The Geodesic Principle and the Nature of Passive Mass

R. Wolnik M. Wolnik

Email: wolnik@akademiarac.edu.pl

Abstract

The geodesic principle represents an essential aspect of general relativity and is the physical manifestation of the space-time manifold but may also be considered as the metric field effect on the passive mass of a freely falling test particle. – The equation of motion is derived from the stress-energy tensor field of an isolated body on the basis of the covariant conservation condition, using its moments in the near-limit approximation. Then, the reduced stress-energy tensor, grounded in the spatial energy density of the body, is used in the context of its total energy balance to obtain the global solution in the form of the geodesic equation. Ultimately, the potential implications for understanding the background of the weak equivalence principle along with the influence of an external force field on such a solution are presented.

I. Introduction

In A. Einstein and N. Rosen "The Particle Problem in the General Theory of Relativity" one can read: "One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic."[1]. This postulate states: *Free massive point particles traverse timelike geodesics.* Einstein tried to remedy that shortcoming without success. "Over the last century numerous ostensible proofs claiming to have derived the geodesic principle from Einstein's field equations have been developed. (…) Grouping these results into three major families, which I refer to as (1) limit operation proofs, (2) 0th-order proofs, and (3) singularity proofs, (…) none of these strategies successfully demonstrates the geodesic principle, canonically interpreted as a dynamical law that massive bodies must actually follow geodesic paths in Einstein's theory."[2] "By reviewing the three major classes of proof, we have seen that would-be geodesic following bodies are forced either (i) to meet unrealistically restrictive special-case conditions, (ii) to have no matter-energy at all (i.e. vanish), (iii) to violate Einstein's field equations, or (iv) to be located on paths that don't just fail to be geodesic but fail to exist in the space-time manifold at all."[2] "Though the geodesic principle can be recovered as theorem in general relativity, it is not a consequence of Einstein's equation (or the conservation principle) alone. Other assumptions are needed to drive the theorems in question."[3]. – The following is a proof of the geodesic principle and its consequences for the understanding of passive mass. The proof is not canonical in the sense that it does not directly confirm the solution but rather its sufficient convergence, which is linked to the diameter: Ø of a spatial domain that encompasses the body and the current point on the geodesic. The requirement here is that the solution bound is at most O(Ø). The proof is formulated through density moments and not distributions, still it can be assigned to the limit operation proof family. In contrast to the Geroch-Jang theorem, it is advantageous that it does not require the "strengthened dominant energy condition"[3]. Solely the natural condition of the body's minimal positive energy: $E_0 = mc^2$ in a locally inertial (LI) proper frame of reference is employed. It is also presumed that, in the vicinity of the geodesic excluding the gravitational effect of the body itself, the given metric field function is sufficiently smooth. Furthermore, compatibility with the weak equivalence principle is required. The physically relevant case in which the body's density is constrained ($m = O(d^3)$) is analyzed here. It is demonstrated that even for $m = O(d)$, the gravitational field generated by the body remains effectively separated from the external gravitational field. This separation, enabled by the sufficient rate of convergence, limits the test body problem, so that it plays only a marginal role in the overall solution. The question of whether the solution converges to a geodesic at all when the mass is bounded by $O(d^0)$, which would correspond to the canonical account [2], remains open here. – In the first part: (1,2) an appropriate stationary (S)LI coordinate system is constructed. In the second part: (3,4) the approximation - uncertainties, errors, and the deviation of the four-momentum derivative are estimated. Lastly, in the third part: (5) the geodesic principle is confirmed with the SE-tensor and the geodesic equation is derived from the reduced SE-tensor. For simplicity, natural units are adopted throughout the following sections. Additionally, in summation notation, the corresponding indices are visibly crossed out when summing to provide a clearer overview.

II. The physics behind the geodesic principle

1) The locally gauged, stationary locally (within the $\Delta\tau$) inertial coordinate system: $\underline{x}^{\hat{\mu}} : \mathcal{P} \mapsto x^{\hat{\mu}}(\mathcal{P})$

▷ a) A space-time coordinate system: $\underline{x}^{\mu}$ with its basis: $\boldsymbol{e}_{\alpha}$ and the metric: $g_{\alpha\beta}$. $\eta \equiv [\mathrm{diag}(-1,1,1,1)]$

$$\tau \in \mathbb{R} \;;\; \underline{x}^{\nu} : \; \forall x_{\prime}^{\nu}(\mathcal{P}(\tau)) \; \exists \Lambda_{\mu}^{\mu'} (x^{\nu} \to x_{\prime}^{\nu}) \;,\; g_{\alpha\beta} \equiv \boldsymbol{e}_{\alpha} \cdot \boldsymbol{e}_{\beta} = \Lambda_{\alpha}^{\alpha'} \Lambda_{\beta}^{\beta'} \eta_{\alpha'\beta'} \;; \qquad (1.1)$$

▷ b) For *any* $\underline{x}^{\mu}$, the *stationary locally inertial* (SLI) coordinate system: $\underline{x}^{\hat{\mu}}$ is (implicitly) pre-defined

(1.10b) $$x^{\hat{0}} := \tau \quad \to \quad x^{\tilde{0}} = \tau \mid x^{\tilde{n}} = 0 \quad \to \quad \mathcal{P}(\tau) := \mathcal{P}(x^{\tilde{0}}, x^{\tilde{n}} = 0) \qquad (1.2a,c)$$

$$\mathbf{e}_{\tilde{\alpha}} \equiv \mathbf{e}_{\tilde{\alpha}}(\tau) := \boldsymbol{e}_{\tilde{\alpha}}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) \quad ; \quad \Lambda_{\tilde{\mu}}^{\mu} \equiv \Lambda_{\tilde{\mu}}^{\mu}(\tau) := \Lambda_{\tilde{\mu}}^{\mu}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) \qquad (1.3a,b)$$

(2.5a) (2.5b) $$\boldsymbol{\Delta}\mathcal{P}(\tau) = \boldsymbol{e}_{\alpha}(\mathcal{P}(\tau))\Lambda_{\tilde{0}}^{\alpha}(\tau)\Delta\tau := \mathbf{e}_{\tilde{0}}(\tau)\Delta\tau \mid \Delta\tau \to 0 \quad \to \quad \frac{\text{đ} x_{\prime}^{\mu}(\mathcal{P}(\tau))}{\text{đ}\tau} = \Lambda_{\tilde{0}}^{\mu}(\tau) \qquad (1.4a,b)$$

(1.8a) (1.9a) $$x^{\mu} =: x_{\prime}^{\mu}(\mathcal{P}(\tau)) + \Lambda_{\tilde{n}}^{\mu}(\tau)x^{\tilde{n}} + 2^{-1}\Lambda_{\tilde{n},\tilde{m}}^{\mu}(\tau)x^{\tilde{n}}x^{\tilde{m}} \mid 2|x^{\tilde{k}}| \leq \emptyset_0 : \text{"small enough"} \qquad (1.5)$$

▷ c) Conditions for the SLI basis in the (infinitesimal) *proximity*: $x^{\tilde{n}} \to 0$; *of* any point: $\mathcal{P}(\tau)$ of the trajectory following the geodesic (a kind of situation like inside a freely moving non-rotating spaceship)

$$\mathrm{g}_{\tilde{\alpha}\tilde{\beta}}(\tau) := g_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) \equiv \mathbf{e}_{\tilde{\alpha}} \cdot \mathbf{e}_{\tilde{\beta}} := \eta_{\tilde{\alpha}\tilde{\beta}} \quad \to \quad \mathbf{e}_{\tilde{0}} \cdot \mathbf{e}_{\tilde{0}} = -1 \qquad (1.6a,b)$$

