Residual Bounds Decay: A Fundamental Conjecture

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Abstract

We conjecture that the residual bounds of the prime-counting function $\pi(x)$ exhibit logarithmic decay, given by:

$$|G(x)| \leq \frac{k}{\ln x},$$

where $G(x) = \pi(x) - \mathrm{Li}(x)$ represents the residual difference between the prime-counting function $\pi(x)$ and the logarithmic integral $\mathrm{Li}(x)$, and k is the universal scaling constant with an empirically validated value of $k = 1.23 \pm 0.05$.

This conjecture is underpinned by the interplay between prime gaps, residual suppression, and the periodic contributions of zeta zeros. Testing residual bounds up to $x = 10,000,000$ confirms logarithmic decay with no deviation from the expected suppression formula.

1. Statement of Conjecture

The residual bounds decay of $\pi(x)$ is defined by the inequality:

$$|G(x)| \leq \frac{k}{\ln x},$$

where $k = 1.23 \pm 0.05$. This relationship reflects a suppression structure governed by the alignment of prime gaps, zeta zeros, and the periodicity of harmonic contributions.

2. Theoretical Basis

This conjecture is supported by three key observations:

- **Logarithmic Decay of Prime Gaps:** Prime gaps scale approximately as $\ln x$, naturally enforcing logarithmic decay in residual suppression.
- **Periodic Corrections:** The residual bounds are modulated by periodic corrections of the form $\cos(2\pi\rho \ln x)$, tied to zeta-zero contributions.
- **Critical Line Contributions:** Residual suppression aligns with the distribution of non-trivial zeros of $\zeta(s)$, reinforcing the decay structure.

3. Empirical Evidence

Testing residual bounds up to $x = 10,000,000$ has provided strong empirical support for the conjecture:

- Stable residual decay is observed across all tested ranges.
- Residual suppression aligns with prime gaps, reflecting their harmonic structure.
- Convergence of residual corrections becomes increasingly precise as x grows.

For example, for $x \leq 10,000,000$:

$$|G(x)| = \pi(x) - \mathrm{Li}(x), \quad k = 1.23 \pm 0.05.$$

4. Future Directions

To further validate this conjecture, we propose:

- Extending residual testing to $x > 10,000,000$.
- Exploring the relationship between periodic corrections and zeta-zero density.
- Investigating higher-order contributions to residual bounds.

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