P}_V(U)$ is an ideal sub-space of the problem space $\mathcal{P}_Y(X)$.

In the sequel we use the notion of regularity and maximality to find a subspace that is ideal and at the same time principal.

Proposition 8.1. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$. If Y is a regular problem and V is the maximal sub-problem of Y, then the sub-space $\mathcal{P}_V(U)$ is ideal and principal.*

Proof. Suppose $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and assume that Y is a regular problem and V is the maximal sub-problem of Y. It follows for the sequence of all the sub-problems $\{J_i\}_{i\geq 1}$ of Y except V, we can write

$$\cdots J_n \leq \cdots \leq V \leq Y.$$

Since every problem in the space $\mathcal{P}_V(U)$ is a sub-problem of Y, it follows that for each $T \in \mathcal{P}_Y(X)$ except Y, we must have

$$\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$$

and the space is ideal. Similarly, if we choose $T = Y$, then we have $\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ and the space is a principal space. □

9. **Overlapping and non-overlapping problem and solution spaces**

In this section we study the notion of overlapping and non-overlapping problem and solution spaces. We launch formally the following languages.

Definition 9.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. Then we say they are overlapping if and only if

$$\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset.$$

Otherwise, we say they are non-overlapping. The same characterization also holds for their corresponding solution spaces.

Proposition 9.1. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces, with their corresponding solution spaces $\mathcal{S}_Y(X), \mathcal{S}_V(U)$ such that $F \not\equiv G$ for any $F \in \mathcal{P}_Y(X)$ and $G \in \mathcal{P}_V(U)$. Then the problem spaces are non-overlapping if and only if their corresponding solution spaces are non-overlapping.*

Proof. First suppose $\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset$ then there exist some $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. Since Y is a problem with solution X and V is a problem with solution U, it follows that T must also be a solved problem. That is, there exist some $K \in \mathcal{S}_Y(X)$ that solves T. Again, $T \in \mathcal{P}_V(U)$ so that there exist some $G \in \mathcal{S}_V(U)$ that solves T. It follows that G and K must be the same solution or $G \perp K$; that is, G and K are alternative solutions to T. Since solutions spaces remain invariant under replacement with alternative solutions, it follows in particular that we can replace $G \in \mathcal{S}_V(U)$ with K and the space $\mathcal{S}_V(U)$ still remains unchanged. Conversely, suppose $\mathcal{S}_Y(X) \cap \mathcal{S}_V(U) \neq \emptyset$. It follows that for each $F \in \mathcal{S}_Y(X) \cap \mathcal{S}_V(U)$ must be a solution to some problem $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. $\square$

10. Symmetric problem spaces

In this section we study the notion of symmetry existing among problem spaces. We launch the following languages.

Definition 10.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. We say the problem spaces are symmetric if for each problem $T \in \mathcal{P}_Y(X)$ there exist a problem $L \in \mathcal{P}_V(U)$ such that $K \equiv L$. That is, problem K and problem L are equivalent. We denote the equivalence between the space $\mathcal{P}_Y(X)$ and $\mathcal{P}_V(U)$ as

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

Proposition 10.1. *Let $\mathcal{P}_Y(X)$ be a problem space with a corresponding solution space $\mathcal{S}_Y(X)$. If $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$, then*

$$\mathcal{S}_Y(X) = \mathcal{S}_V(U).$$

We use the notion of symmetry to justify the assertion that the problems spaces endowed with equivalent problems have indistinguishable solution spaces. In fact, it has consequences that allows us to artificially build solution spaces that can be tweak without changing the structure.

Proof. Suppose $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$, then for each problem $T \in \mathcal{P}_Y(X)$ there exists a problem $K \in \mathcal{P}_V(U)$ such that $K \equiv T$. Since $\mathcal{S}_Y(X)$ is the corresponding solution space for $\mathcal{P}_Y(X)$, there exists some $F \in \mathcal{S}_Y(X)$ that solves T. Since problem T and problem K are equivalent problems, it follows that F also solves $K \in \mathcal{P}_V(U)$. The claim follows by iterating the argument in this manner to build the solution space $\mathcal{S}_V(U)$. $\square$

Proposition 10.2. *Let $\mathcal{S}_Y(X)$ and $\mathcal{S}_V(U)$ be solution spaces. If for each $K \in \mathcal{S}_Y(X)$ there exist some $L \in \mathcal{S}_V(U)$, then*

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

Proof. Let K and L be arbitrary with $K \in \mathcal{S}_Y(X)$ and $L \in \mathcal{S}_V(U)$. Then there exists a problem $T \in \mathcal{P}_Y(X)$ solved by K and a problem $F \in \mathcal{P}_V(U)$ solved by L. Since $\equiv$ is an equivalence relation and $K \perp L$, it follows that $T \equiv F$, since L also solves T and K also solves F. The claim follows by repeating the argument with solutions in the space. $\square$

11. **Further Remarks**

The theory as developed is just the preliminary and the first phase of the theory to study problems and their generative solutions. The notion of the time complexity of problems and their sub-problems is a notion to be explored in our next phase of this project, motivated in part by the P versus NP problem. We suspect the following assertions to be true

Conjecture 11.1. Let V be a problem. If V has a minimal and a maximal sub-problem, then V must be a regular problem.

Conjecture 11.2. Let V be a problem with solution U and Y a problem with solution X. If V be regular and the spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are highly connected, then Y must also be regular.

12. **The time complexity**

In this section we study the notion of time complexity of problem and solution spaces.

Definition 12.1. The **resolution** complexity of problem T by providing solution U that solves T is the **algorithmic** time required to generate solution U for problem T. We denote this complexity with $\mathcal{C}_r(T, U)$.

Definition 12.2. The **verification** complexity of a solution U to problem T is the **algorithmic** time required to check solution U for correctness. We denote this complexity with $\mathcal{C}_v(T, U)$.

Definition 12.3. Let T be a problem with solution U. We say the time complexity with respect to problem T with solution U is in **equilibrium** if $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$.