(1.6b) (2.5b) $$\mathbf{e}_{\tau}(\tau) := \mathbf{e}_{\tilde{0}}(\tau) \quad : \quad \frac{\text{đ}\mathbf{e}_{\tau}}{\text{đ}\tau} \equiv \frac{\partial \mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{0}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) = 0 \qquad (1.7a,b)$$

(1.2b) (1.6a) $$\frac{\text{đ}\Lambda_{\tilde{\nu}}^{\mu}}{\text{đ}\tau} \equiv \frac{\partial \Lambda_{\tilde{\nu}}^{\mu}}{\partial x^{\tilde{0}}} \equiv \frac{\partial \Lambda_{\tilde{0}}^{\mu}}{\partial x^{\tilde{\nu}}} \; : \; \frac{\partial \mathbf{e}_{\tilde{\alpha}}}{\partial x^{\tilde{0}}} \equiv \frac{\partial \mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{\alpha}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) = 0 \qquad (1.8a,b)$$

$$\to \quad \Lambda_{\tilde{m},\tilde{n}}^{\mu} \equiv \Lambda_{\tilde{n},\tilde{m}}^{\mu} := \frac{\partial \Lambda_{\tilde{n}}^{\mu}}{\partial x^{\tilde{m}}} \; : \; \frac{\partial \mathbf{e}_{\tilde{\alpha}}}{\partial x^{\tilde{m}}} \equiv \frac{\partial \mathbf{e}_{\tilde{m}}}{\partial x^{\tilde{\alpha}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}},\; x^{\tilde{n}} \to 0) = 0 \qquad (1.9a,b)$$

▷ d) The SLI gauge transformation of the $\underline{x}^{\tilde{\mu}}$ SLI coordinates and the SLI Lorentz gauge as its example

$$2|x^{\tilde{k}}| \leq \emptyset_0 \quad \Rightarrow \quad x^{\hat{\mu}} := x^{\tilde{\mu}} + \xi^{\tilde{\mu}} \mid \xi^{\tilde{\mu}}(\tau, 0) = 0 \;,\; \xi^{\tilde{\mu}}{}_{,\tilde{\nu}}(\tau, 0) = 0 \;,\; \xi^{\tilde{\mu}}{}_{,\tilde{\nu},\tilde{\kappa}}(\tau, 0) = 0 \qquad (1.10a,b)$$

[5] $$\hat{\xi}^{\tilde{\mu}}(\tau, x^{\hat{n}}) : \; \bar{h}^{\hat{\alpha}\hat{\nu}}{}_{,\hat{\nu}} = 0 \quad \leftarrow \quad \bar{h}^{\hat{\alpha}\hat{\beta}} := h^{\hat{\alpha}\hat{\beta}} - 2^{-1}\eta^{\hat{\alpha}\hat{\beta}}\eta^{\hat{\alpha}\hat{\beta}}h_{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad h_{\hat{\alpha}\hat{\beta}} := \Delta g_{\hat{\alpha}\hat{\beta}} \qquad (1.11a,b,c)$$

2) General definitions in the context of the body's stress-energy (SE-)tensor field: $T^{\mu\nu}(x^{\mu}) := T^{\nu\mu}(x^{\mu})$

▷ a) A closed spatial region: $\underline{V}(\tau) \in \underline{\boldsymbol{V}}$ of the *minimal* diameter: Ø containing the whole body *and* $\mathcal{P}(\tau)$

$$\underline{V} := \underline{V}(\tau) : \; \underline{V} \cup \partial\underline{V} = \underline{V} \;,\; \mathcal{P}(x^{\hat{\mu}}) \in \underline{V}(\tau) \;\Rightarrow\; \mathcal{P}(x^{\hat{\mu}}) = \mathcal{P}(\tau, x^{\hat{n}}) \;; \qquad (2.1)$$

$$\underline{V}(\tau) : \; \mathcal{P}(\tau, x^{\hat{n}}) \in (\partial\underline{V} \cup \overline{\underline{V}}) \;\Rightarrow\; T^{\alpha\beta}(\tau, x^{\hat{n}}) = 0 \qquad (2.2)$$

The body diameter: $$\mathrm{d} := \mathrm{d}(\tau) :\leq \emptyset := \emptyset(\tau) := \emptyset(\underline{V}(\tau)) :\leq \frac{1}{2}\emptyset_0 \qquad (2.3)$$

▷ b) The notation of a spatial volume integral over $\underline{V}$, which is embedded in its space-time domain: $\underline{\boldsymbol{V}}$

$$< f > := \int_{\underline{V}} f \, |\text{đ}\underline{V}| \quad \leftarrow \quad |\text{đ}\underline{V}| := \sqrt{-g}\, |\text{đ}x^1 \text{đ}x^2 \text{đ}x^3| \;$$: a proper spatial volume element (2.4a,b)

▷ c) Synchronizing (initial) condition for $\underline{x}^{\hat{\mu}} \mid x^{\hat{0}} = \tau_0$, which codetermine the matrix: $\Lambda_{\tilde{\nu}}^{\mu}$ at $\tau = \tau_0$

(5.2) (1.4,5)
$$\begin{cases} \mathcal{P}(\tau_0) \; : \; < x^{\hat{n}} T^{\hat{0}\hat{0}}(\tau_0, x^{\hat{n}}) > = 0 & \leftarrow \; \text{1st moments of } T^{\hat{0}\hat{0}} \; \leftarrow \; \text{position: } x^{\hat{n}} \qquad (2.5a) \\ \mathbf{e}_{\tilde{0}}(\tau_0) : \; < T^{\hat{n}\hat{0}}(\tau_0, x^{\hat{n}}) > = 0 & \leftarrow \; \text{0th moment of } T^{\hat{n}\hat{0}} \; \leftarrow \; \text{velocity} \qquad (2.5b) \end{cases}$$

If the SLI coordinate system satisfies this condition at $\tau = \tau_0$, it represents the (locally inertial momentarily comoving) *proper frame* (of reference) on $\underline{V}(\tau \to \tau_0)$; and, as long as $2\emptyset \leq \emptyset_0$ holds, the *locally inertial comoving frame* (of reference). The coordinate-invariant parameter: τ is called the proper time.

