THE ELECTROWEAK SYNCHROTRON RADIATION

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December 15, 2024

Abstract

The power spectral formula is derived for radiation of neutral electroweak bosons from an electron in a weak external homogenous electromagnetic field. The calculation method is based on the source theory formulation of quantum field theory introduced by Schwinger. For ultra-relativistic electrons, it is derived that the Z^0 emission rate is suppressed relative to photon emission. This implies that the decay of electron into W-boson and neutrino is also exponentially small under similar conditions. The article is written in the form of mathematical simplicity and the Schwinger pedagogical clarity.

1 Introduction

In modern physics, Schwinger (1945, 1949) used the relativistic generalization of the Larmor formula to get the total synchrotron radiation. Schwinger also obtained the spectrum of the synchrotron radiation from the method which was based on the electron work on the electromagnetic field, $P = -\int(\mathbf{j}\cdot\mathbf{E})d\mathbf{x}$, where the intensity of electric field he expressed as the subtraction of the retarded and advanced electric field of a moving charge in a magnetic field, $\mathbf{E} = \frac{1}{2}(\mathbf{E}_{ret} - \mathbf{E}_{adv})$, (Schwinger, 1949).

In quantum electrodynamics description of the motion of electron in a homogeneous magnetic field, the stationarity of the trajectories is broken by including the mass operator into the wave equation. Then, it is possible from the mass operator to derive the power spectral formula (Schwinger, 1973). Different approach is involved in the Schwinger et al. article (1976).

2 The electroweak synchrotron radiation

The power spectral formula is derived here for radiation of neutral electroweak bosons from an electron in a weak external homogenous electromagnetic field. The calculation method is based on the source theory formulation of quantum field theory introduced by Schwinger. For ultra-relativistic electrons, it is derived that the Z^0 emission rate is suppressed relative to photon emission. This implies that the decay of electron into W-boson and neutrino is also exponentially small under similar conditions.

In the standard unified electroweak theory (Abers et al., 1973) the electron couples to massive neutral weak boson Z^0 as well as the photon. In this treatment we generalize the synchrotron radiation in such a way that instead of the process

$$e^- \to e^- + \gamma \tag{1}$$

we will compute

$$e^- \to e^- + Z^0 \tag{2}$$

in an external homogenous electromagnetic field expressed by the tensor $F_{\mu\nu}$ using the method of Tsai and Yildiz (1973) and following the article of Chen and Noble (1985). We avoid the Klein catastrophe which is the spontaneous pair creation by an electric field, because we will consider the weak field limit

$$\left|\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right| = \left|E^2 - H^2\right| \ll F_c = \frac{m_e^2}{c} \approx 4.4 \times 10^{13}\text{G} = 1.3 \times 10^{16}\text{V/cm}, \tag{3}$$

where the pair creation effects are negligible.

The starting point of our calculation of the total Z^0-emission is the action corresponding to the exchange of the virtual Z^0 particle, i. e.

$$W = -\frac{1}{2}\int (dx)dx')\bar{\psi}(x)M(x,x')\psi(x') \tag{4}$$

where ψ is the total electron wave function:

$$\psi = \frac{1}{2}(1+\gamma_5)\psi + \frac{1}{2}(1-\gamma_5)\psi \equiv \psi_R + \psi_L \tag{5}$$

and we will see that he distinction between left and right handed couplings will be transparent in the computation. We present $M(x,x')$ as $\langle x|M|x'\rangle$ and use the standard el.-weak boson coupling (Abers et al., 1973) in order to get

$$\begin{aligned} M = i\int \frac{(dk)}{(2\pi)^4}\Big\{ & g_R^2\frac{1}{2}(1-\gamma_5)\gamma^\mu \frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\gamma_\mu\frac{1}{2}(1+\gamma_5) \quad + \\ & g_R g_L\frac{1}{2}(1-\gamma_5)\gamma^\mu \frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\gamma_\mu\frac{1}{2}(1-\gamma_5) \quad + \\ & \frac{1}{2}(1+\gamma_5)\gamma^\mu \frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\gamma_\mu\frac{1}{2}(1+\gamma_5) \quad + \\ & g_L^2\frac{1}{2}(1+\gamma_5)\gamma^\mu \frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\gamma_\mu\frac{1}{2}(1-\gamma_5)\Big\} \quad + C.T., \end{aligned} \tag{6}$$