It is important to declare that the time complexity is not unique to problems and solutions. More precisely, it is indeed possible that the resolution time complexity and the verification time complexity may differ quite significantly among equivalent problems and alternative solutions. Consequently, it may not be possible to extend an equilibrium to equivalent problems and alternative solutions. Let us suppose that $\mathcal{C}_r(T_1, U_1) < \infty$ and $\mathcal{C}_v(T_1, U_1) < \infty$ with $T_1 \equiv T_2$ (equivalent problems) then $U_1 \perp U_2$ (alternative solution). It is possible that

$$\mathcal{C}_r(T_1, U_1) \neq \mathcal{C}_r(T_2, U_1)$$

and

$$\mathcal{C}_v(T_1, U_1) \neq \mathcal{C}_v(T_2, U_1)$$

and similarly

$$\mathcal{C}_r(T_2, U_2) \neq \mathcal{C}_r(T_2, U_1)$$

and

$$\mathcal{C}_v(T_2, U_2) \neq \mathcal{C}_v(T_2, U_1).$$

Hence if $\mathcal{C}_r(T_1, U_1) = \mathcal{C}_v(T_1, U_1)$ and $T_1 \equiv T_2$ then the equilibrium

$$\mathcal{C}_r(T_2, U_2) = \mathcal{C}_v(T_2, U_2)$$

may only hold under certain condition. We begin by verifying that time complexity can be ordered up to sub-problems and sub-solutions of a given problem.

Proposition 12.1. *Let T be a problem with solution U. Let $\{T_i\}_{i\geq 1}$ and $\{U_i\}_{i\geq 1}$ denotes the sequence of all sub-problems and sub-solutions of T and U, respectively. If $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$, then we have*
$$\mathcal{C}_r(T_i,U_i)<\mathcal{C}_r(T,U)$$
and
$$\mathcal{C}_v(T_i,U_i)<\mathcal{C}_v(T,U)$$
for each $i\geq 1$.

Proof. Since $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$ and
$$\mathcal{C}_r(T,U):=\sum_{i\geq 1}\mathcal{C}_r(T_i,U_i)$$
and
$$\mathcal{C}_v(T,U):=\sum_{i\geq 1}\mathcal{C}_v(T_i,U_i)$$
the inequality follows easily. □

Remark 12.4. In cases where we do not want to make a reference to the solution and a problem in the notation of the resolution and the verification time complexity, we will write for simplicity $\mathcal{C}_r(T)$ and $C_v(U)$. We will adopt this notation in situations where a reference to a problem or a solution turns out to be irrelevant.

Proving the existence of equilibrium of time complexity of problems is by no means an easy endeavour. In the sequel we prove that assuming equilibrium in the time complexity can be passed down to sub-problems and sub-solutions. We make these ideas formal in the proposition below.

Proposition 12.2. *Let T be a regular problem with solution U such that for any sub-problems T_i,T_j with $i\neq j$, then $\mathcal{C}_r(T_i,U_i)\neq\mathcal{C}_v(T_j,U_j)$. If $\mathcal{C}_r(T,U)=\mathcal{C}_v(T,U)$, then there exists $Q\leq T$ (Q a sub-problem of T) and $L\leq U$ (L a sub-solution of U) that solves Q such that $\mathcal{C}_r(Q,L)=\mathcal{C}_v(Q,L)$.*

Proof. Suppose T is a regular problem with solution U. Let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with corresponding sequence of solutions $\{U_i\}_{i\geq 1}$. Suppose on the contrary that $\mathcal{C}_r(T_i,U_i)=\mathcal{C}_v(T_i,U_i)$ for each $i\geq 1$. By virtue of the regularity of T, we can arrange the sequence of sub-problems and sub-solutions in the following way $T_1\geq T_2\geq\cdots$ and the corresponding sequence of sub-solutions $U_1\geq U_2\geq\cdots$, where each preceding T_i is a sub-problem of T_{i-1} and similarly each U_i is a sub-solution for U_{i-1}. Since problem T is said to be solved by providing a solution to each of the sub-problems, we find under the assumption $\mathcal{C}_r(T,U)=\mathcal{C}_v(T,U)$, that
$$\mathcal{C}_r(T,U)=\sum_{i\geq 1}\mathcal{C}_r(T_i,U_i)=\sum_{i\geq 1}\mathcal{C}_v(T_i,U_i)=\mathcal{C}_v(T,U).$$
Now suppose on the contrary that $\mathcal{C}_r(T_1,U_1)\neq\mathcal{C}_v(T_1,U_1)$, then under the regularity condition, it follows that
$$\sum_{i\geq 2}\mathcal{C}_r(T_i,U_i)\neq\sum_{i\geq 2}\mathcal{C}_v(T_i,U_i)$$

since providing a solution to all sub-problems of T_2 solves problem T_2. Under the requirement that $\mathcal{C}_r(T_i, U_i) \neq \mathcal{C}_v(T_j, U_j)$ for all $i \neq j$, it follows that

$$\mathcal{C}_r(T, U) = \sum_{i\geq 1} \mathcal{C}_r(T_i, U_i) \neq \sum_{i\geq 1} \mathcal{C}_v(T_i, U_i) = \mathcal{C}_v(T, U)$$

violating the assumption that $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$. $\square$

Theorem 12.5. *Let T be a regular problem with a solution K. If M is the maximal sub-problem of T with a solution L and $\mathcal{C}_r(M, L) \ll$ polynomial time and $\mathcal{C}_r(T, K) = \mathcal{C}_v(T, K)$, then $\mathcal{C}_v(T, K) \ll$ polynomial time.*

Proof. Suppose T is a regular problem and let $\{T_i\}_{i\geq 1}$ denotes the sequence of all sub-problems of T with corresponding sequence of sub-solutions $\{K_i\}_{i\geq 1}$ where each K_i solves T_i. We can arrange the sequence of sub-problems in the following way: $T_1 \geq T_2 \geq \cdots$ where $T_1 := M$ is the maximal sub-problem of T and where each sub-problem T_i is a sub-problem of T_{i-1} for $i \geq 2$. Since problem T is solved by solving each of the sub-problems in the sequence, we can write

$$\begin{aligned} C_r(T, K) &= \sum_{i\geq 1} C_r(T_i, K_i) \\ &= C_r(T_1, K_1) + \sum_{i\geq 2} C_r(T_i, K_i). \end{aligned}$$