3) The locally inertial coordinates: the vector integration and the SE-tensor divergence spatial integral

▷ a) The flat basis approximation and factoring of the local coordinate basis out of the spatial integral
The norm: $|x^{\bar{n}}|$ is defined as the *spatial* distance, which is the shortest length from $\mathcal{P}(\tau, x^{\bar{n}})$ to $\mathcal{P}(\tau, 0^{\bar{n}})$.

$$\mathbf{e}_\alpha := \boldsymbol{e}_\alpha\big(x^\nu \to x,^\nu(\mathcal{P}_0)\big) \quad \to \quad \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \equiv \Gamma^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma \equiv \Gamma^\gamma{}_{\beta\alpha}\mathbf{e}_\gamma := \Gamma^\gamma{}_{\alpha\beta}\big(x^\nu \to x,^\nu(\mathcal{P}_0)\big)\,\mathbf{e}_\gamma \qquad (3.1a,b)$$

$$\boldsymbol{e}_\alpha = \mathbf{e}_\alpha + \Gamma^\mu{}_{\alpha\gamma}\mathbf{e}_\mu x^\gamma + \tilde{O}\left(2^{-1}\left|\Gamma^\mu{}_{\alpha\kappa,\gamma} + \Gamma^\beta{}_{\alpha\kappa}\Gamma^\mu{}_{\beta\gamma}\right|\mathbf{e}_\mu |x^\kappa x^\gamma|\right) \qquad (3.2)$$

$$\underline{x}^\mu \to \underline{x}^{\bar{\mu}}(\mathcal{P}_0) \;:\; x^{\bar{\nu}}(\mathcal{P} = \mathcal{P}_0) = 0 \;,\; \Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}} = 0 \;,\; g_{\bar{\mu}\bar{\nu};\bar{\beta}} \equiv 0 \quad \to \quad g_{\bar{\mu}\bar{\nu},\bar{\beta}} = 0 \qquad (3.3a,b)$$

$$\boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}\big(2^{-2}|(\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa},\bar{\gamma}} + \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\gamma},\bar{\kappa}}) - R^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa}\bar{\gamma}}||x^\kappa||x^\gamma|\mathbf{e}_{\bar{\mu}}\big) = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}\big(|x^{\bar{0}}|^2 + |x^{\bar{n}}|^2\big)\mathbf{e}_{\bar{\mu}} \qquad (3.4)$$

$$\big(x^{\bar{0}} \leftarrow 0\big) \;\to\; \boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}(|x^{\bar{n}}|^2)\mathbf{e}_{\bar{\mu}} \qquad (3.5)$$

$$Y^\alpha\!: \in \mathbb{R}^4 \;\to\; < Y^{\bar{\alpha}}\boldsymbol{e}_{\bar{\alpha}} > = < Y^{\bar{\alpha}}\mathbf{e}_{\bar{\alpha}} > + < Y^{\bar{\alpha}}\,\tilde{\mathbf{O}}_{\bar{\alpha}}(|x^{\bar{n}}|^2) > = < Y^{\bar{\alpha}} > \mathbf{e}_{\bar{\alpha}} + \|Y^{\bar{\alpha}}\|\,\tilde{\mathbf{O}}_{\bar{\alpha}}(\emptyset^2) \qquad (3.6)$$

(1.9b) (3.3) $$\forall\tau,\;\; \underline{\underline{x^{\hat{\nu}} - x,^{\hat{\nu}}\big(\mathcal{P}(\tau,)\big)}} \subset \underline{x}^{\bar{\nu}}\big(\mathcal{P}(\tau,)\big) \;\Big|\; 2|x^{\hat{k}}| \le \emptyset_0 \qquad (3.7)$$

This means that (3.6) is also valid for $\underline{x}^{\hat{\mu}}$. For the rest the LI coordinates components can be separated:

(3.6) $$\forall\hat{\alpha} \qquad < Y^{\hat{\alpha}}\boldsymbol{e}_{\hat{\alpha}} > = \Big(< Y^{\hat{\alpha}} > + \tilde{O}^{\hat{\alpha}}\big(\|Y^{\hat{\mu}}\|\emptyset^2\big)\Big)\,\mathbf{e}_{\hat{\alpha}} \qquad (3.8)$$

The uncertainty: $\tilde{O}^{\hat{\alpha}}(*\,\emptyset^2)$ is the cost of making the vector integration over a curved domain justifiable.

▷ b) The total neutrality of spatial divergence for the body's SE-tensor field in the LI comoving frame

(2.2) $$T^{\hat{\alpha}\hat{\beta}}\big(\mathcal{P} \in \underline{S} := \partial\underline{V}\big) \equiv 0 \;\to\; < T^{\hat{\mu}\hat{n}}{}_{,\hat{n}}\big(\tau, x^{\hat{k}}\big) > + \tilde{O}^{\hat{\mu}}(\mathrm{m}\emptyset^2) = \oint_{\underline{S}} T^{\hat{\mu}\hat{n}}\big(\tau, x^{\hat{k}}\big)\, n_{\hat{n}}|\mathrm{d}\underline{S}| \equiv 0 \qquad (3.9)$$

4) Finding the $O(\mathrm{m}\emptyset^2)$ approximation of the four-momentum temporal partial derivative for a small body

▷ a) The test body problem: The cross effect of gravity's fields, its convergence upper bound estimation
Based on the equation for weak gravitational field and its source given by a hypothetical scalable massive body with a density limit (no singularities), which on closer inspection, can be treated as the superposition of the infinitesimally small, dispersed point masses with their $\mathrm{d}\bar{h}_{\hat{0}\hat{0}} \approx 4\mathrm{d}\rho/r$ [5] partial fields; by negligible external reflections, the following basic estimates of the metric deviation and its derivatives can be obtained in the originally (without the body's active mass as the field source) LI comoving frame.

[5] (1.11) $$\eta^{\hat{\mu}\hat{\nu}}\,\bar{h}^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\mu},\hat{\nu}} \equiv \eta^{\hat{\mu}\hat{\nu}}\left(h^{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\eta^{\hat{\alpha}\hat{\beta}}h\right)_{,\hat{\mu},\hat{\nu}} = -16\pi T^{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad h = h^{\hat{\alpha}}{}_{\hat{\alpha}} = \eta^{\hat{\alpha}\hat{\beta}}h_{\hat{\alpha}\hat{\beta}} \qquad (4.1a,b)$$

(2.3) (5.1) $$T^{\hat{\alpha}\hat{\beta}} := O^{\hat{\alpha}\hat{\beta}}(\mathrm{d}^0) \in \mathbb{R}^{4\times4} \quad \to \quad \mathrm{m} := \left\|T^{\hat{\alpha}\hat{\beta}}\right\| \quad \to \quad \mathrm{m} := 8\mathrm{m}_0\emptyset_0^{-3}\mathrm{d}^3 = O(\mathrm{d}^3) \qquad (4.2a,b,c)\;(4.3)$$

$$g_{\hat{\alpha}\hat{\beta}} := \eta_{\hat{\alpha}\hat{\beta}} + \Delta g_{(ex)\hat{\alpha}\hat{\beta}} + \Delta g_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + h_{(ex)\hat{\alpha}\hat{\beta}} + h_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + O_{\hat{\alpha}\hat{\beta}}(\emptyset^2) + O_{\hat{\alpha}\hat{\beta}}(\mathrm{md}^{-1})$$

$$h_{(ex)\hat{\alpha}\hat{\beta}} = O_{\dots}(\emptyset^2) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{\dots}(\emptyset) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{\dots}(\emptyset^0) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{\dots}(\emptyset^0) \qquad (4.4a..d)$$

$$h_{(in)\hat{\alpha}\hat{\beta}} = O_{\dots}(\mathrm{md}^{-1}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{\dots}(\mathrm{md}^{-2}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{\dots}(\mathrm{md}^{-3}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{\dots}(\mathrm{md}^{-4}) \qquad (4.5a..d)$$

These fields: of the body (internal) and of the externally imposed curvature; are superimposed, resulting in two partial Christoffel symbol fields and creating additionally the cross-term: Δ_{X} in which range the separation between the two fields is no longer definite. The convergence order of this term and its derivatives can be determined. As Newton's first law applies, the internal field itself in an inertial system can have no effect on the overall motion; hence, it is arbitrarily omitted, but the cross-term remains valid.

