Research on Infinite series involving circular cotangent function and Fourier Series

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Proof of Convergence

$$\sum_{n=1}^{\infty} \frac{\cot(\pi\sqrt{2}n)}{n^3} \quad \text{(Wolfram Alpha says diverges)}$$

Now, let's prove it converges:

$$\left|\frac{\cot(\pi\sqrt{2}n)}{n^3}\right| = \frac{1}{n^3}\frac{\left|\cos(\pi\sqrt{2}n)\right|}{\left|\sin(\pi\sqrt{2}n)\right|}$$

$$\leq \frac{1}{n^3} \cdot \frac{1}{\left|\sin(\pi\sqrt{2}n)\right|} \quad \{|\cos\theta| \leq 1\}$$

Consider $\theta \in [0, \frac{\pi}{2}]$,

$$\frac{d^2}{d\theta^2}\sin\theta = -\sin\theta \leq 0 \quad \Rightarrow \sin\theta \text{ is concave on } [0, \frac{\pi}{2}]$$

$$\sin(\lambda\frac{\pi}{2} + (1-\lambda)0) \geq \lambda\sin\frac{\pi}{2} + (1-\lambda)\sin 0 = \lambda, \quad \lambda \in [0,1]$$

Or,

$$\sin(\lambda\frac{\pi}{2}) \geq \lambda, \quad \lambda \in [0,1]$$

Let $x = \frac{\lambda}{2}$, $x \in [0, \frac{1}{2}]$,

$$\therefore \sin(\pi x) \geq 2x, \quad x \in [0, \frac{1}{2}]$$

$$\therefore |\sin(\pi\sqrt{2}n)| = |\sin(\pi(\sqrt{2}n - m))|, \, m \in \mathbb{Z}$$

Choosing m such that:

$$-\frac{1}{2} \leq \sqrt{2}n - m \leq \frac{1}{2} \quad \text{(Closest integer to } \sqrt{2}n)$$

$$|\sin(\pi(\sqrt{2}n - m))| \geq 2|\sqrt{2}n - m|$$

$$\therefore \frac{|\cot(\pi\sqrt{2}n)|}{n^3} \leq \frac{1}{n^3} \cdot \frac{1}{\left|\sin(\pi(\sqrt{2}n))\right|} = \frac{1}{n^3} \cdot \frac{1}{\left|\sin(\pi(\sqrt{2}n - m))\right|}$$

$$\leq \frac{1}{2n^3} \cdot \frac{1}{|\sqrt{2}n - m|} = \frac{1}{2n^4} \cdot \frac{1}{|\sqrt{2} - \frac{m}{n}|}$$

The last step is to employ Liouville theorem/equality: For irrational algebraic number x of degree d, there exists a constant C such that

$$\left|x - \frac{p}{q}\right| \geq \frac{C}{q^d}, \quad \forall p, q \in \mathbb{Z}, q \neq 0$$

$$\therefore |\sqrt{2} - \frac{m}{n}| \geq \frac{C}{n^2} \Rightarrow \frac{1}{|\sqrt{2} - \frac{m}{n}|} \leq \frac{n^2}{C}$$

$$\therefore \frac{|\cot(\pi\sqrt{2}n)|}{n^3} \leq \frac{n^2}{2n^4 C} = \frac{1}{2n^2 C}$$

$$\sum_{n=1}^{\infty} \frac{1}{2Cn^2} \text{ converges to } \frac{\zeta(2)}{2C} = \frac{\pi^2}{12C}$$

$$\therefore \sum_{n=1}^{\infty} \frac{\cot(\pi\sqrt{2}n)}{n^3} \text{ also converges (hence, proved its convergence).}$$