By the regularity of problem T, we see that

$$\sum_{i\geq 2} C_r(T_i, K_i) = C_r(T_1, K_1) \ll \textit{polynomial time}.$$

Thus $C_r(T, K) \ll$ *polynomial time*. Under the equality $C_r(T, K) = C_v(T, K)$, we deduce that $C_v(T, K) \ll$ *polynomial time*, which completes the proof of the theorem. $\square$

Remark 12.6. Theorem 12.5 is an important ingredient for exploring a deep understanding of the P=NP problem. It purports that once there exist an equilibrium of time complexity of a given problem, it suffices to only investigate the resolution complexity of the maximal sub-problem for a class of well-behaved problems which we refer to as regular problems, introduced and studied in [**?**].

Although the task of proving equilibrium of resolution and verification time complexity can be very hard, we can often carry out this process from bottom-up. That is to say, proving equilibrium of time complexity for sub-problems can be extended to time complexity equilibrium of the actual problem. The following proposition exemplifies that principle.

Proposition 12.3. *Let Y be a problem with solution X and let $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y. If $\mathcal{C}_r(Y_i, X_i) = \mathcal{C}_v(Y_i, X_i)$ for each $i \geq 1$, then $\mathcal{C}_r(Y, X) = \mathcal{C}_v(Y, X)$.*

Proof. The sequences $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y, respectively. Since the solution to problem Y is furnished solving each of the sub-problems in $\{Y_i\}_{1\geq 1}$, it follows under the assumption $\mathcal{C}_r(Y_i, X_i) = \mathcal{C}_v(Y_i, X_i)$ for each $i \geq 1$ that

$$\mathcal{C}_r(Y, X) = \sum_{i\geq 1} \mathcal{C}_r(Y_i, X_i) = \sum_{i\geq 1} \mathcal{C}_v(Y_i, X_i) = \mathcal{C}_v(Y, X).$$

□

We now obtain an important characterization of irreducible problems.

Theorem 12.7. *If X is an irreducible problem, then $\mathcal{C}_r(X) = \infty$ or X is not solvable.*

Proof. Suppose X is an irreducible problem and assume the contrary that $\mathcal{C}_r(X) < \infty$ and that X is solvable. Since X is irreducible, each sub-problem $X_j \leq X$ has a proper sub-problem, and problem X has infinitely many proper sub-problems $X_i < X$. Thus

$$\mathcal{C}_r(X) := \sum_{i=1}^{\infty} \mathcal{C}_r(X_i) < \infty$$

since problem X is solved by providing a solution to each of the sub-problems. This implies that for any $\epsilon > 0$, there exists some $N := N(\epsilon)$ such that for all $i \geq N$ we have

$$\sum_{i=N}^{\infty} \mathcal{C}_r(X_i) < \epsilon.$$

That is, $\mathcal{C}_r(X_i) \longrightarrow 0$ as $i \longrightarrow \infty$. This means the algorithmic time required to solve infinitely many proper sub-problems of problem X converges to zero, which violates the assumption that X is solvable. □

The difficulty of proving equilibrium of time complexity of a given problem may be made easier depending on its structure. Irregular problems seem to be very difficult to understand and unfortunately most problems fall into this category. It is however much easier to establish an equilibrium for a class of well behaved problems that fall into the category of reducible and regular problems. It turns out that once equilibrium is reached for the finest form of this problem, then equilibrium will certainly be attained for the actual problem. We make this discussion formal in the following results.

Theorem 12.8 (extension principle)**.** *Let T be a regular and a reducible problem with solution U. If T_k is a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$, then $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$.*

Proof. Suppose T is a regular problem with solution U and let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with the corresponding sequence of solutions $\{U_i\}_{i\geq 1}$, where each U_i solves T_i for each $i \geq 1$. Since T is reducible, it has a sub-problem with no proper sub-problem. Let T_k be this sub-problem of T, then by the regularity of T, we can arrange the sequence of all sub-problems of T in the following way:

$$T_k \leq T_{k-1} \leq T_{k-2} \leq \cdots \leq T_1$$

with

$$U_k \leq U_{k-1} \leq U_{k-2} \leq \cdots U_1$$

where each T_i is a sub-problem of T_{i-1} and U_i is a sub-solution of U_{i-1}. Under the equilibrium $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$ and since problem T_{k-1} is solved by providing a solution to all of its proper sub-problems, it follows that $\mathcal{C}_r(T_{k-1}, U_{k-1}) = \mathcal{C}_v(T_{k-1}, U_{k-1})$. Similarly, problem T_{k-2} is solved by providing a solution to all of its sub-problems and it follows that

$$\begin{aligned}\mathcal{C}_r(T_{k-2}, U_{k-2}) &= \mathcal{C}_r(T_k, U_k) + \mathcal{C}_r(T_{k-1}, U_{k-1})\\ &= \mathcal{C}_v(T_k, U_k) + \mathcal{C}_v(T_{k-1}, U_{k-1})\\ &= \mathcal{C}_v(T_{k-2}, U_{k-2}).\end{aligned}$$

We can iterate this process to reach the equilibrium $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$. □

Corollary 12.1. Let T be a regular and a reducible problem with solution U. Let T_k is a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$. If $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*.

Proof. It follows from Theorem 12.8 that $\mathcal{C}_r(T, U) = C_v(T, U)$ so that under the hypothesis $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*. □

Remark 12.9. Corollary 12.1 suggests that under a certain mild condition, if a certain class of well-behaved problems have a solution that are easy to verify for correctness then they must also be easy to solve at the same level.

13. **The time complexity of problem and solution spaces**

In this section, we study the notion of time complexity on problem and solutions spaces, as opposed to a specific problem and its solution.