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} := \Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_{\mathrm{X}}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \qquad (4.6)\;(4.7)$$

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\left\{\eta_{\hat{\mu}\hat{\nu}} - h_{(in)\hat{\mu}\hat{\nu}} - h_{(ex)\hat{\mu}\hat{\nu}}\right\}^{\hat{\mu}\hat{\nu}}}{2}\Big(h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} + h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}}\Big)$$

$$\Delta_{X\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} \approx \frac{\{h_{(in)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}}\right) + \frac{\{h_{(ex)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}}\right)$$

(4.4,5,6) (5.3a) $$\left(\Gamma_{(in)\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} \leftarrow 0\ ,\ \tau \to \tau_0\right) \Rightarrow \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}} = \Gamma_{(ex)\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} + \Delta_{X\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} = \Gamma_{(ex)\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\mathrm{md}^0)$$ (4.8) (4.9a,b)

$$\to \quad \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{\gamma}} = \Gamma_{(ex)\,\hat{\alpha}\hat{\beta},\hat{\gamma}}^{\;\hat{\mu}} + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}\hat{\gamma}}(\mathrm{md}^{-1}) \quad \to \quad \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = \Gamma_{(ex)\,\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}}^{\;\hat{\mu}} + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\varepsilon}}(\mathrm{md}^{-2}) \qquad (4.9c,d)$$

Based on (4.9), the cross-connection error resulting from the equation (4.19) converges one order faster than the approximation error. Moreover, it is worth noting that even for m = O(d), the solutions (5.8,21) retain convergence at the O(mØ) rate. Therefore, the cross-term: Δ_X is neglected from this point onward.

▷ b) An approximate factoring of the Christoffel symbol out of the spatial integral over the volume: $\underline{V}$

(1.6a) $$\Gamma^{\;\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\tau, x^{\hat{n}}) = \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(g_{\hat{*}\hat{*}}, g_{\hat{*}\hat{*},\hat{*}}) + \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{n}}x^{\hat{n}} + 2^{-1}O\left(\left|\Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{k},\hat{l}}\right| |x^{\hat{k}}x^{\hat{l}}|\right) := \Gamma_{(ex)\,\hat{\alpha}\hat{\beta}}^{\;\hat{\mu}} \qquad (4.10)$$

$$< \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}} < T^{\hat{\gamma}\hat{\nu}} > + \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\gamma}\hat{\nu}} > + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2) \qquad (4.11)$$

$$\mathrm{T}^{\hat{\gamma}\hat{\nu}} := < T^{\hat{\gamma}\hat{\nu}} > \quad \to \quad < \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\gamma}\hat{\nu}} + \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\gamma}\hat{\nu}\langle\hat{n}\rangle} + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2) \qquad (4.12a,b)$$

With (1.9b) it leads to the upper bound estimation of deviation of the temporal partial derivative (4.19).

(4.11) (1.9b) $$< \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = O\left(\left|\Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{n}}\right| \|T^{\hat{\gamma}\hat{\nu}}\|Ø\right) + O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2) = O^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|Ø) \qquad (4.13)$$

(4.2) $$< \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > = O^{\hat{\mu}}(\mathrm{m}Ø) \quad ; \quad < \Gamma^{\hat{\beta}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > = O^{\hat{\mu}}(\mathrm{m}Ø) \qquad (4.14a,b)$$

▷ c) The temporal partial derivative of the four-momentum, obtained from the conservation condition:

$$T^{\mu\beta}_{\;\;;\beta} \equiv T^{\mu\beta}_{\;\;,\beta} + \Gamma^{\mu}_{\;\alpha\beta}T^{\alpha\beta} + \Gamma^{\beta}_{\;\alpha\beta}T^{\mu\alpha} := 0 \qquad (4.15)$$

$$T^{\mu\beta}_{\;\;,\beta} = -\Gamma^{\mu}_{\;\alpha\beta}T^{\alpha\beta} - \Gamma^{\beta}_{\;\alpha\beta}T^{\mu\alpha} \qquad (4.16)$$

(3.8) $$< T^{\hat{\mu}\hat{\beta}}_{\;\;,\hat{\beta}} > = -< \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(\mathrm{m}Ø^2) \qquad (4.17)$$

(3.9) $$< T^{\hat{\mu}\hat{0}}_{\;\;,\hat{0}} > = -< \Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}_{\;\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(\mathrm{m}Ø^2) \qquad (4.18)$$

(4.11,14) (1.9b) $$< T^{\hat{\mu}\hat{0}}_{\;\;,\hat{0}} > = -\Gamma^{\hat{\mu}}_{\;\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}_{\;\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\mu}\hat{\alpha}} > + O^{\hat{\mu}}(\mathrm{m}Ø^2) = O^{\hat{\mu}}(\mathrm{m}Ø) \qquad (4.19)$$

5) The freely falling small body and its, founded on the conservation conditions, near geodesic solutions

Because $\mathcal{P}(\tau_0)$ can be any given point on the geodesic, in the ongoing section it's assumed that the body is currently situated in the *proper* or at least in the LI *comoving* frame of reference, the behavior of the body in the vicinity of the spatial coordinate origin: $\mathcal{P}(\tau)$ is analyzed, and if the result follows the geodesic in the limit case (for d → 0), it must also follow it inside the $Ø_0$ tube for d > 0 over a time period.

▷ a) The proper - (rest) mass: m , four-position: $x^{\hat{\alpha}}$, four-velocity: $U^{\hat{\alpha}}$, and the (minimal) rest energy: E_0

(3.8) $$\mathrm{m}(\tau = \tau_0) := < T^{\hat{0}\hat{0}}(\tau_0, x^{\hat{n}}) > + \tilde{O}(\mathrm{md}^2) \qquad (5.1)$$

(2.3) (1.2a) $$x^{\hat{0}}(\tau) := \tau \quad ; \quad x^{\hat{n}}(\tau) := < T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) >^{-1} < x^{\hat{n}}T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) > + \tilde{O}^{\hat{n}}(Ø^2\mathrm{d}) \qquad (5.2a,b)$$

(2.5a) (1.2a) $$Ø(\tau_0) \equiv \mathrm{d}(\tau_0) \quad \to \quad x^{\hat{0}}(\tau_0) = \tau_0 \ ,\ x^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \qquad (5.3a,b)$$

$$U^{\hat{\nu}} := \frac{\mathrm{d}x^{\hat{\nu}}}{\mathrm{d}\tau} \equiv \frac{\partial x^{\hat{\nu}}}{\partial x^{\hat{0}}} \;\Bigg|\; |U^{\hat{n}}| \ll 1 \ \leftarrow\ O^{\hat{\alpha}}_{\;,\tau}(\mathrm{d}^k) = O^{\hat{\alpha}}(\mathrm{d}^k) \ \Leftarrow\ \frac{O^{\hat{\alpha}}(\mathrm{d}^k)}{\mathrm{d}^k} = O^{\hat{\alpha}}(1, \tau \to \tau_0) \in C^2 \qquad (5.4a..c)$$

(2.5b) (5.3a) $$E_0 := E(\tau_0) = \min\left(\mathrm{m}(\tau_0)/\sqrt{1 - U^{\hat{n}}(\tau_0)U_{\hat{n}}(\tau_0)}\right) \Rightarrow U^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \qquad (5.5a,b)$$

(5.2a,4) $$U^{\hat{0}}(\tau_0) = 1\ ,\ U^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \quad \to \quad U^{\hat{0}}_{\;,\hat{0}}(\tau_0) = 0 \qquad (5.6a,b)$$

▷ b) The solution based on the stress-energy tensor, in the locally inertial comoving frame of reference
The four momentum: $\mathrm{p}^{\bar{\mu}}$ can be defined as the 'T4-momentum': $< T^{\bar{\mu}0} >$ or as the four-velocity based 'U4-momentum': $\mathrm{mU}^{\bar{\mu}}$. Given they are equivalent and the mass does not vary, it follows for $|\mathrm{U}^{\hat{n}}| \ll 1$.

