On the Elliptic Integral

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Abstract

$\int \sqrt{f(x)^2 + g(x)^2}dx$ and $\int \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ are integral equations that appear in the process of calculating the length of an ellipse. They do not provide an exact solution, and already known solutions are somewhat complicated. The solutions both the two elliptic integral equations above can be obtained by the method of eliminating the square root. The arc length of an ellipse is given by $l = a\theta E(k)$, and $E(k)$ is expressed as a power series regarding the eccentricity of the ellipse.

In mathematics, the distance between coordinates is calculated using the Pythagorean theorem.

The distance between Cartesian coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{1}$$

In the parametric equation $x = f(t)$, $y = g(t)$, the distance between the interval $a < t < b$ can be written as follows by using Green's theorem,

$$l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\,. \tag{2}$$

If the given curve is a circle, the distance can be calculated easily. In case of an ellipse, the arc length can be calculated using the elliptic integral presented by A.M. Legendre[2] or "On the arc length of an ellipse"

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[2] Exercices de Calcul Intégral, Première partie des fonctions elliptiques

presented by the author[3].

However, in cases other than the above, it is not easy to find a solution to the equation for finding the distance above if the square root is not removed. Therefore, we need to find a solution to the above equation using an expansion series by removing the square root of the given integral.

In equation (2), if $dx/dt = dy/dt$, we have the following solution,

$$l = \int_a^b \sqrt{2}dx\,, \qquad or \qquad l = \int_a^b \sqrt{2}dy\,. \tag{3}$$

If $dx/dt \neq dy/dt$, then

$$l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{1 + f^2}\left(\frac{dx}{dt}\right) dt, \tag{4}$$

where,

$$f = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}. \tag{5}$$

Eliminating the square root of the above, we get an expansion series,

$$l = \int_a^b \sqrt{1 + f^2}dx \tag{6}$$

$$= \int_a^b \left(1 + \frac{1}{2}f^2 - \frac{1}{8}f^4 + \frac{1}{16}f^6 - \frac{5}{128}f^8 + \frac{7}{256}f^{10} - \frac{21}{1024}f^{12} + \cdots\right) dx$$

$$= \int_a^b \left(1 - (-1)^n \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{f^{2n}}{2n-1}\right) dx.$$

where $(2n-1)!!$ is the double factorial for odd numbers and $(2n)!!$ the double factorial for even numbers.

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3 https://vixra.org/abs/2407.0173, Jul. 29, 2024

For instance, let $f = k\sinh x$, where $0 < k < 1$, we may get a solution as follows

$$l = \int_0^a \sqrt{1 + k^2 \sinh^2 x}\, dx, \qquad 0 < k < 1, \tag{7}$$

$$= \int_0^a \left(1 + \frac{1}{2}k^2 \sinh^2 x - \frac{1}{8}k^4 \sinh^4 x + \frac{1}{16}k^6 \sinh^6 x - \frac{5}{128}k^8 \sinh^8 x + \cdots\right) dx$$

$$= \left[\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{2^{14}}k^8 - \frac{441}{2^{16}}k^{10} - \cdots\right) x\right]_0^a$$
$$+ \left[\left(\frac{1}{8}k^2 + \frac{1}{32}k^4 + \frac{15}{2^{10}}k^6 + \frac{35}{2^{12}}k^8 + \frac{735}{2^{17}}k^{10} + \cdots\right)\sinh(2x)\right]_0^a$$
$$- \left[\left(\frac{1}{256}k^4 + \frac{3}{2^{10}}k^6 + \frac{35}{2^{14}}k^8 + \frac{105}{2^{16}}k^{10} + \cdots\right)\sinh(4x)\right]_0^a$$
$$+ \left[\left(\frac{1}{3072}k^6 + \frac{5}{12288}k^8 + \frac{105}{2^{18}}k^{10} + \cdots\right)\sinh(6x)\right]_0^a$$
$$- \left[\left(\frac{5}{2^{17}}k^8 + \frac{35}{2^{19}}k^{10} + \cdots\right)\sinh(8x)\right]_0^a + \cdots$$
$$= aE(k) + \left(\frac{1}{8}k^2 + \frac{1}{32}k^4 + \frac{15}{2^{10}}k^6 + \frac{35}{2^{12}}k^8 + \frac{735}{2^{17}}k^{10} + \cdots\right)\sinh(2a)$$
$$- \left(\frac{1}{256}k^4 + \frac{3}{2^{10}}k^6 + \frac{35}{2^{14}}k^8 + \frac{105}{2^{16}}k^{10} + \cdots\right)\sinh(4a)$$
$$+ \left(\frac{1}{3072}k^6 + \frac{5}{12288}k^8 + \frac{105}{2^{18}}k^{10} + \cdots\right)\sinh(6a)$$
$$- \left(\frac{5}{2^{17}}k^8 + \frac{35}{2^{19}}k^{10} + \cdots\right)\sinh(8a) + \cdots,$$

where $E(k)$ is the eccentric expansion series

$$\begin{aligned} E(k) &= 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{2^{14}}k^8 - \frac{441}{2^{16}}k^{10} - \cdots \\ &= 1 - \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{2n-1}. \end{aligned} \tag{8}$$

which is given when $a \to 0$ of the above solution. As n increases, $E(k)$ decreases gradually but does not converge. As a increases, it diverges quickly.

