Power Sums Via Odd Sequences

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Power sums can be reinterpreted as weighted sums of odd sequences using a simple transformation of Riemann sums into Lebesgue sums. This reformulation introduces a self-referential recursive framework in which power sums are expressed as linear combinations of power sums in descending order.

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I. INTRODUCTION

Consider a sequence of n rectangles on a plane, each with a base of 1 and heights that increase according to the powers of the integers: $1^k, 2^k, 3^k, \ldots, n^k$, where $k = 0, 1, 2 \ldots$. The cumulative area of these rectangles serves as a visual interpretation of a power sum of powers, denoted as follows:

$$S_n^k \equiv \sum_{i=1}^{n} i^k, \quad k = 0, 1, 2, \ldots. \tag{1}$$

In particular, when $k = 0$, we encounter the simple case:

$$S_n^0 = \sum_{i=1}^{n} 1 = n, \tag{2}$$

which is the total area of n square units. The sum S_n^k can be interpreted as the Riemann sum (and integral) of a step function $f(x) = \lfloor x \rfloor^k$ defined in the interval $[1, n]$, where $\lfloor x \rfloor$ represents the floor function. Our primary goal was to derive a closed-form expression for an unknown power sum S_n^k using a recursive framework. We show that this problem can be solved by representing S_n^k as a linear combination of the previously known power sums S_n^i, for $i < k$. This approach leads to the following final representation:

$$S_n^k = \sum_{i=0}^{k-1} a_i S_n^i, \tag{3}$$

where a_i are either rational numbers or polynomials in n with rational coefficients. This formula can then be reduced to the known formulas. Note that we have included $i = 0$ in the sum, as $S_n^0 = n$ is a known quantity that may appear in the linear combination. This result can then be reduced to its ordinary representation.

A. The general case

Our method starts with a formal trick to manipulate the structure of a generic power sum by decomposing it into an equivalent expression. This is a different way of encoding the information about the power sums in their sequence of terms. Specifically, we express any power sum S_n^k in what we call its (k, h)-form:

$$S_n^k = \sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^h i^{k-h}, \quad h = 0, 1, 2 \ldots, \quad h \leq k. \tag{4}$$

This move favors the possibility of a fractional representation of the power sum if we look for a polynomial $p_h(j)$ such that:

$$\sum_{j=1}^{i} p_h(j) = i^h. \tag{5}$$

By the inverse relation between summations and finite differences, the degree of the polynomial must be $(h-1)$, so that its summation yields the i^h term. By substituting this into Eq. (**??**), we obtain the double sum:

$$S_n^k = \sum_{i=1}^{n} \sum_{j=1}^{i} p_h(j) i^{k-h}. \tag{6}$$

To simplify this double sum, we have to change the order. Considering that $1 \leq j \leq i \leq n$, we allow j to range from 1 to n, and for each j, variable i ranges from j to n. Finally, because $p_h(j)$ depends only on j, we can factor it out of the inner sum, yielding a more manageable expression:

$$S_n^k = \sum_{j=1}^{n} p_h(j) \sum_{i=j}^{n} i^{k-h}. \tag{7}$$

The inner sum, $\sum_{i=j}^{n} i^{k-h}$, can evidently be expressed as the difference between the power sums S_n^{k-h} and S_{j-1}^{k-h} (assumed known), and we denote this difference as μ_j^k:

$$\sum_{i=j}^{n} i^{k-h} = S_n^{k-h} - S_{j-1}^{k-h} \equiv \mu_j^k. \tag{8}$$

So the power sum S_n^k can now be expressed as:

$$S_n^k = \sum_{j=1}^{n} p_h(j) \mu_j^k. \tag{9}$$

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By developing the product in the sum, we get a polynomial $p_k(j)$ of order k containing all the decreasing powers of j. Executing the summation and shifting the j^k summation to the left, we obtain the desired result.