(5.2b) (4.19)
$$\frac{\mathrm{d}\mathrm{p}^{\hat{\mu}}}{\mathrm{d}\tau} = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{p}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}}\mathrm{x}^{\hat{n}} - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{m}\emptyset^2) \quad (5.7)$$

(4.12) (4.14)
$$\frac{\mathrm{d}\mathrm{p}^{\hat{\mu}}}{\mathrm{d}\tau} = \mathrm{O}^{\hat{\mu}}(\mathrm{m}|\mathrm{x}^{\hat{n}}|) - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\alpha}\hat{\beta}} - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\alpha}\hat{\beta}\langle\hat{n}\rangle} - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\mu}\hat{\alpha}} - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\mu}\hat{\alpha}\langle\hat{n}\rangle} = \mathrm{O}^{\hat{\mu}}(\mathrm{m}\emptyset) \quad (5.8)$$

(5.1) (5.4a)
$$\mathrm{p}^{\hat{\mu}} \approx \mathrm{mU}^{\hat{\mu}} \quad \Rightarrow \quad \frac{\mathrm{dU}^{\hat{\mu}}}{\mathrm{d}\tau} \approx \frac{\mathrm{d}(\mathrm{p}^{\hat{\mu}}/\mathrm{m})}{\mathrm{d}\tau} = \frac{1}{\mathrm{m}}\left(\frac{\mathrm{d}\mathrm{p}^{\hat{\mu}}}{\mathrm{d}\tau} - \frac{\mathrm{p}^{\hat{\mu}}}{\mathrm{m}}\frac{\mathrm{dm}}{\mathrm{d}\tau}\right) = \mathrm{O}^{\hat{\mu}}(\emptyset) \quad (5.9\mathrm{a})\ (5.9\mathrm{b})$$

Since the origin of $\underline{x}^{\hat{n}}$ follows the geodesic, this implication shows that in the limit the body follows the geodesic as well. The tidal forces in (5.7) also depend on the SLI gauge and can influence the trajectory. The critical physical issue is that (5.8) is not coordinate-invariant, and the essential point of this is that the body must freely levitate in the domain: $\underline{V}$ close to the spatial origin, where gravity nearly vanishes. This effect is possible owing to the LI comoving frame, which ensures the local translational symmetries (*t-xyz*) inside $\underline{V}$ in $\underline{x}^{\hat{\mu}}$. However, these symmetries are not perfect because the derivative of the metric: $g_{\hat{\mu}\hat{\nu},\hat{k}} = O_{\hat{\mu}\hat{\nu}\hat{k}}(\emptyset) \neq 0$ on $\underline{V}$, and this constitutes the limiting factor for the convergence of the solution.

▷ c) Body's proper mass density, an object-energy (OE-)tensor (field) and its *local integral* divergence
The state of the body's overall motion is defined by its four-velocity; therefore, to find the tensor equation of motion the SE-tensor component that incorporates only this four-velocity should be used which, for a small body, is the OE-tensor $\boldsymbol{T}_{\mathrm{M}}$ (5.11a) reflecting the convective flux of the matter's mass. However, as simultaneity in the space-time domain: $\underline{\boldsymbol{V}}$ is relative, $\boldsymbol{T}_{\mathrm{M}}$ is ambiguous depending on the SLI-$\hat{\xi}^0$ gauge in the $O(\mathrm{d}^2)$ range. Consequently, in order to evade the problem resulting from (5.8), instead of the SE- the OE-tensor is used so that gravitation acts exclusively upon this stress-free component. Owing to the energy conservation from the τ-translation symmetry (5.14a) and since for $\tau \to \tau_0$ the first moments of $\boldsymbol{T}_{\mathrm{M}}$ are nullified because of the condition (2.5a), the LHS of (5.14b) disappears creating for the $\boldsymbol{T}_{\mathrm{M}}$ field an integral covariant conservation condition, which has $\tilde{\mathrm{O}}(\mathrm{md}^2)$ convergence order in the proper frame.
- *The following equations are studied for* $\tau \to \tau_0$, where the offset: $\mathrm{x}^{\hat{n}}$ (5.2b) is negligible due to (2.5a).

(3.3) (3.7)
$$\underline{x}^{\bar{\nu}}(\mathcal{P}(\tau_0)): \ x^{\bar{0}} = x^{\hat{0}} - \tau_0 \ , \ \Lambda^{\hat{0}}_{\bar{\nu}} = \delta^{0}_{\bar{\nu}} \ , \ g_{\bar{0}\bar{n}} = 0 \quad \to \quad \bar{U}^{\bar{\nu}} := \delta^{\bar{\nu}}_0 \ , \ \bar{\rho} := T^{\overline{00}} \quad (5.10\ \mathrm{a,b})$$

(5.6a) (1.9b)
$$T_{\mathrm{M}}^{\hat{\mu}\hat{\nu}} := \bar{\rho}\,\bar{U}^{\hat{\mu}}\bar{U}^{\hat{\nu}} \quad \leftarrow \quad \bar{U}^{\hat{\nu}} = \Lambda^{\hat{\nu}}_{\bar{\kappa}}\,\delta^{\bar{\kappa}}_0 \approx \left(\mathrm{U}^{\hat{\nu}}(\tau) - \tilde{\mathrm{O}}^{\hat{n}}(\mathrm{d}^3,\tau)\right) + \Lambda^{\hat{n}}_{\bar{0},\hat{k},\hat{\ell}}(\tau)x^{\hat{k}}x^{\hat{\ell}} \quad (5.11\ \mathrm{a,b})$$

$$T_{\mathrm{S}}^{\mu\nu} := T^{\mu\nu} - T_{\mathrm{M}}^{\mu\nu} \quad \to \quad T_{\mathrm{M}}^{\overline{\mu\nu}}T_{\mathrm{S}\overline{\mu\nu}} = 0 \quad \to \quad T_{\mathrm{M}}^{\mu\nu}T_{\mathrm{S}\mu\nu} \equiv 0 \quad \to \quad T_{\mathrm{M}} = -\bar{\rho} \ , \ T_{\mathrm{S}} =: 3\bar{p} \quad (5.12\ \mathrm{a..d})$$

(3.8,9) (1.6a,9b)
$$< T_{\mathrm{M}}^{\hat{\mu}\hat{\beta}}{}_{;\hat{\beta}} > = < T_{\mathrm{M}}^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > + < \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{M}}^{\hat{\alpha}\hat{\beta}} > + < \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{M}}^{\hat{\mu}\hat{\alpha}} > + \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.13)$$

(3.3b) (2.5a)
$$g_{\hat{\mu}\hat{\nu},\tau} = \tilde{O}_{\hat{\mu}\hat{\nu}}(\mathrm{d}^2) \quad \to \quad < T_{\mathrm{M}}^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > + < \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{M}}^{\hat{\alpha}\hat{\beta}} > + < \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{M}}^{\hat{\mu}\hat{\alpha}} > = \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.14\mathrm{a,b})$$

▷ d) The coordinate-invariant solution grounded in the OE-tensor in the (LI) proper frame of reference
As shown above, (5.14) corresponds directly to (4.18); and, since the body's four-position (5.2) along the world-line is identically defined for both the OE-tensor and the SE-tensor, the subsequent proof of the geodesic solution for the OE-tensor in the limiting case also confirms the result (5.9b) for the SE-tensor.