where C. T. is the contact term determined by eq. $M = 0$ and $\partial M/\partial\gamma\Pi = 0$ at point $\gamma\Pi = -m$ for $F_{\mu\nu} = 0$ and

$$g_R = e\tan\Theta_W, \quad g_L = \frac{1}{2}e(\tan\Theta_W - \cot\Theta_W), \tag{7}$$

where Θ_W is the electroweak mixing angle and

$$\Pi_\mu = -i\partial_\mu - eqA_\mu \quad q = \pm 1. \tag{8}$$

According to the optical theorem the total decay rate $\Gamma(e^- \to e^- + Z^0)$ is related to the imaginary part of the matrix element M by

$$\Gamma = -\left(\frac{2m}{\mathcal{E}}\right) \mathrm{Im}\, M. \tag{9}$$

The γ–Z^0 interference does not occur because of real Z^0 emission. Using the proper-time technique and by replacement of the momentum integration by the algebraic procedures, we get in accordance with Chen and Noble (1985):

$$M = -\frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s}\int_0^1 du\left(\det\frac{2eqFs}{D}\right)^{1/2} e^{-is\Phi}\left\{\frac{1}{2}\left[(g_R^2+g_L^2)-(g_R^2-g_L^2)\gamma_5\right] \times \right.$$
$$\left[(-4-\mathrm{tr}A+2i\sigma A)\gamma\frac{2(1-u)eqFs}{D}\Pi + 2\gamma(1+A^T)\frac{2(1-u)eqFs}{D}\Pi\right] +$$
$$\left. g_R g_L(-4-\mathrm{tr}A+2i\sigma A)m\right\} + C.T., \tag{10}$$

where

$$A = \exp(2ueqFs) - 1 \tag{11}$$

$$D = A + 2(1-u)eqFs \tag{12}$$

$$\Phi = u(\Pi^2 + m^2 - eq\sigma F) + (1-u)M_Z^2 + \Pi\left[-\frac{1}{2eqFs}\ln\left(-\frac{D}{D^T}\right)\right]\Pi \tag{13}$$

$$\sigma A = \frac{1}{2}\sigma_{\mu\nu}A^{\mu\nu} \tag{14}$$

$$C.T. = -m_c - \zeta_c(m + \gamma\Pi) \tag{15}$$

with

$$m_c = \frac{m}{(4\pi)^2}\int_0^\infty \frac{ds}{s}\int_0^1 du e^{-is(m^2u^2+(1-u)M_Z^2)}\left[(g_R^2+g_L^2)(u-1)+4g_Rg_L\right] \tag{16}$$

and the explicit form of ζ_c it is not necessary to calculate since M will be approximated on condition $(m + \gamma M)\psi = 0$.

Now, let us consider Z^0 production in pure magnetic and pure electric field.

2.1 The homogenous magnetic field

In this case we consider the H-field in the z-direction, or, $F_{12} = -F_{21} = H$. Assuming $\Pi_3 = 0$ without loss of generality and taking the expectation values between fields obeying $(m + \gamma M)\psi = 0$, we have the matrix element M in the form

$$M = \frac{m}{(4\pi)^2} \int_0^\infty \frac{dx}{x} \int_0^1 du \exp\left(-i\frac{m^2}{eH}ux - i\frac{M_Z^2}{eH}\frac{1-u}{u}\right) \quad \times$$

$$\left\{\Delta^{-1/2} \exp\left\{i[\beta - (1-u)x]\frac{(\mathcal{E}^2 - m^2)}{eH}\right\} \left[(g_R^2 + g_L^2)\left(e^{-i\zeta(\beta+x)}(u-1) \quad + \right.\right.\right.$$

$$\left.(1-u)\left(\frac{\mathcal{E}^2 - m^2}{m^2}\right)\left(\frac{1-u}{\Delta}\cos(\beta - x) + \frac{u}{\Delta}\frac{\sin x}{x}\cos\beta - \cos(\beta + x)\right) \quad \right) +$$