1. *The form* $(k,h) = (1,1)$

To implement the method, one must find the polynomial $p_h(j)$, and this obviously requires knowledge of power sums with exponents less than k. To make the method is self-referential and recursive, we have to start from the scratch. For this reason it is essential to arrange the form $(k,h) = (1,1)$ first, without resorting to the known result. With $p_1(j) = j^0 = 1$, Eq. (**??**) becomes:

$$S_n^1 = \sum_{i=1}^{n} i = \sum_{i=1}^{n}\sum_{j=1}^{1} 1 \cdot i. \tag{10}$$

Reversing the order of summation, we get:

$$S_n^1 = \sum_{j=1}^{n}\sum_{i=j}^{n} 1 = \sum_{j=1}^{n}(n-j+1) = -S_n^1 + n^2 + n, \tag{11}$$

from which:

$$S_n^1 = \frac{(n+1)n}{2} = \frac{n+1}{2} S_n^0. \tag{12}$$

This demonstrates how changing the summation order is critical to obtaining the correct end result.

2. *The form* $(k,h) = (k,2)$

The next polynomial to determine must to be of the form $p_2(j) = aj + b$ so that:

$$\sum_{j=1}^{i} p_2(j) = a\sum_{j=1}^{i} j + b\sum_{j=1}^{i} 1 = a\frac{i^2+i}{2} + bi = i^2. \tag{13}$$

This implies that $a = 2$ and $b = -1$, and the sum operates on the first i odd numbers $2j-1$, a well known result. It follows that we can manage the $(k,2)$-form of the power sum with $p_2(j) = 2j-1$ and get:

$$S_n^k = \sum_{j=1}^{n}(2j-1)\sum_{i=j}^{n} i^{k-2}, \tag{14}$$

where

$$\sum_{i=j}^{n} i^{k-2} = S_n^{k-2} - S_{j-1}^{k-2} \equiv \mu_j^k. \tag{15}$$

So the power sum μ_j^k can now be expressed as:

$$S_n^k = \sum_{j=1}^{n}(2j-1)\mu_j^k. \tag{16}$$

This formulation decomposes the sum of powers of order k into sums over the sums of lower-order powers S_n^{k-2} and S_{j-1}^{k-2} for $k \geq 2$, providing a recursive structure for expressing higher-order sums in terms of lower-order ones. To go on, we need to arrange the form $(k,h) = (2,2)$ of the power sum.

3. *The form* $(k,h) = (2,2)$

The case $(2,2)$ has a significant interpretation. We have:

$$\mu_j^2 = S_n^0 - S_{j-1}^0 = n - j + 1, \tag{17}$$

which satisfies the "boundary conditions":

$$\mu_1^2 = n, \quad \mu_n^2 = 1. \tag{18}$$

Thus, the double sum can be formally reduced to a simpler form, interpretable as a Lebesgue sum of the step function $2j-1$ over a discrete measure space. The measure μ_j^2 "counts" the contributions of each odd number, weighted by itself [**?**]:

$$S_n^2 = \sum_{j=1}^{n}(2j-1)\mu_j^2 = \sum_{j=1}^{n}(2j-1)(n-j+1). \tag{19}$$

Expanding, yields:

$$S_n^2 = \sum_{j=1}^{n}[-2j^2 + (2n+3)j - (n+1)]. \tag{20}$$

Solving for S_n^2, we obtain:

$$S_n^2 = \frac{2n+3}{3}S_n^1 - \frac{n+1}{3}S_n^0, \tag{21}$$

expressing the sum of the squares in terms of lower power sums. Substituting known values of S_n^1 and S_n^0 gives:

$$S_n^2 = \frac{2n+3}{3}\left(\frac{n^2+n}{2}\right) - \frac{n+1}{3}(n) = \frac{2n^3+3n^2+n}{6}. \tag{22}$$

This approach reveals a connection between S_n^2 and Lebesgue summation. By shifting focus from indices to values of odd numbers, we gain a new perspective. The table below illustrates the transformation from double to single sum for $n = 5$:

S_5^2	σ_1	σ_3	σ_5	σ_7	σ_9	μ_j^2	$\sum_L$
					9	1	9
				7	7	2	14
			5	5	5	3	15
		3	3	3	3	4	12
	1	1	1	1	1	5	5
$\sum_R$	1^2	2^2	3^2	4^2	5^2		55