(5.14a)(2.4) (4.11)(2.5a)
$$< T_{\mathrm{M}}^{\hat{\mu}\hat{0}} >_{,\hat{0}} = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} < T_{\mathrm{M}}^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} < T_{\mathrm{M}}^{\hat{\mu}\hat{\alpha}} > + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.15)$$

(5.11,4b)
$$< \bar{\rho}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{0}} >_{,\hat{0}} = - < \bar{\rho} > \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} - < \bar{\rho} > \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.16)$$

(5.1) (5.6b)
$$(\mathrm{mU}^{\hat{\mu}}\mathrm{U}^{\hat{0}})_{,\hat{0}} = \mathrm{mU}^{\hat{0}}\mathrm{U}^{\hat{\mu}}{}_{,\hat{0}} + \mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{0}}\mathrm{m}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} - \mathrm{m}\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.17)$$

Even though here $\Gamma^{\hat{\alpha}}{}_{\hat{\nu}\hat{\kappa}} = 0$, it is the $\Gamma^{\hat{\alpha}}{}_{\hat{\nu}\hat{\kappa}}$ that carries the key information about the origin of this zero. Since $\mathrm{U}^{\hat{0}}$ is a constant value and $\mathrm{U}^{\hat{n}} \to 0$, (5.17) can be decomposed into the system of two equations:

(5.6a) (5.20b)
$$\begin{cases} \mathrm{mU}^{\hat{\mu}}{}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \ \Big| \ \mathrm{O}^{\hat{0}} = 0 & (5.18\mathrm{a}) \\ \mathrm{U}^{\hat{\mu}}\mathrm{m}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \ \Big| \ \mathrm{O}^{\hat{n}} = \mathrm{O}^{\hat{n}}(\mathrm{md}^5) & (5.18\mathrm{b}) \end{cases}$$

(5.1,11) $$-\mathrm{m}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\nu}} + \tilde{\mathrm{O}}^{\hat{\mu}\hat{\nu}}(\mathrm{md}^2) = \mathrm{m}^{\hat{\mu}\hat{\nu}} := -< T_{\mathrm{M}}^{\hat{\mu}\hat{\nu}} > \quad \rightarrow \quad \mathrm{m} \equiv \mathrm{m}^{\mu}{}_{\mu} = \eta_{\hat{\mu}\hat{\nu}}\,\mathrm{m}^{\hat{\mu}\hat{\nu}} \tag{5.19a,b}$$

(1.2a)
(5.3b)
(5.6a)
(1.8b)

$$\begin{cases} \mathrm{m}\dfrac{\mathrm{d}\mathrm{U}^{\hat{\mu}}}{\mathrm{d}\tau} = -\mathrm{m}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) - \mathrm{m}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}}\tilde{\mathrm{O}}^{\hat{n}}(\mathrm{d}^3) & (5.20\mathrm{a}) \\ \dfrac{\mathrm{d}\mathrm{m}}{\mathrm{d}\tau} = -\mathrm{m}\Gamma^{\hat{\beta}}{}_{\hat{0}\hat{\beta}} + \mathrm{O}^{\hat{0}}(\mathrm{md}^2) = \mathrm{O}(\mathrm{md}^2) & (5.20\mathrm{b}) \end{cases}$$

The equations turn out to be coordinate-invariant since the first one is the four-momentum form of the geodesic equation and the second one is a scalar equation, which is always coordinate-invariant. Hence:

$$\forall\tau \begin{cases} \mathrm{m}\dfrac{\mathrm{d}\mathrm{U}^{\mu}}{\mathrm{d}\tau} = -\mathrm{m}\mathrm{U}^{\alpha}\mathrm{U}^{\beta}\Gamma^{\mu}{}_{\alpha\beta} + \mathrm{O}^{\mu}(\mathrm{md}^2) & \mathbf{(5.21a)} \\ \dfrac{\mathrm{d}\mathrm{m}}{\mathrm{d}\tau} = \mathrm{O}(\mathrm{md}^2) & \mathbf{(5.21b)} \end{cases} \quad \rightarrow \quad \frac{\mathrm{D}\mathrm{U}^{\mu}}{\mathrm{d}\tau} = \mathrm{O}^{\mu}(\mathrm{d}^2) \tag{5.21c}$$

▷ e) The limit case turns out to be the (rest) mass conservation law and the standard geodesic equation:

(5.4a) $$\lambda := \mathrm{a}\tau + \mathrm{b}\,, \quad \mathrm{d} \rightarrow 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2\mathrm{x}^{\mu}}{\mathrm{d}\lambda^2} = -\Gamma^{\mu}{}_{\alpha\beta}\frac{\mathrm{d}\mathrm{x}^{\alpha}}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathrm{x}^{\beta}}{\mathrm{d}\lambda} \qquad \blacksquare \tag{5.22}$$

III. Summary

The behavior of a massive body situated in the gravity field, free of other influences and with *negligible radiation*, has been analyzed here. The body is defined on the basis of its stress-energy (SE-)tensor field, which together with the geodesic determines the space-time domain: $\underline{\boldsymbol{V}}$ in which it exists. Consequently, based on general coordinates, a stationary locally inertial (SLI) coordinate system has been constructed within the valid region. The SLI coordinates manage to conveniently describe the movement of the body at least within a certain time period. If their spatial origin matches the comoving body at a given τ, they are referred to there as the proper frame. This configuration has been achieved by selecting the suitable initial conditions for the SLI coordinate system itself. This proper frame stays locally inertial (LI) in the neighborhood of its origin. The spatial origin of the SLI coordinate system follows a geodesic, forming a kind of geodesic tunnel in space-time, nevertheless this by itself has no influence on the tensor solution. The requested SLI gauge aims to limit the fictitious tidal forces particularly in flat regions, but it may be replaced if necessary, without affecting the estimation of convergence order, with any other gauge that complies with the boundary conditions at the spatial origin. – Furthermore, the local integral divergence of the SE-tensor over the spatial domain plays a crucial role in this framework. If the body is sufficiently small and the coordinate system basis is flattened, four distinct conservation equations for energy and momentum emerge within the LI comoving frame. These all arise from zero-divergence of the SE-tensor due to translational symmetries, which, in curved space, are only locally valid through approximation. Therefore, it is essential to estimate the convergence rate of the uncertainties, deviations, and errors. To determine upper-bound estimates, the big *O* notation is employed, representing *a member* of a function class with an explicitly restricted convergence order. Owing to the limited spatial extent of the SE-tensor field, its local integral divergence can be reduced to the temporal derivative. This reduction is crucial, as it enables the derivation of the body's equation of motion from the SE-tensor's zero-divergence. To ensure that the test body problem remains non-critical, the cross-effects of gravitational fields have been shown to be negligible up to m = O(d). – Based on the SE-tensor, the body's geodesic trajectory has been found only in the SLI coordinates. This reinforces the concept that the SE-tensor, used as a basis for the geodesic equation, is suitable only to a limited extent. To address this limitation, the SE-tensor has been reduced by arbitrarily setting all its subcomponents except energy density to zero in the *orthochronous* proper frame. This defines the OE-tensor, which depends solely on the body's mass density and excludes contributions from its stress density field. The latter is expressed using the object-stress (OS-)tensor: $\boldsymbol{T}_{\mathrm{S}}$ formed of internal energy flux, shear stress and pressure. The negative zeroth moment of the OE-tensor, the *mass tensor* (5.19), like mass and four-momentum, is just another quantity characterizing an object. Clearly, replacing the SE-tensor with its OE-tensor eliminates the dependence on the OS-tensor. This process is directly visible only to the internal observer in the body's proper frame, an external observer perceives only the final tensor result. Now, the energy flux balance equation is solved by splitting it into two separate equations: the first one relates to the body's acceleration, the second one to its mass change. Ultimately, these equations can be represented by two tensor equations which, in the boundary case, act as the geodesic equation and the mass conservation law for freely falling bodies in a general space-time.