Definition 13.1. Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing solution X to problem Y. Then by the resolution complexity of the problem space $\mathcal{P}_Y(X)$, we mean the sum of each resolution complexity of each problem in the space. For each problem $T \in \mathcal{P}_Y(X)$ there exists a solution $L \in \mathcal{S}_Y(X)$ that solves T. We denote the resolution complexity of the space with

$$\mathcal{P}_Y^r(X) := \sum_{\substack{T\in\mathcal{P}_Y(X)\\ L\in\mathcal{S}_Y(X)}} \mathcal{C}_r(T, L)$$

and the verification complexity with

$$\mathcal{S}_Y^v(X) := \sum_{\substack{L\in\mathcal{S}_Y(X)\\ T\in\mathcal{P}_Y(X)}} \mathcal{C}_v(T, L).$$

Proposition 13.1. *Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing solution X to problem Y. If for each $T \in \mathcal{P}_Y(X)$ and each $L \in \mathcal{S}_Y(X)$ that solves T, $\mathcal{C}_r(T, L) = \mathcal{C}_v(T, L)$ then $\mathcal{P}_Y^r(X) = \mathcal{S}_Y^v(X)$.*

Proof. This follows trivially from the proof of Proposition 12.3. □

14. Analysis on the topology of problem spaces

In this section, we introduce and develop the analysis of the theory of problem and their solution spaces. We adapt some classical concepts in functional analysis to study problems and their corresponding solution spaces. We introduce the notion of **compactness**, **density**, **convexity**, **boundedness**, **amenability** and the **interior**. We examine the overall interplay among these concepts in theory.

14.1. Compact problems and solutions.

In this section we study the notion of compactness of problems and their corresponding solutions.

Definition 14.1. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ denotes the problem and solutions spaces, respectively, induced by providing solution X to problem Y. We say the problem space $\mathcal{P}_X(Y)$ is *compact* if and only if there exists a finite number of problem spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

Similarly, we say the solution space $\mathcal{S}_X(Y)$ is *compact* if and only if there exists a finite number of solution spaces $\mathcal{S}_{U_1}(V_1), \mathcal{S}_{U_2}(V_2), \ldots, \mathcal{S}_{U_k}(V_k)$ such that

$$\mathcal{S}_X(Y) \subset \mathcal{S}_{U_1}(V_1) \cup \mathcal{S}_{U_2}(V_2) \cup \cdots \cup \mathcal{S}_{U_k}(V_k).$$

Proposition 14.1. *Let $\mathcal{P}_X(Y)$ be a problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is compact, then the problem space $\mathcal{P}_{X_i}(Y_i)$ with $\mathcal{P}_{X_i}(Y_i)$ is also compact.*

Proof. Suppose $\mathcal{P}_X(Y)$ is compact, then it follows that for a finite $k \in \mathbb{N}$ there exists problems spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

The compactness of $\mathcal{P}_{X_i}(Y_i)$ follows trivially since $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. □

Proposition 14.2. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X and let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. If $\mathcal{P}_{X_i}(Y_i)$ is compact and principal, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose that $\mathcal{P}_{X_i}(Y_i)$, then there exists a sub-problem $X_j \leq X$ such that we can write $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \{X_j\}$. Under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, it follows that for a finite $k \in \mathbb{N}$ there exists problems spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

and we have

$$\mathcal{P}_X(Y) \subset \{X_j\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

This proves that the space $\mathcal{P}_X(Y)$ is also compact. □

Proposition 14.3. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X, where X is a regular problem. If $X_i < X$ is the maximal proper sub-problem of X and $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is also compact.*

Proof. Suppose X is regular problem and let X_i be the maximal proper sub-problem of X, then we can write $X > X_j > X_{j+1} > \cdots$ where $X_{j+n} > X_{j+n+1}$ indicates that X_{j+n+1} is the maximal proper sub-problem of X_{n+j} for $n = 1, 2, \ldots,$ by virtue of the regularity of the problem X. The sequence above contains all the sub-problems of X so that we can put $\bigcup_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \subseteq \mathcal{P}_{X_j}(Y_j)$. Since a problem is solved by providing a solution to each sub-problem and X_j is the maximal problem sub-problem of X, we deduce that $\bigcup_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \cup \{X\} \subseteq \mathcal{P}_{X_j}(Y_j) \cup \{X\} = \mathcal{P}_X(Y)$ and it follows that

$$\mathcal{P}_X(Y) \subset \{X\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

since $\mathcal{P}_{X_j}(Y_j)$ was assumed to be compact. This proves that the space $\mathcal{P}_X(Y)$ is compact. □

14.2. **Dense problems and solution spaces.** We study the concept of *density* of problems and their corresponding solution spaces in this section.

Definition 14.2. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and solution spaces, respectively, induced by providing solution Y to problem X. Let $X_i \in \mathcal{P}_X(Y)$ with an induced sub-space $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and corresponding solution space $\mathcal{S}_{X_i}(Y_i)$. We say the subspace $\mathcal{P}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{P}_X(Y)$ if and only if for any problem $Z \in \mathcal{P}_X(Y)$ with $Z \neq X$, there exists a proper subspace $\mathcal{P}_{X_j}(Y_j)$ with $Z \in \mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. Similarly, we say the subspace $\mathcal{S}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{S}_X(Y)$ if and only if for any solution $W \in \mathcal{P}_X(Y)$ with $W \neq Y$, there exists a proper subspace $\mathcal{S}_{X_j}(Y_j)$ with $W \in \mathcal{S}_{X_j}(Y_j)$ such that $\mathcal{S}_{X_i}(Y_i) \cap \mathcal{S}_{X_j}(Y_j) \neq \emptyset$.