4. The cases $(k,2)$ for $k>2$

When $k>2$, in constructing the corresponding Lebesgue sum for these cases, we can still use the function $(2j-1)$, but the function μ_j^k is no longer a simple counting measure as seen in the case of μ_j^2. Instead, μ_j^k becomes a more complicated arithmetic function. To better understand this, we will outline the step-by-step construction for the specific case $k=3$, with $n=5$. In this case, we have:

$$\mu_j^3 = \frac{n^2+n}{2} - \frac{j^2-j}{2}. \tag{23}$$

The generalized sum for $k=3$ becomes:

$$S_n^3 = \sum_{j=1}^{n}\left[-j^3+\frac{3}{2}j^2+\frac{2n^2+2n-1}{2}j-\frac{n^2+n}{2}\right],$$

which simplifies to:

$$\begin{aligned} S_n^3 &= \frac{3}{4}S_n^2+\frac{2n^2+2n-1}{4}S_n^1-\frac{n+1}{4}S_n^0 \\ &= \frac{n^4+2n^3+n^2}{4} = (S_n^1)^2. \end{aligned} \tag{24}$$

This result expresses the cube sum S_n^3 as a function of lower-order sums that matches the square of the linear sum S_n^1. The following table summarizes the process of transforming the double sum S_5^3 into a single sum:

S_5^3	j	$(2j-1)$	μ_j^3	$(2j-1)\mu_j^3$
	5	9	5	45
	4	7	9	63
	3	5	12	60
	2	3	14	42
	1	1	15	15
$\sum_{j_1}^5$	15	25	55	225

B. The form $(k,h)=(k,3)$

Our method suggests the possibility of a broader hierarchy of sum representations of the form $(k,1),(k,2),(k,3),\ldots,$ each of which can be analyzed using the approach outlined above. The non-uniqueness of these decompositions becomes apparent from the fact that different associated polynomials must be used. To illustrate this point, we can explore the specific case of the form $(k,h)=(k,3)$ using the established methodological framework. This investigation will reveal the fundamental malleability of power sum representations. To determine $p_3(j)$ in the form aj^2+bj+c, we set:

$$\begin{aligned} &\sum_{j=1}^{i}(aj^2+bj+c) = i^3, \\ &a\frac{2i^3+3i^2+i}{6}+b\frac{i^2+i}{2}+ci = i^3, \end{aligned}$$

from which we easily deduce: $a=3, b=-3, c=1$. It follows that we can express the sum S_n^k for $k\geq 3$ in this way:

$$S_n^k = \sum_{j=1}^{n}(3j^2-3j+1)\sum_{i=j}^{n} i^{k-3}, \tag{25}$$

which simplifies to:

$$S_n^k = \sum_{j=1}^{n}(3j^2-3j+1)\mu_j^k, \tag{26}$$

where:

$$\mu_j^k = S_n^{k-3} - S_{j-1}^{k-3}. \tag{27}$$

In particular, for $k=3$, we have:

$$\mu_j^3 = S_n^0 - S_{j-1}^0 = n-j+1. \tag{28}$$

Thus, the generalized sum for $k=3$ becomes:

$$\begin{aligned} S_n^3 &= \sum_{j=1}^{n}(3j^2-3j+1)(n-j+1) \\ &= \sum_{j=1}^{n}\left[-3j^3+3j^2n+6j^2-3jn-4j+n+1\right]. \end{aligned} \tag{29}$$

Simplifying this expression leads to:

$$\begin{aligned} S_n^3 &= \frac{3(n+2)}{4}S_n^2-\frac{3n+4}{4}S_n^1+\frac{n+1}{4}S_n^0 \\ &= \frac{n^4+2n^3+n^2}{4}. \end{aligned} \tag{30}$$

Comparison of Eq. (**??**) with Eq. (**??**) shows that the decomposition of S_n^3 into combinations of lower sums is not unique. While the final result is fixed, the route taken to arrive at it can vary. We can then think of obtaining representations of sums of progressively higher orders that provide equivalent decompositions for the same sum but with different coefficients.

