Derivation of a Generalized Wave Equation for Photons and Its Connection to Maxwell's Equations

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Abstract

This work presents a generalized wave equation for photons derived from their relativistic energy-momentum relation. Using a scalar potential, the equation extends to include spatial and temporal variations, providing a comprehensive framework for photon dynamics. A parametric formulation simplifies the solution process, and in free space, the equation naturally reduces to Maxwell's equations. The step-by-step derivation elucidates the connections between classical electrodynamics and quantum-inspired wave mechanics.

1 Introduction

Schrödinger's equation describes the quantum behavior of massive particles but is unsuitable for photons due to its reliance on a non-relativistic dispersion relation. For massless particles like photons, the relativistic energy-momentum relation:

$$E^2 = p^2c^2,$$

forms the foundation of a wave equation tailored to photons. This paper derives such an equation, extending it to include spatially varying potentials. A parametric form simplifies the solution process, and in free space, the equation reduces to Maxwell's equations, providing a unifying perspective between quantum and classical descriptions of light.

2 Derivation of the Generalized Wave Equation

2.1 Energy-Momentum Relation for Photons

The energy and momentum of a photon are related by:

$$E = \hbar\omega, \quad p = \frac{E}{c} = \frac{\hbar\omega}{c}.$$

The relativistic dispersion relation:

$$E^2 = p^2c^2,$$

governs photon behavior.

2.2 Operator Substitutions

Promote energy and momentum to operators in the quantum framework:

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -i\hbar\nabla.$$

Substituting these into the energy-momentum relation gives:

$$\hat{E}^2\psi = c^2\hat{\mathbf{p}}^2\psi,$$

or equivalently:

$$-\hbar^2c^2\nabla^2\psi = \hbar^2\frac{\partial^2\psi}{\partial t^2}.$$

This is the free-space wave equation for photons.

2.3 Incorporating a Scalar Potential

To describe interactions, we introduce a scalar potential $V(\mathbf{r})$, modifying the total energy:

$$E_{\text{tot}} = E - V(\mathbf{r}).$$

Squaring both sides gives:

$$E_{\text{tot}}^2 = (E - V(\mathbf{r}))^2 .$$

Substituting quantum operators yields:

$$\hat{E}^2\psi = (\hbar\omega - V(\mathbf{r}))^2 \psi.$$

Thus, the generalized wave equation becomes:

$$-\hbar^2 c^2 \nabla^2 \psi = (\hbar\omega - V(\mathbf{r}))^2 \psi.$$

2.4 Time-Independent Form

For stationary states where $\psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-i\omega t}$, the time-independent form is:

$$-\hbar^2 c^2 \nabla^2 \psi(\mathbf{r}) = (E - V(\mathbf{r}))^2 \psi(\mathbf{r}).$$

3 Parametric Formulation

3.1 Separation of Variables

Assume a separable solution:

$$\psi(r, \theta, \phi, t) = R(r)\Theta(\theta, \phi)T(t).$$

Substituting into the wave equation and dividing through by ψ, we obtain:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r^2\Theta}\nabla_\Omega^2\Theta - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0,$$

where ∇_Ω^2 is the angular Laplacian. Separate variables:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{\ell(\ell+1)}{r^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2.$$

3.2 Radial and Angular Equations

The radial equation becomes:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]R = 0.$$

The angular equation is:

$$\nabla_\Omega^2\Theta + \ell(\ell+1)\Theta = 0,$$

with solutions given by spherical harmonics $Y_\ell^m(\theta, \phi)$.

4 Connection to Maxwell's Equations

4.1 Vector Form of the Wave Equation

The generalized wave equation for photons can be extended to a vector field $\mathbf{\Psi}(\mathbf{r}, t)$, which represents the electromagnetic wave:

$$-\hbar^2 c^2 \nabla^2 \mathbf{\Psi} = \hbar^2 \frac{\partial^2 \mathbf{\Psi}}{\partial t^2}.$$

Dividing through by $\hbar^2$ gives:

$$\nabla^2 \mathbf{\Psi} - \frac{1}{c^2} \frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = 0,$$

where $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the speed of light, and μ_0 and ε_0 are the permeability and permittivity of free space, respectively.

This is the vector wave equation, describing the propagation of $\mathbf{\Psi}$ in free space.

4.2 Defining the Electromagnetic Fields

The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the potentials:

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

where: - $\phi(\mathbf{r}, t)$ is the scalar potential, - $\mathbf{A}(\mathbf{r}, t)$ is the vector potential.

In the Lorenz gauge:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0,$$

the wave equations for ϕ and $\mathbf{A}$ decouple:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0.$$

4.3 Defining the Riemann-Silberstein Vector

The complex vector $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$ (the Riemann-Silberstein vector) combines $\mathbf{E}$ and $\mathbf{B}$. Substituting $\mathbf{\Psi} = \mathbf{F}$ into the vector wave equation gives:

$$\nabla^2 \mathbf{F} - \frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} = 0.$$

Separating real and imaginary parts: - The real part ($\mathbf{E}$) satisfies:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

- The imaginary part ($\mathbf{B}$) satisfies:

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$

4.4 Recovering Maxwell's Equations

Maxwell's equations in free space are:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

From the wave equation for $\mathbf{\Psi}$, we can derive these step by step.

4.4.1 Gauss's Law for E

The wave equation for $\mathbf{E}$:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

implies $\nabla \cdot \mathbf{E} = 0$ in free space because there are no sources (charges).

4.4.2 Gauss's Law for B

Similarly, the wave equation for $\mathbf{B}$:

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0,$$

implies $\nabla \cdot \mathbf{B} = 0$ in free space because magnetic monopoles do not exist.

4.4.3 Faraday's Law

Using the definition of $\mathbf{B} = \nabla \times \mathbf{A}$, take the curl of $\mathbf{E}$:

$$\nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t} \right).$$

Because the curl and time derivative commute:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t}.$$

4.4.4 Ampère-Maxwell Law

From the wave equation for $\mathbf{B}$:

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$

Substituting $c^2 = \frac{1}{\mu_0 \varepsilon_0}$:

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

4.4.5 Summary of Results

The vector wave equation for photons in free space produces:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

These are Maxwell's equations in free space.

5 Conclusion

We derived a generalized wave equation for photons based on their energy-momentum relation and extended it to include scalar potentials. By adopting a parametric approach, we simplified the analysis of spatial and temporal components. In free space, the wave equation reduces to Maxwell's equations, bridging quantum mechanics and classical electromagnetism. This framework provides a foundation for further exploration of photon dynamics in complex environments.