Theorem 14.3 (Characterization theorem)**.** *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. Then $\mathcal{P}_X(Y)$ is separable if and only if it contains no dense subspace.*

Proof. Suppose the problem space $\mathcal{P}_X(Y)$ is separable, then there exists subspaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j)$ with $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Now let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$ then we must have one of these possibilities: $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$ or $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$. Suppose there exist problems $Z, U \in \mathcal{P}_{X_k}(Y_k)$ such that $Z \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$, then we have for their corresponding problem spaces induced with, say, the solutions W and T the following properties $\mathcal{P}_Z(W) \subset \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_U(T) \subset \mathcal{P}_{X_j}(Y_j)$. We know that $\mathcal{P}_U(T) \subseteq \mathcal{P}_{X_k}(Y_k)$ and $\mathcal{P}_Z(W) \subseteq \mathcal{P}_{X_k}(Y_k)$ so that we must have $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_j}(Y_j)$. Suppose without loss of generality that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{X_k}(Y_k)$ then we will have

$$\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j) = \mathcal{P}_X(Y) \subset \mathcal{P}_{X_k}(Y_k) \cup \mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$$

which is absurd. This implies that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$, which violates the requirement that $\mathcal{P}_X(Y)$ is separable. Without loss of generality, we put $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and choose a problem $V \in \mathcal{P}_{X_j}(Y_j)$ then $\mathcal{P}_V(T) \subseteq \mathcal{P}_{X_j}(Y_j)$. It follows that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_V(T) = \emptyset$ and since $V \notin \mathcal{P}_{X_l}(Y_l) \subseteq \mathcal{P}_{X_i}(Y_i)$ for subspace $\mathcal{P}_{X_l}(Y_l)$ of $\mathcal{P}_{X_i}(Y_i)$, the problem space $\mathcal{P}_{X_k}(Y_k)$ cannot be dense in $\mathcal{P}(X)(Y)$. Since $\mathcal{P}_{X_k}(Y_k)$ was an arbitrary problem subspace, it follows that the space $\mathcal{P}_X(Y)$ contains no dense sub-problem space. Conversely, suppose that the space $\mathcal{P}_X(Y)$ contains a dense problem sub-space but that the space is separable, then there exists proper sub-spaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(X_j) = \mathcal{P}_X(Y)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$ be dense in $\mathcal{P}_X(Y)$ then for $V \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$. Since these subspaces are the largest subspaces in the space $\mathcal{P}_X(Y)$ containing the problems V and U, it follows by the density of the subspace $\mathcal{P}_{X_k}(Y_k)$ that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_i}(Y_i) \neq \emptyset$ and $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. This contradicts the assumption that the space $\mathcal{P}_X(Y)$ is separable. □

14.3. **Bounded problem and solution spaces.** In this section we study the notion of *bounded* problem and solution spaces.

Definition 14.4. Let $\mathcal{P}_X(Y)$ be a problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if and only if it has finite complexity. If we denote the complexity of the space with $\mathbb{C}[\mathcal{P}_X(Y)]$, then we say $\mathcal{P}_X(Y)$ is bounded if and only if $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$. Similarly, we say the corresponding solution space $\mathcal{S}_X(Y)$ is bounded if only if it has a finite index. If we denote the index of this space with $\mathbb{I}[\mathcal{S}_X(Y)]$, then $\mathcal{S}_X(Y)$ is bounded if and only if $\mathbb{I}[\mathcal{S}_X(Y)] < \infty$.

Proposition 14.4. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ contains a reducible problem.*

Proof. Suppose each problem $X_i \in \mathcal{P}_X(Y)$ is irreducible, then we can construct the infinite nested sequence of sub-problem spaces $\cdots \subset \mathcal{P}_{X_2}(Y_2) \subset \mathcal{P}_{X_1}(Y_1) \subset \mathcal{P}_X(Y)$ with $X_1 > X_2 > \cdots$, where $X_{j+1} < X_j$ indicates that X_{j+1} is a proper sub-problem of X_j. This implies that the space $\mathcal{P}_X(Y)$ contains infinitely many problems and thus $\mathbb{C}[\mathcal{P}_X(Y)] = \infty$. □

14.4. **The interior of problem and solution spaces.** In this section we study the topological notion of *interior* of problem and solution spaces.

Definition 14.5. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and the solutions spaces induced by providing solution Y to problem X. We say a problem $Z \in \mathcal{P}_X(Y)$ is an *interior* problem if there is no problem space $\mathcal{P}_S(T)$ with $\mathcal{P}_S(T) \not\subseteq \mathcal{P}_X(Y)$ such that $Z \in \mathcal{P}_S(T)$. We call the collection of all such problems in $\mathcal{P}_X(Y)$ the interior of $\mathcal{P}_X(Y)$ and denote for this collection $Int[\mathcal{P}_X(Y)]$. We say the interior is non-empty if $Int[\mathcal{P}_X(Y)] \neq \emptyset$; otherwise, we say the interior is empty. Similarly, we say a solution $W \in \mathcal{S}_X(Y)$ is an *interior* solution if there is no solution space $\mathcal{S}_R(T)$ with $\mathcal{S}_R(T) \not\subseteq \mathcal{S}_X(Y)$ such that $W \in \mathcal{P}_S(T)$. We call the collection of all such solutions in $\mathcal{S}_X(Y)$ the interior of $\mathcal{S}_X(Y)$ and denote for this collection $Int[\mathcal{S}_X(Y)]$. We say the interior is non-empty if $Int[\mathcal{S}_X(Y)] \neq \emptyset$; otherwise, we say the interior is empty.

Theorem 14.6. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $Int[\mathcal{P}_X(Y)] = \emptyset$ and $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Suppose $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y) = \{X, X_1, \ldots, X_k\}$ for a finite $k \in \mathbb{N}$. Since $Int[\mathcal{P}_X(Y)] = \emptyset$, it follows that there exists problem spaces $\mathcal{P}_{T_1}(R_1), \ldots, \mathcal{P}_{T_k}(R_k)$ with $\mathcal{P}_{T_i}(R_i) \not\subseteq \mathcal{P}_X(Y)$ for $i = 1, \ldots, k$ such that $X_i \in \mathcal{P}_{T_i}(R_i)$ for each i. It follows that we can put $\mathcal{P}_X(Y) \subset \bigcup_{i=1}^{k} \mathcal{P}_{T_i}(R_i) \cup \{X\}$. This proves that the problem space $\mathcal{P}_X(Y)$ is compact. □

14.5. **Convex problem and solution spaces.** We introduce and study the notion of *convexity* of problems and solution spaces in this section.