II. STACKED SUMMATIONS

The concept of "stacked summations," also referred to as "nested summations," offers a systematic method for expressing and evaluating power sums. By representing the sum of k-th powers of the first n natural numbers, $S_n^k = \sum_{i=1}^n i^k$, as a series of k nested summations, we gain insights into their recursive and polynomial structures. For any positive integer k, the power sum S_n^k can be expressed as k nested summations, where each layer corresponds to a level of summation:

$$S_n^k = \underbrace{\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_k=1}^{i_{k-1}}}_{k \text{ summations}} 1\,. \tag{31}$$

In this formulation:
- The number of nested summations equals the power k.
- The limits of the inner summations depend on the indices of the outer summations.
- The innermost summation is always equal to 1, reflecting the recursive nature of the process. This process avoids the need of predetermine any polynomial. We illustrate the method by calculating S_n^k explicitly for $k = 2, 3$. For $k = 1$, we accept our precedent derivation.

$$S_n^2 = \sum_{i=1}^{n} i^2 = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1, \tag{32}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n} \frac{i(i+1)}{2}, \tag{33}$$

$$= \frac{n(n+1)(2n+1)}{6}. \tag{34}$$

$$S_n^3 = \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{l=1}^{k} 1, \tag{35}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} k, \tag{36}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{j(j+1)}{2}, \tag{37}$$

$$= \sum_{i=1}^{n} \frac{i(i+1)(i+2)}{6}, \tag{38}$$

$$= \frac{n^2(n+1)^2}{4}. \tag{39}$$

A. Advantages and Insights

The stacked summations approach offers several advantages:
1. *Systematic Representation*: Each power k adds a new summation layer, simplifying higher-order computations.
2. *Recursive Structure*: The method highlights how S_n^k builds upon S_n^{k-1}.
3. *Polynomial Nature*: It naturally explains why power sums result in polynomial expressions.
4. *Pattern Recognition*: The coefficients and structures of the resulting polynomials exhibit clear patterns.
5. *Theoretical Insights*: This method connects power sums to combinatorial and algebraic frameworks.

While stacked summations may not always be the most computationally efficient, their systematic nature and recursive insights make them a valuable tool for theoretical exploration. This approach deepens our understanding of the relationships between powers, nested iterations, and the resulting polynomial forms, bridging the gap between computation and mathematical theory.

II. CONCLUSION

Our reformulation provides a new perspective on power sums, revealing hidden structures that go beyond their apparent simplicity. By introducing the weighting factors μ_j^2, we express power sums as weighted sums of sequences of odd numbers in the form $(k, h) = (k, 2)$ with $k \geq 2$. This provides a consistent framework for analyzing power sums of different orders. In particular:
1) For $k = 2$, the weights simplify to: $\mu_j^2 = n - j + 1$. This allows us to interpret S_n^2 as a Lebesgue sum of the step function $2j - 1$ over a discrete measure space. The measure μ_j^2 "counts" the contributions of each odd number.
2) For $k > 2$, the weighting factors μ_j^k become more complex, assigning decreasing weights to odd numbers and reflecting the influence of the higher-order power sum.
3) Unlike the case of $k = 2$, which involves multiple sequences, the method for $k > 2$ is based on a single sequence of odd numbers.

While this reformulation does not improve computational efficiency, it does provide a systematic self-referential way to compute and decompose power sums into lower-order sums that may reveal new patterns, relationships, and identities. Using this approach with $h = 3$ demonstrates the non-uniqueness of power sum decompositions. This flexibility allows for multiple valid decompositions, depending on the desired outcome–whether computational efficiency or theoretical insight. The non-uniqueness highlights the richness of power sums and suggests new avenues of investigation, as different decompositions can provide alternate ways to understand their properties.

[1] H. Lebesgue, *Measure and the Integral*, (Holden Day, San Francisco, 1966, pag. 180).