$$\left.\left. 2g_R g_L \left(e^{-i\zeta(\beta-x)} + e^{-i\zeta(\beta+x)}\right)\right] - \left[(g_R^2 + g_L^2)(u-1) + 4g_R g_L\right]\right\}, \tag{17}$$

where

$$\Delta = \det\left(\frac{D}{2eqFs}\right) = (1-u)^2 + u(1-u)\frac{\sin 2x}{x} + u^2\left(\frac{\sin x}{x}\right)^2 \tag{18}$$

$$\zeta = q\sigma_3 \tag{19}$$

$$x = eHus \tag{20}$$

$$\tan\beta = (1-u)\frac{\sin x}{(1-u)\cos x + u\frac{\sin x}{x}} \tag{21}$$

$$\mathcal{E}^2 = m^2 + (2n + 1 - \xi')eH; \quad \xi' = \pm 1, \tag{22}$$

where $\mathcal{E}$ is the energy eigenvalue of the Dirac equation, ξ' is the eigenvalue of $\gamma^0\xi$ and n = 0, 1, 2, .. .

2.2 The homogenous electric field

This field is considered in the z direction with the consequence $F_{30} = -F_{03} = E$ and $E \ll F_c$ so that the spontaneous pair creation is negligible. Using the fact that the eigenvalues of the matrix $F^{\mu}_{.\nu}$ are $0, \pm E$ instead of $0, \pm iH$ for a magnetic field we can use the following substitution in eq. (18):

$$H \to iE; \quad \sin x \to \sinh x; \quad \cos x \to \cosh x; \quad \tan\beta \to \tanh\beta \tag{23}$$

and

$$x = eEus; \quad \xi = q\sigma_{03} \tag{24}$$

$$\mathcal{E}^2 - m^2 \quad \to -(p_\perp^2 + m^2), \tag{25}$$

where $p_\perp$ is the eigenvalue of Π_2.

2.3 The electron decay rate

The first step is to evaluate imaginary part of M. We will restrict the situation to the high-energy regime $\mathcal{E}/m \gg 1$ and for the weak field limit $eH/m^2 \ll 1$. Because of the presence of the exponential factor under the integrand, the x-integration is dominated by small x. Analyzing the exponential structure

$$\exp\left\{-i(m^2ux + M_Z^2(1-u)x/u) + [\beta - (1-u)x]\,\mathcal{E}^2/eH\right\} \tag{26}$$

for small x, we get that the dominant range is for

$$x \sim \frac{M_Z}{\mathcal{E}u(1-u)}\left(1-u+\frac{u^2m^2}{M_Z^2}\right)^{1/2}; \quad 0 \le u \le 1. \tag{27}$$

We divide the u-integration in three regions

i)

$$0 \le u < u_0; \quad 1 \gg u_0 \gg \frac{M_Z}{\mathcal{E}} \tag{28}$$

ii)

$$u_0 < u < 1-\varepsilon \tag{29}$$

iii)

$$1-\varepsilon < u \le 1 \tag{30}$$

where

$$\varepsilon \gg (M_Z/\mathcal{E})^2 \quad \text{if} \quad \mathcal{E}/m \ll (M_Z/m)^2; \quad \varepsilon \gg m/\mathcal{E} \quad \text{if} \quad \mathcal{E}/m \gg (M_Z/m)^2. \tag{31}$$

Only the contribution ii) are important and iii) contribution are negligible.

In order to compute the $\Gamma\left(e^- \to e^- + Z^0\right)$, we expand integrand of M for small x and define the new variables

$$x = \frac{mz}{\mathcal{E}(1-u)}\left(1+\left(\frac{M_Z}{m}\right)^2\frac{(1-u)}{u^2}\right)^{1/2} \tag{32}$$

$$\xi = \frac{2u}{3\mathcal{Y}(1-u)}\left(1+\left(\frac{M_Z}{m}\right)^2\frac{(1-u)}{u^2}\right)^{3/2}, \tag{33}$$

where

$$\mathcal{Y} = \frac{\mathcal{E}}{m}\frac{eH}{m^2}. \tag{34}$$

After some calculation we get (Chen at al., 1985)