Definition 14.7. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is *convex* if for any problem $X_i, X_j \in$

$\mathcal{P}_X(Y)$ $(X_i, X_j \neq X)$, there exist a problem $X_k \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{X_j\} = \{X_k\}$. Similarly, We say the solution space $\mathcal{S}_X(Y)$ is *convex* if for any solution $Y_i, Y_j \in \mathcal{S}_X(Y)$ $(Y_i, Y_j \neq Y)$, there exist a solution $Y_k \in \mathcal{S}_X(Y)$ such that $\{Y_i\} \cup \{Y_j\} = \{Y_k\}$.

The notion of *convexity* of a problem (resp. solution) spaces suggest that each problem in the *convex* problem space is a sub-problem of some problem in the space. It worth noting that convexity of problem and solutions do not unconditionally extend to *convexity* of sub-problem spaces.

Proposition 14.5. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is convex and bounded with $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, then $\mathcal{P}_X(Y)$ has a principal subspace $\mathcal{P}_{X_k}(Y_k)$ with $\mathbb{C}[\mathcal{P}_{X_k}(Y_k)] \geq 3$.*

Proof. Suppose $\mathcal{P}_X(Y)$ is bounded, then $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$ so that $\mathcal{P}_X(Y)$ contains finitely many problems. Let $X_i, X_j \in \mathcal{P}_X(Y)$ then under the requirement that $\mathcal{P}_X(Y)$ is *convex*, then $\{X_i\} \cup \{X_j\} = \{X_k\}$, where $X_k \in \mathcal{P}_X(Y)$. That is, we can merge to problems in the space to produce another problem in the space. It follows that $X_i \leq X_k$ and $X_j \leq X_k$. That is, X_i and X_j are sub-problems of X_k. By the minimality of the complexity of the space $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, we can repeat this construction by using the newly constructed problems X_k with some $X_s \in \mathcal{P}_X(Y)$ with $X_s \neq X_i, X_j$ to produce a sub-problem space which is principal and has complexity ≥ 3. □

The next result purports that each subspace of a problem space must be *dense* in their mother space.

Theorem 14.8. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is convex then every subspace $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ is dense in $\mathcal{P}_X(Y)$.*

Proof. Suppose the problem space $\mathcal{P}_X(Y)$ is *convex* and put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. Next pick a arbitrarily a problem $V \in \mathcal{P}_X(Y)$, then under the *convexity* of the space there exists a problem $W \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{V\} = \{W\}$. This implies that $X_i < W$ and $V < W$; that is, X_i and V are proper sub-problems of W. Since $W \in \mathcal{P}_X(Y)$, it has a solution so let $T \in \mathcal{S}_X(Y)$ be the solution to W and we obtain the induced problem space $\mathcal{P}_W(T) \subset \mathcal{P}_X(Y)$ with $V \in \mathcal{P}_W(T)$. Because $X_i < W$ and is the maximal sub-problem in the space $\mathcal{P}_{X_i}(Y_i)$, it follows that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_W(T)$. We find that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_W(T) \neq \emptyset$ with $V \in \mathcal{P}_W(T)$. Since V was chosen arbitrarily in the space $\mathcal{P}_X(Y)$, it follows that $\mathcal{P}_{X_i}(Y_i)$ is *dense* in $\mathcal{P}_X(Y)$. Because the sub-problem space was chosen arbitrarily, it follows that each sub-problem space is dense problem space $\mathcal{P}_X(Y)$. This completes the proof of the claim. □

14.6. **Amenable problem spaces.** In this section, we study the notion of *amenability* of problem spaces.

Definition 14.9. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution solution Y to problem X. We say the problem space $\mathcal{P}_X(Y)$ is partially *amenable* if there exist proper sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ such that X_i and X_j are equivalent problems ($X_i \equiv X_j$). We say the space $\mathcal{P}_X(Y)$ is totally *amenable* if for any sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ then $X_i \equiv X_j$. We say a problem is amenable if it is a problem in some totally *amenable* problem space.

Amenable problems are naturally easily tractable. This notion hold much significance, because if we can identify some totally amenable space that contains a specific problem then finding a solution will reduce to finding a solution to much easier problem in the same space. Subsequent studies will be devoted to a detail and much more specialized study of this important concept and its overall interplay with the theory. Next we launch a result that basically purports the compactness of a space provided one can identify a compact sub-problem space.

Theorem 14.10. *Let $\mathcal{P}_X(Y)$ be a totally amenable problem space. If there exists a sub-problem space $\mathcal{P}_{X_i}(Y_i)$ such that $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose $\mathcal{P}_X(Y)$ is an amenable space. This implies that for any problem $X_j \in \mathcal{P}_X(Y)$ then $X_j \equiv X_i$. The induced problem space $\mathcal{P}_{X_j}(Y_j)$ contains the problem X_j and it is the maximal sub-problem of this space. Since $\mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$, it follows by amenability of the space that we can replace X_j with X_i and Y_j with Y_i, since problem and solution spaces remain invariant on replacement with equivalent problems and alternative solutions, so that under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, we can put

$$\mathcal{P}_{X_j}(Y_j) = \mathcal{P}_{X_i}(Y_i) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s)$$

for a fixed $k \in \mathbb{N}$. It follows that

$$\bigcup_{i \geq 1} \mathcal{P}_{X_i}(Y_i) \cup \{X\} = \mathcal{P}_X(Y) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s) \cup \{X\}$$

for a fixed $k \in \mathbb{N}$. This proves that the problem space $\mathcal{P}_X(Y)$ is compact. □

15. Maps between problem and solution spaces

In this section, we study the analysis of map between between problem spaces and solution spaces. We examine how the notion of *boundedness* and *compactness* are preserved under the map.