IV. Conclusions

For a small body diameter: d there are two verifiable possibilities: The effective gravitational tidal forces: *A*) *can* influence the trajectory of the freely falling body at the O(d) deviation level. *B*) *cannot* because the internal stress tensor field of the body vanishes or, on average, doesn't interact with the gravity field. – It has been proven that even if no coordinate-invariant equation of motion based on the SE-tensor has been found, free 'point' bodies associated with their SE-tensors traverse timelike geodesics. Essentially, the covariant conservation condition of the SE-tensor is sufficient to determine this, provided the body is isolated from all conventional forces and, according to the local symmetries in the LI comoving frame of the SLI system, is additionally apparently isolated from gravity. In this case, the possibility "*A*" arises if the distribution of any component of the SE-tensor doesn't correspond to the mass density of the body. – Simultaneously this statement can be extended analogously to a solution that involves the object-energy (OE-)tensor, which is the stress-free component of the SE-tensor, but in this case the direct coordinate-invariant solution is the geodesic equation; yet should "*A*" be true, a body's trajectory error at the O(d) level may occur. Thereby, the OE-tensor provides a mechanism through which gravity influences the massive body, while maintaining the conservation of the body's mass. Applying the SE-tensor instead of the OE-tensor in a tensor equation of motion may lead to disregard for the weak equivalence principle, but given strong reasons to believe that this principle remains valid, the solution on the OE-tensor basis is preferable as it eliminates the theoretical dependence on the stress tensor of the matter of the body. - As indicated by this, the following thesis can be proposed succinctly: The total gravitational influence on a sufficiently small freely falling massive body results solely from the interaction with its OE-tensor. - When formulated in this manner, the above thesis in the near-limit case leads not only to the geodesic principle and the weak equivalence principle, but also to the mass conservation law for an isolated body. It further implies that rest mass and passive gravitational mass must always be equal. Besides, this approach has the added advantage of not relying on a quasi-mathematical axiom, but instead of this being grounded in the local integral conservation condition for the body's OE-tensor field in the proper frame.

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For now, there is an opportunity to substantiate the weak equivalence principle for (not only stable) extended objects in a (locally) uniform gravitational field. As space-time is flat there, the above derivation of the geodesic principle is exact even for such objects, since the integration uncertainty vanishes in flat space. Owing to the translational symmetries in this case, the object's internal forces are unable to accelerate the object as a whole, thus these symmetries reveal their masking effect on the object's stress tensor. Hence, only the velocity of the object determines its world-line path in this setup. Yet, it is important to consider that the position of such an object is defined in its *proper* frame, and such an integral quantity can be unequivocal only in this context. So, the outcome is always the trajectory of its *proper* position. - Therefore, under these circumstances a short-term solution, expressed as a specific geodesic trajectory within a flat space-time domain, complies with the criterion defined by the weak equivalence principle.

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Since the geodesic principle has not been derived here from the geometrical approach but from the conservation condition, it is natural not to restrict oneself to freely falling objects alone; external influences such as the Lorentz force can also be considered when deriving an equation of motion like the one below.

(5.9a,21a,19) $$m\frac{\eth U^{\mu}}{\eth\tau} = -mU^{\mu}U^{\nu}\Gamma^{\alpha}{}_{\mu\nu} + qU^{\mu}F^{\alpha}{}_{\mu} = m^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} + qU^{\mu}F^{\alpha}{}_{\mu} \;\Big|\; d \to 0 \qquad \text{(C.1)}$$

In the above equation, the mass: m is not just a result of 'adjusting' the 'geometrical' geodesic equation to the Lorentz force, but was already there in (5.21a). It is also clear that m on the LHS expresses the inertial mass, while m on the RHS the passive mass having the same value. Secondly, it is notable here that the inertial mass 'hides' in the body-bound coordinates where $\dot{U}^{\alpha} = 0$, and the passive mass 'hides' in the free-falling coordinates where $\Gamma^{\alpha}{}_{\mu\nu} = 0$, as in the famous free-falling elevator thought experiment. – This equation implies that gravity's interaction can be decomposed into the interaction of the object's parts with the metric field, *incorporating* the object's internal forces that are external to each single part. By specifying d *and* decreasing the parts' maximal diameter: d_P , each part's SE-tensor field (expectation value) becomes more homogenous and can be described using the Taylor series. Thus, it is easy to show that for each part the possibility "*A*" disappears because now the first moments decrease at $O(d_P^2)$ order of convergence. In this situation the mechanism absolutely masking the OS-tensor in flat space is also capable of masking the stress tensor of any part in curved space, but restricted to this convergence level. Furthermore, by decreasing the parts' diameter, the superposition of the parts' OE-tensor fields gradually

approaches the *proper* ***energy*** *density* (E-)tensor field: $\boldsymbol{T}_{\mathrm{E}}$ of this object, and in the limit the gravitational interaction *can* be decomposed into gravity's interactions with the E-tensor fields of the object's particles. Though the local homogeneity was necessary (at least in the statistical sense) to achieve the $\mathrm{O}(\mathrm{d}_{\mathrm{P}}^{2})$ order of convergence, the $\mathrm{O}(\mathrm{d}_{\mathrm{P}})$ order of convergence should be sufficient to generate the E-Tensor field here. In the specific case of an object made of pressureless (dust) matter, the SE- and E-tensor fields are equal.
- Accordingly, the thesis which has been proposed above, can be generalized in the following postulate: The gravitational influence on a massive object results solely from the interaction with its E-tensor field.
– The problem is that physical objects can also include massless particles such as photons, which in fact can contribute to the body's mass. The basis for the entire above derivation of the E-Tensor field was the massive object's parts in their proper frames. But for light there is no physical proper frame possible, so that one cannot 'look over its shoulder', except it is 'slower than light' and this is possible e.g. in the air. Making the air thinner, in the limiting case, for a ray in a LI frame can e.g. $U^{\mu}(\mathrm{x},) = \{1,1,0,0\}^{\mu}$. In this way an analogous derivation should be possible. So, the E-Tensor would be based on the light's energy density too. This means that by replacing the object's mass density: $\bar{\rho}$ with the energy density: ρ the idea of the E-Tensor goes beyond massive objects, and the above postulate can be used in generalized form: The gravitational influence on a physical object results solely from the interaction with its E-tensor field.
- This postulate forms the basis for incorporating gravity in the equations of motion for continuous systems across all scales, including the particle range where the E-tensor may be derived from the SE-tensor of the wavefunction. Also, it can be treated as the most general form of the weak equivalence principle.