$$\Gamma\left(e^- \to e^- + Z^0\right) =$$

$$\frac{2m^2}{\sqrt{3}(4\pi)^2\mathcal{E}}\int_0^1 du\left\{\left[(g_L^2+g_R^2)(u-1)+g_Rg_L\right]\int_\xi^\infty K_{5/3}(\eta)d\eta \quad +\right.$$

$$\left[(g_L^2+g_R^2)\left(\frac{4+2u^2-16u/3}{1-u}+\frac{2-4u/3}{u^2}\frac{M_Z^2}{m^2}\right)-8g_Rg_L\right]K_{2/3}(\xi) \quad +$$

$$\xi'\left[(g_L^2+g_R^2)(u-2)+4g_Rg_L\right]\left(1+\frac{1-u}{u^2}\frac{M_Z^2}{m^2}\right)^{1/2}K_{1/3}\Bigg\}, \tag{35}$$

where $K_\nu(\eta)$ is the modified Bessel function of the second kind.

For the special case $\mathcal{Y} \ll (M_Z/m)^2$, it is $\xi \gg 1$ and the Bessel function may be approximated by their asymptotic forms ($\sim (\pi/2\xi)^{1/2}\exp(-\xi)$). The u-integration can be done in the steepest descent approximation, with $\xi(u)$ having a minimum value $\sqrt{3}\,(M_Z^2/m^2)\,/\mathcal{Y}$ for $u \simeq 1-2m^2/M_Z^2$. The decay rate is then

$$\Gamma(e^- \to e^- + Z^0) = \frac{m^4}{4\sqrt{3}\pi\mathcal{E}M_Z^2}(g_L^2+g_R^2)\mathcal{Y}\exp\left\{-\sqrt{3}\left(M_Z^2/m^2\right)/\mathcal{Y}\right\} \tag{36}$$

where higher order terms in m^2/M_Z^2 have been neglected. This expression is valid for $\mathcal{E}/m \gg (M_Z/m)^2$.

2.4 The Z^0 power spectrum

The power spectrum $P(\omega)$ for radiation of Z^0 where ω is the Z^0 energy and $M_Z \le \omega \le \mathcal{E}$ can be determined by a simple modification of the method used to calculate the decay rate. By inserting a unit factor

$$1 = \int_{-\infty}^{\infty} d\omega\delta(\omega-k^0) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{i(\omega-k^0)\tau} \tag{37}$$

into the matrix element M in eq. (6), the spectrum $P(\omega)$ is identified with ω-integrand . It is the same procedure as one of the photon case. The result is (Chen at al., 1985):

$$P(\omega) = -\frac{2m}{\mathcal{E}}\mathrm{Im}\left(\frac{m}{(4\pi)^2}\int_0^\infty\frac{ds}{s}\int_0^1\exp\left\{-is\left[m^2u^2+M_Z^2(1-u)\right]\right\}\right. \quad \times$$

$$\left\{\Delta^{-1/2}\exp -i[\beta-(1-u)x](\mathcal{E}^2-m^2)/eH\right\} \quad \times$$

$$\left[(g_L^2+g_R^2)\left(e^{-i\xi(\beta+x)}(u-1)+(1-u)\left(\frac{\mathcal{E}^2-m^2}{m^2}\right)\right.\right. \quad \times$$

$$\left(\frac{1-u}{\Delta}\cos(\beta-x)+\frac{u}{\Delta}\frac{\sin x}{x}\cos\beta-\cos(\beta+x)\right) \quad +$$

$$\left.\frac{i}{2ms\Pi^0}e^{-i\xi(\beta+x)}\frac{d}{du}\gamma^0\right)+2g_Rg_L\left(e^{-i\xi(\beta-x)}+e^{-i\xi(\beta+x)}\right)\Big] \quad -$$

$$\left.\left[(g_L^2+g_R^2)\left(u-1+\frac{i}{2ms\Pi^0}\frac{d}{du}\gamma^0\right)+4g_Rg_L\right]\right\} \quad \times$$

$$\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{i(\omega-u\Pi^0)\tau}e^{-\frac{i\tau^2}{4s}}. \tag{38}$$