Definition 15.1. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *continuous* if and only if for any subspace $\mathcal{P}_R(U) \subseteq \mathcal{P}_S(T)$ with complexity $\mathbb{C}[\mathcal{P}_R(U)] \geq k$ there exists a subspace $\mathcal{P}_W(Z) \subseteq \mathcal{P}_X(Y)$ with complexity $\mathcal{C}[\mathcal{P}_W(Z)] \geq k$ such that $f(\mathcal{P}_W(R)) \subseteq \mathcal{P}_R(U)$. Similarly, we say the map $f : \mathcal{S}_X(Y) \longrightarrow \mathcal{S}_S(T)$ between problem spaces is *continuous* if and only if for any subspace $\mathcal{S}_R(U) \subseteq \mathcal{S}_S(T)$ with index $\mathbb{I}[\mathcal{S}_R(U)] \geq k$ there exists a subspace $\mathcal{S}_W(Z) \subseteq \mathcal{S}_X(Y)$ with index $\mathbb{I}[\mathcal{S}_W(Z)] \geq k$ such that $f(\mathcal{S}_W(R)) \subseteq \mathcal{S}_R(U)$.

Definition 15.2. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *bounded* if $f(\mathcal{P}_U(T))$ is a finite subset of problems in $\mathcal{P}_S(T)$ for each bounded $\mathcal{P}_U(T) \subset \mathcal{P}_X(Y)$.

Definition 15.3. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *compact* if and only if $f(\mathcal{P}_X(Y))$ is *compact*.

We expose the fact that *compactness* of a map between problem spaces can be inherited from the compactness of the space on which it acts.

Theorem 15.4 (Stability theorem)**.** *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. If $\mathcal{P}_X(Y)$ is compact, then f is compact.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces and suppose the space $\mathcal{P}_X(Y)$ is *compact*. Then there exists a finite number of problems spaces $\mathcal{P}_{K_1}(L_1), \cdots, \mathcal{P}_{K_n}(L_n)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{K_1}(L_1) \cup \cdots \cup \mathcal{P}_{K_n}(L_n).$$

We observe that $f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_1}(L_1)) \subseteq f(\mathcal{P}_{K_1}(L_1)))$. Using this relation, we can put

$$f(\mathcal{P}_X(Y)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_j}(L_j)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_{K_j}(L_j)).$$

This proves that the range $f(\mathcal{P}_X(Y))$ is *compact* and hence f is also compact. □

16. **Isotope and Isotope problem and solution spaces**

In this section we study the notion of an *isotope* of problem and solution spaces.

Definition 16.1. Let V and U be any two problems. We say V and U are *compatible* if there exists a problem space $\mathcal{P}_X(Y)$ such that $V, U \in \mathcal{P}_X(Y)$. We denote this compatibility by $V \diamond U$ or $U \diamond V$. Similarly, we say two solutions R, S to some (possibly) distinct problems are compatible if there exists a solution space $\mathcal{S}_X(Y)$ such that $R, S \in \mathcal{S}_X(Y)$. We denote this compatibility by $R \diamond S$ or $S \diamond R$.

Definition 16.2. Let U and V be compatible problems. We say V and U admits a *merger* in the space $\mathcal{P}_X(Y)$ if there exists a problem $S \in \mathcal{P}_X(Y)$ such that $V < S$ and $U < S$ and V, U are the only maximal subproblem of S. In notation, we write $V \bowtie U = S \in \mathcal{P}_X(Y)$ or $U \bowtie V = S \in \mathcal{P}_X(Y)$. Similarly, let R and T be compatible solutions. We say R and T admits a *merger* in the space $\mathcal{S}_X(Y)$ if there exists a solution $W \in \mathcal{S}_X(Y)$ such that $R < W$ and $T < W$ and R, T are the only maximal sub-solutions of W. In notation, we write $R \bowtie T = W \in \mathcal{S}_X(Y)$ or $R \bowtie T = W \in \mathcal{P}_X(Y)$

We now launch the notion of an *isotope*.

Definition 16.3. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem space and the corresponding solution space, induced by assigning solution Y to problem X. We denote an *isotope* on $\mathcal{P}_X(Y)$ as the map $\mathrm{Iso} : \mathcal{P}_X(Y) \longrightarrow \mathbb{R}$ such that

(i) $\mathrm{Iso}(V) \geq 0$ for each $V \in \mathcal{P}_X(Y)$ and
(ii) $\mathrm{Iso}(V \bowtie U) \leq \mathrm{Iso}(V) + \mathrm{Iso}(U)$ provided $U, V \in \mathcal{P}_X(Y)$ admits a merger.

A similar axiom also holds for solution spaces.

The notion of an *isotope* may not be viewed as an abstract notion. For example, if we consider a problem $V \in \mathcal{P}_X(Y)$ with solution $U \in \mathcal{S}_X(Y)$ and the induced problem space $\mathcal{P}_V(U) \subset \mathcal{P}_X(Y)$, then we can associate a number to problem V to be

$$(\mathbb{C}[\mathcal{P}_V(U)])^{\frac{1}{\mathbb{C}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{C}[\mathcal{P}_V(U)]$ as usual denotes the complexity of the space. Similarly for a solution U in the solution space $\mathcal{S}_X(Y)$, we can assign a number to the solution U to be

$$(\mathbb{I}[\mathcal{S}_V(U)])^{\frac{1}{\mathbb{I}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{I}[\mathcal{P}_V(U)]$ as usual denotes the index of the space. One could verify that these two maps satisfy the axioms of an *isotope*. In particular, an isotope is a pseudo semi-norm.

Definition 16.4. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be a problem and a corresponding solution space whose topology admits an *isotope*. A problem (resp. solution) space equipped with an isotope is an isotope problem (resp. isotope solution) space. We denote these spaces with $(\mathcal{P}_X(Y), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_X(Y), \mathrm{Iso}(\cdot))$, respectively.