Using Fourier Series

By using Fourier series:

$$\cot(\alpha\theta) = \frac{\sin(\alpha\pi)}{\pi}\left[\frac{1}{\alpha} + 2\alpha\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2 - n^2}\cos(n\theta)\right], \quad \cos(n\pi) = (-1)^n$$

$$\Rightarrow \cot(\alpha\pi) = \frac{1}{\alpha\pi} + 2\alpha\pi\sum_{n=1}^{\infty}\frac{1}{(\alpha\pi)^2 - n^2\pi^2}$$

$$\Rightarrow \cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2 - n^2\pi^2}$$

$$\Rightarrow \frac{1}{x} - \cot(x) = -\sum_{n=1}^{\infty}\frac{2x}{x^2 - n^2\pi^2}$$

$$\Rightarrow 1 - x\cot(x) = \sum_{n=1}^{\infty}\frac{2x^2}{n^2\pi^2}\left[\frac{1}{1 - \frac{x^2}{n^2\pi^2}}\right]$$

$$\Rightarrow 1 - x\cot(x) = \sum_{n=1}^{\infty}\frac{2x^2}{n^2\pi^2}\left[1 + \frac{x^2}{n^2\pi^2} + \frac{x^4}{n^4\pi^4} + \dots\right]$$

$$\Rightarrow \frac{x^2}{3} + \frac{x^4}{45} + \dots = \left[\frac{2x^2}{\pi^2}\zeta(2) + \frac{2x^4}{\pi^4}\zeta(4) + \dots\right]$$

Comparing coefficients of x^2 and x^4,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}$$

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2}$$

Finding The Sum

Define $f(\alpha) = \sum_{n=1}^{\infty} \frac{\cot(\alpha\pi n)}{n^3}$, Target sum $= f(\sqrt{2})$.
Since $\cot((\alpha+1)\pi n) = \cot(\alpha\pi n)$,

$$\therefore f(\alpha+1) = f(\alpha) = f(\alpha-1)$$

Thus,

$$\therefore f(\sqrt{2}+1) = f(\sqrt{2}) = f(\sqrt{2}-1)$$

Now,

$$f(\sqrt{2}+1) = \sum_{n=1}^{\infty} \frac{\cot((\sqrt{2}+1)\pi n)}{n^3} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2}+1)\pi n^4} + \frac{2(\sqrt{2}+1)}{\pi} \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^2((\sqrt{2}+1)^2 n^2 - k^2)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2}+1)\pi n^4} + \frac{2(\sqrt{2}+1)}{\pi} \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left(-\frac{1}{k^2 n^2} + \frac{(\sqrt{2}+1)^2}{k^2((\sqrt{2}+1)^2 n^2 - k^2)} \right)$$

$$= \frac{\pi^3}{90(1+\sqrt{2})} + \frac{2(1+\sqrt{2})}{\pi} \left\{ -\zeta^2(2) \right\} + \frac{2(1+\sqrt{2})^3}{\pi} \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{k^2((1+\sqrt{2})^2 n^2 - k^2)}$$

$$= \frac{\pi^3}{90(1+\sqrt{2})} - \frac{(\sqrt{2}+1)\pi^3}{18} - \frac{2(1+\sqrt{2})}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2}-1)^2 k^2 - n^2}$$

$$= \frac{\pi^3}{90(1+\sqrt{2})} - \frac{(\sqrt{2}+1)\pi^3}{18} + \frac{2(1+\sqrt{2})}{2\pi(\sqrt{2}-1)^2} \sum_{k=1}^{\infty} \frac{1}{k^4} - \frac{\sqrt{2}+1}{\sqrt{2}-1} \sum_{k=1}^{\infty} \frac{\cot((\sqrt{2}-1)\pi k)}{k^3}$$

$$\therefore f(\sqrt{2}) = \frac{\pi^3}{90(\sqrt{2}+1)} - \frac{(\sqrt{2}+1)\pi^3}{18} + \frac{(\sqrt{2}+1)^3\pi^3}{90} - (\sqrt{2}+1)^2 f(\sqrt{2}-1)$$

$$\Rightarrow f(\sqrt{2}) \left\{ 1 + (\sqrt{2}+1)^2 \right\} = \frac{\pi^3(\sqrt{2}-1) - \pi^3(\sqrt{2}+1)^3 5 + \pi^3(\sqrt{2}+1)^3}{90}$$

$$\therefore f(\sqrt{2}) = \frac{\pi^3\sqrt{2}}{360}$$

Or,

$$\sum_{n=1}^{\infty} \frac{\cot(\pi\sqrt{2}n)}{n^3} = \frac{\pi^3\sqrt{2}}{360}$$

Thus, we found the required sum.
Done By Pratham Prasad.