In the high-energy and the weak-field limit the τ-dependent term is

$$\int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \exp\left\{ i(\omega - u\Pi^0)\tau - \frac{i\tau^2}{4s} \right\} \approx \delta(\omega - uE), \tag{39}$$

since the Gaussian function is close to unity. The u- and z-integration give

$$P(\omega) = \frac{2m^2}{\sqrt{3}(4\pi)^2\mathcal{E}} \frac{\omega}{\mathcal{E}} \left\{ \left[(g_L^2 + g_R^2)\left(\frac{M_Z^2}{2m^2} - 1 \right) + 4g_R g_L \right] \int_{\xi'}^{\infty} K_{5/3}(\eta) d\eta \quad + \right.$$

$$\left[(g_L^2 + g_R^2)\left(4 + \left(\frac{\omega}{\mathcal{E}}\right)^2 \left(1 - \frac{\omega}{\mathcal{E}}\right)^{-1} + 2\left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right) - 8g_R g_L \right] K_{2/3}(\xi') \quad +$$

$$\left. \xi' \left[(g_L^2 + g_R^2)\left(\frac{\omega}{\mathcal{E}} - 2 \right) + 4g_R g_L \right] \left[1 - \left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right]^{1/2} K_{1/3}(\xi') \right\} \tag{40}$$

where

$$\xi' = \frac{2\omega}{3\mathcal{E}\mathcal{Y}(1 - \omega/\mathcal{E})} \left[1 + \frac{M_Z^2}{m^2} \left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \right]^3. \tag{41}$$

For $M_Z = 0$, $g_R = g_L = e$ in eq. (40) we get the current power spectrum for photon emission in a weak magnetic field. The same formula applies in a weak electric field with the replacement

$$\mathcal{Y} \to \frac{p_\perp}{m} \frac{eE}{m^2}. \tag{42}$$

For the special case $Y \ll M_Z^2/m^2$, we get the simplified formula as follows:

$$P(\omega) = \frac{2m^2}{\sqrt{3}(4\pi)^2\mathcal{E}} \frac{\omega}{\mathcal{E}} \quad \times$$

$$\left\{ (g_L^2 + g_R^2) \left[3 + \frac{M_Z^2}{2m^2} + \frac{\omega^2}{\mathcal{E}^2} \left(1 - \frac{\omega}{\mathcal{E}}\right)^{-1} + 2\left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right] - 4g_R g_L \quad + \right.$$

$$\left. \xi' \left[(g_L^2 + g_R^2)\left(\frac{\omega}{\mathcal{E}} - 2 \right) + 4g_R g_L \right] \left[1 + \left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z}{m^2} \right]^{1/2} \right\} \left(\frac{\pi}{2\xi'} \right)^{1/2} e^{-\xi'}. \tag{43}$$

This spectrum is sharply peaked near

$$\frac{\omega}{\mathcal{E}} \approx 1 - \frac{2m^2}{M_Z^2}. \tag{44}$$

Because of the large mass of the neutral weak boson synchrotron radiation by Z^0 is exponentially suppressed relative to photon emission for $\mathcal{Y} \ll (M_Z/m)^2 \sim 10^{10}$. The further discussion concerning radiation of Z^0 boson can be found in ref. (Chen at al., 1985).

3 Discussion

The quantum mechanical problem of calculating the synchrotron radiation of photons from relativistic electrons in a homogeneous external magnetic field has been solved by various authors (Sokolov et al., 1953, 1966, 1968; Klepikov, 1954; Tsai et al., 1973). Sokolov et al. (1953) utilized the Dirac wave functions of an electron in a constant magnetic field. The transition amplitude for $electron \longrightarrow electron + photon$ was computed by perturbation theory (i.e. to first order in the fine structure constant α) and the power spectrum obtained by squaring the amplitude and summing over final states. Tsai and Yildiz (1973)have presented a more efficient method for calculating radiation in external fields based on Schwingers source theory formulation of quantum field theory (Schwinger, 1969; 1970; 1973; 1989). This latter approach, which was used in this paper, eliminated the need for using wave functions by replacing the sum over final states by expectation values obtained directly from the Dirac equation.

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