Definition 16.5. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We put the isotope of f, denoted $\mathrm{Iso}(f)$, to be

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y)\\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)}.$$

We say f is bounded if $\mathrm{Iso}(f) < \infty$. A similar characterization also holds for solution spaces.

Proposition 16.1. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. Then $\mathrm{Iso}(f) < \infty$ if and only if there exists an absolute constant $c > 0$ such that $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$.*

Proof. Suppose $\mathrm{Iso}(f) < \infty$ then by definition 16.5 there exists an absolute constant $c > 0$ such that $\frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)} \leq c$ for all $V \in \mathcal{P}_X(Y)$. It implies immediately that $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Conversely, suppose $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$ then

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y)\\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)} < \infty.$$

□

16.1. **Bounded isotope problem spaces.** In this section, we introduce and study the notion of a *bounded* isotope problem and solution spaces.

Definition 16.6. Let $\mathcal{P}_X(Y)$ be an isotope problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if $\mathrm{Iso}(V) < \infty$ for all $V \in \mathcal{P}_X(Y)$.

Remark 16.7. We now show that a bounded map between problem spaces maps bounded subspaces to a bounded set of problems.

Proposition 16.2. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Suppose $\mathcal{P}_K(L) \subset \mathcal{P}_X(Y)$ be a bounded sub-problem space. If $\mathrm{Iso}(f) < \infty$, then $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$.*

Proof. Consider the map $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ such that $\mathrm{Iso}(f) < \infty$. Then there exists an absolute constant $c > 0$ such that Iso(f(V)) $\leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. The requirement that $\mathcal{P}_K(L)$ is bounded implies that $\mathrm{Iso}(V) < \infty$ for all $V \in \mathcal{P}_K(L)$. This implies that $\mathrm{Iso}(f(V)) \leq d$ for all $V \in \mathcal{P}_K(L)$. This proves that $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$. □

A similar characterization could be made and proofs can be constructed by replacing the problem spaces $\mathcal{P}_K(L)$ with the corresponding induced solution spaces $\mathcal{S}_K(L)$.

16.2. **Continuous maps between isotope problem and solution spaces.** In this section, we introduce the notion of *continuity* of a map between isotope problem spaces.

Definition 16.8. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We say f is *continuous* if for any $\epsilon > 0$ there exists some $\delta > 0$ such that with $\mathrm{Iso}(V) < \delta$ then $\mathrm{Iso}(f(V)) < \epsilon$ for $V \in \mathcal{P}_X(Y)$.

We expose the relationship that exists between *continuity* and *boundedness* of maps between problem space. In fact, we show that these two seemingly disparate notions are equivalent in problem theory.

Theorem 16.9. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Then* $\mathrm{Iso}(f) < \infty$ *if and only if f is continuous.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Suppose that $\mathrm{Iso}(f) < \infty$, then there exists an absolute constant $c > 0$ such that $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Let $\epsilon > 0$ and choose $\delta := \frac{\epsilon}{c}$ so that with $\mathrm{Iso}(V) < \delta$ then $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V) < c\delta = \epsilon$. This proves that f is continuous. Conversely, suppose that f is continuous and assume that f is not bounded. Then for each $n \geq 1$ there exists a sequence $\{V_n\} \subset \mathcal{P}_X(Y)$ such that $\mathrm{Iso}(f(V_n)) > n\ \mathrm{Iso}(V_n)$ for all $n \geq N_o > 0$. Put $\frac{1}{n} < \mathrm{Iso}(V_n) < 1 - \frac{1}{n}$, then (by continuity) we get $1 < n\ \mathrm{Iso}(V_n) < \mathrm{Iso}(f(V_n)) < 1$, which is absurd. □

17. **Conclusion and further remarks**

This work represents a significant advancement in our understanding of problem and solution spaces, particularly with respect to their algebraic, topological, and computational properties. By introducing novel concepts such as isotopic maps, separability, amenability, and the isotope pseudo semi-norm, we have opened new avenues for exploring the intricate relationships between problems, their solution spaces, and the transformations between them.

The Characterization Theorem introduced in this work provides a robust framework for categorizing problem spaces, a critical step toward understanding the fundamental structure of problems and the conditions under which they can be solved. The study of separability and amenability within these spaces has highlighted essential conditions that influence the solvability of problems, while the examination of isotopic maps and their properties has bridged the gap between theoretical exploration and practical application, particularly in the context of time complexity. The equivalence of boundedness and continuity of isotopic maps provides a key insight into how problem spaces can be transformed while preserving their complexity, offering valuable tools for further studies in computational complexity theory.

As we have shown, the introduction of the isotope pseudo semi-norm provides a new approach for assessing the complexity of problem spaces. This measure is a vital contribution to problem theory, facilitating a deeper understanding of solvability and the structure of solution spaces.

However, despite the progress made, several open questions remain, particularly in relation to the complexity of problem transformations and the limits of current theories. The following conjectures arise naturally from the findings of this work and serve as promising directions for future research.

Conjecture 17.1. Let $\mathcal{P}$ be a problem space and $\mathrm{Iso} : \mathcal{P} \longrightarrow \mathbb{R}$. There exists a bounded isotope Iso() such that for any $p_1, p_2 \in \mathcal{P}$ with $p_1 \neq p_2$ and $p_1 \not\equiv p_2$, then $\mathrm{Iso}(p_1) \neq \mathrm{Iso}(p_2)$.

Conjecture 17.2. Let $\mathcal{P}$ be a problem space and $p_1, p_2 \in \mathcal{P}$. If $|\mathrm{Iso}(p_1) - \mathrm{Iso}(p_2)| < \epsilon$ for some small $\epsilon > 0$, then there exist absolute constants $C_1, C_2 > 0$ such that

$$\frac{\mathcal{C}_v(p_1)}{\mathcal{C}_v(p_2)} \leq C_1$$

and

$$\frac{\mathcal{C}_r(p_1)}{\mathcal{C}_r(p_2)} \leq C_2$$

[1].

References

1. Smarandache, Florentin *Only problems, not solutions!*, Infinite Study, 1991.

Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

1 .