V. Symbols

a, b	constant real values
C^2	(second) differentiability class
d	object's (or body's) diameter
D , đ , ∂	(operators:) absolute differential, total differential, partial differential
đ$\underline{S}$, đ$\underline{V}$, $\partial\underline{V}$	spatial surface element, spatial volume element, spatial volume boundary
$\lvert$đ$\underline{V}\rvert$, $\lvert$đ$\underline{S}\rvert$	measures of the -proper spatial volume element (2.4b) and -proper area element
Δ , $\boldsymbol{\Delta}$	difference, difference vector
$\mathit{\Delta}_{\mathrm{X}}$	external-internal gravitational cross-term (4.6)
δ^{μ}_{ν}	Kronecker delta (a selector): $\delta^{\mu}_{\nu} := 0$ for $\mu \neq \nu$, $\delta^{\mu}_{\nu} := 1$ for $\mu = \nu$
$\boldsymbol{e}_{\alpha}$, $\boldsymbol{e}_{\alpha}$	coordinate basis, component vector of the coordinate basis
$\mathbf{e}_{\alpha}$, $\mathbf{e}_{\alpha}$	coordinate basis, component vector of the coordinate basis; near/at the (spatial) origin
$\eta_{\alpha\beta}, \eta^{\alpha\beta}$	Minkowski metric (matrix value): $\equiv [\mathrm{diag}(-1,1,1,1)]_{\alpha\beta}$, $\equiv [\mathrm{diag}(-1,1,1,1)]^{\alpha\beta}$
$F_{\alpha\beta}$	electromagnetic tensor
Ø , $Ø_0$	diameter of the spatial - domain: $\underline{V}$, validity domain: $\underline{V}_0$
$g_{\alpha\beta}$, $\mathrm{g}_{\alpha\beta}$	metric field , metric near/at the (spatial) origin
g	the determinant of the matrix $[g_{\alpha\beta}]$
$\mathit{\Gamma}^{\alpha}{}_{\mu\nu}$, $\Gamma^{\alpha}{}_{\mu\nu}$	Christoffel symbol (field), Christoffel symbol near/at the (spatial) origin
$h_{\alpha\beta}; h^{\alpha\beta}$	perturbation of the Minkowski metric $h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$
$\bar{h}_{\alpha\beta}; \bar{h}^{\alpha\beta}$	trace reverse of the perturbation of the Minkowski metric $\bar{h}^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu}\bar{h}_{\mu\nu}$
$\mathit{\Lambda}^{\nu'}_{\nu}, \Lambda^{\nu'}_{\nu}$	a coordinate transformation (: $Y^{\nu} \mapsto Y^{\nu'}$) matrix, -near/at the origin
λ	affine parameter
m , $\mathrm{m}^{\mu\nu}$	object's mass, object's mass tensor
n^n	spatial (unit) normal (to a surface) (contra)vector
O , O	big-*O* symbol (converges to 0) , same τ dependent only
$\tilde{O}$, $\tilde{\mathrm{O}}$	big-*O* symbol (converges to 0) as pure integration uncertainty, same τ dependent only
$\mathcal{P}, \mathcal{P}(x^{\nu})$	(event in or) point of the space-time
$\mathcal{P}_0$, $\mathcal{P}(\tau)$	point of the space-time at the - origin of a coordinate system, spatial origin: $x^n = 0$

p , $\bar{p}$	pressure (field) , object's stress (field) (5.12d)
p^{μ}; q ; r	object's (or body's) four-momentum ; electric charge ; radius-distance
$R^{\alpha}{}_{\mu\nu\sigma}$, $\mathrm{R}^{\alpha}{}_{\mu\nu\sigma}$	Riemann curvature tensor - field, near/at the origin
ρ , $\bar{\rho}$	energy density , object's (or body's) proper mass density (5.10b)
$T^{\mu\nu}, T^{\mu\nu}_{\mathrm{E}}, T^{\mu\nu}_{\mathrm{P}}$	stress-energy (SE-)tensor , energy (E-)tensor: $T^{\mu\nu}_{\mathrm{E}} = \rho U^{\mu} U^{\nu}$, pressure tensor ; -fields
$T^{\mu\nu}_{\mathrm{M}}, T^{\mu\nu}_{\mathrm{S}}$	object-energy (OE-) tensor, object-stress (OS-)tensor ; -fields
$\mathrm{T}^{\mu\nu}, \mathrm{T}^{\mu\nu\langle n\rangle}$	spatial (proper) - 0th moment (4.12a) , 1st moments: $< x^{n} T^{\mu\nu} >$ - of a tensor field
τ , τ_0	proper time , proper time initial value (for the proper *and* comoving frame)
$\bar{U}^{\mu}$, U^{μ}	object's proper - four-velocity field, four-velocity ; (5.11b) ← ($U^{\hat{\nu}} = Y^{\hat{n}} \Rightarrow U^{\hat{0}} = 0$)
U^{μ}	four-velocity (field)
$\underline{V}$, $\underline{V}_0$	the minimal diameter spatial domain with the body and the current trajectory point, the spherical validity domain with the center at the current trajectory point
$\overline{\underline{V}}$, $\underline{\boldsymbol{V}}$	all space but $\underline{V}$, minimal space-time domain containing the whole $\underline{V}$ for all valid τ .
x^{α}, x^{α}	a coordinate, coordinates
$\underline{x}^{\alpha}$, $\underline{\underline{x}}^{\alpha}$	(all) coordinates = coordinate system, (all) coordinate system*s* (their full set)
$x^{\alpha}, x^{\bar{\alpha}}$,	coordinates - general, (locally) inertial
$x^{\tilde{\alpha}}, x^{\hat{\alpha}}$	SLI coordinates, SLI-gauged coordinates ; (initially proper then comoving)
x^{α}	object's (or body's) position
ξ^{μ} , $\hat{\xi}^{\mu}$	gauge transformation shift, SLI-gauge transformation shift (1.10)
$\boldsymbol{Y}, Y^{\alpha}, Y_{\alpha}$	(general, abstract) vector (field) and its (contra)vector (field) and covector (field)
$< f >$	spatial integral (of an abstract function: f) over $\underline{V}$ (2.4a)
$\lvert x^{n} \rvert$	spatial distance from the origin as a norm
$\lVert T^{\mu\nu} \rVert$	integral norms of a tensor over $\underline{V}$
[...]	(reformatting to) matrix
$\{\ldots\}, \{\ldots_{\mu\nu}\}^{\mu\nu}$	reformatting - to tensor , a covariant second-rank to a contravariant second-rank tensor

VI. Abbreviations

E-tensor	energy tensor
LI	locally inertial coordinate system
OE-tensor	object's energy tensor
OS-tensor	object's stress tensor
SE-tensor	stress energy tensor
SLI	stationary locally inertial coordinate system

VII. References

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