In case the equation (2) is an elliptic integral, the arc length of an ellipse can be obtained as follows.

Let $x = a\cos\theta$, and $y = b\sin\theta$, then the elliptic integral becomes

$$l = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta = a \int_0^{2\pi} \sqrt{1 - k^2 \cos^2 \theta}\, d\theta \tag{9}$$

where a represents the semi-major axis and b the semi-minor axis, $k = \frac{\sqrt{a^2-b^2}}{a}$ the eccentricity of an ellipse. From the equation (6), if $f = i\, k\cos\, \theta$, we get the same result. The expansion series $F(k)$ of the above is given as

$$\begin{aligned} F(k) &= \sqrt{1 - k^2 \cos^2 \theta} \\ &= 1 - \frac{1}{2}k^2 \cos^2 \theta - \frac{1}{8}k^4 \cos^4 \theta - \frac{1}{16}k^6 \cos^6 \theta - \frac{5}{128}k^8 \cos^8 \theta - \cdots \\ &= 1 - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{k^{2n} \cos^{2n} \theta}{2n-1}. \end{aligned} \tag{10}$$

By integrating the equation (10), we have

$$\begin{aligned} l &= a \int_0^{2\pi} \sqrt{1 - k^2 \cos^2 \theta}\, d\theta \\ &= a \int_0^{2\pi} F(k)\, d\theta \end{aligned} \tag{11}$$

$$\begin{aligned} &= a \left[\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{2^{14}}k^8 - \frac{441}{2^{16}}k^{10} - \frac{4851}{2^{20}}k^{12} - \cdots \right) \theta \right]_0^{2\pi} \\ &\quad - \left[\left(\frac{2}{2^4}k^2 + \frac{8}{2^8}k^4 + \frac{60}{2^{12}}k^6 + \frac{560}{2^{16}}k^8 \ldots \right) \sin 2\theta \right]_0^{2\pi} \\ &\quad - \left[\left(\frac{1}{2^8}k^4 + \frac{12}{2^{12}}k^6 + \frac{140}{2^{16}}k^8 + \frac{1680}{2^{20}}k^{10} \ldots \right) \sin 4\theta \right]_0^{2\pi} \\ &\quad - \left[\left(\frac{1}{3072}k^6 + \frac{5}{12288}k^8 + \frac{105}{262144}k^{10} + \frac{385}{1048576}k^{12} \ldots \right) \sin 6\theta \right]_0^{2\pi} - \cdots \\ &= 2a\pi E(k), \end{aligned}$$

where,

$$\begin{aligned} E(k) &= 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{2^{14}}k^8 - \frac{441}{2^{16}}k^{10} - \frac{4851}{2^{20}}k^{12} - \cdots \\ &= 1 - \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{k^{2n}}{2n-1}. \end{aligned} \tag{12}$$

Therefore, the full arc length of an ellipse is given as $l = 2a\pi E(k)$. And the eccentric expansion series $E(k)$ is given by the above.

As k lies between $0 < k < 1$, we can have the limit for k in Eq. (12)

$$\lim_{k\to+0} l = \lim_{k\to+0}(2a\pi E(k)) = 2a\pi,$$
$$\lim_{k\to+1} l = \lim_{k\to+1}(2a\pi E(k)) \cong 1.279558a\pi. \tag{13}$$

As the limit value $k = 1$ of $E(k,n)$ is $E(1,0) = 1$, $E(1,1) = 0.75$, and $E(1,100) \cong 0.639779$ when $n = 100$, the plot does not converge but decreases gradually as n increases.

Reference

[1] A. M. Le Gendre, ''Exercices de Calcul Intégral sur Divers Ordres de Transcendantes et sur les Quadratures", 1811

[2] T. C. Yoon, ''On the arc length of an ellipse" https://vixra.org/abs/2407.0173, Jul. 29, 2024

[3] Elliptic integral, https://en.wikipedia.org/wiki/, Access Date: Apr. 25, 2024

[4] Ellipse, https://en.wikipedia.org/wiki/, Access Date: Apr. 25, 2024

[5] Double factorial, https://en.wikipedia.org/wiki/, Access Date: Apr. 25, 2024