Notes on Category Theory - I

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Abstract

These are some notes on category theory that might be useful to a first year graduate student in mathematics. This is the first of a three part series including references to literature, on the theme of identity, examples of objects defined using the universal property and a section on monomorphisms and epimorphisms.

1 Some History

- 1930-40s Formulation of the universal mapping problem and the solution satisfying it, example Free modules, Bourbaki series, Theory of sets , original Hermann publications.
- 1945 Category Theory, Maclane and Eilenberg[3]
- 1956 Cartan and Eilenberg[2]
- 1958 Kan Adjoint functors[5]
- 1964 Freyd adjoint functor theorem[4]

2 Identity

Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}\ \mathcal{C}$. By definition[6], X has an identity morphism $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$. A morphism $f : X \to Y$ in $\mathcal{C}$ is called an isomorphism if there exists another morphism $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a morphism $\theta : F \to G$ of functors parametrized by objects in $\mathcal{C}$ is defined such that if $A \xrightarrow{\alpha} B$ then the following diagram commutes,

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\alpha)} & F(B) \\ \downarrow{\scriptstyle \theta(A)} & & \downarrow{\scriptstyle \theta(B)} \\ G(A) & \xrightarrow{G(\alpha)} & G(B) \end{array}$$

. So, there is a category of functors from $\mathcal{C}$ to

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$\mathcal{D}$. An isomorphism in this category is called an isomorphism of functors.

There is a notion similar to isomorphism of categories called "equivalence of categories". Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F(G(D)) = D'$ and $G(F(C)) = C'$ where D and D' are same upto isomorphism and similarly for C and C' for all objects $C \in \text{Obj } \mathcal{C}$ and $D \in \text{Obj } \mathcal{D}$. This can also be written as $F \circ G = 1_{\mathcal{D}}$ and $G \circ F \in 1_{\mathcal{C}}$.

3 Examples of an object that are defined using a Universal Property

3.1 Product

Let $\mathcal{C}$ be a category in which objects called products exist(to be defined now). Let $E_1, E_2 \in \text{Obj } \mathcal{C}$. A product of E_1, E_2 is the triple P, π_1, π_2 such that P(short for the triple) is unique upto unique isomorphism. That is to say, it satisfies a universal property. More precisely, it is a final object in some suitable category. The maps in the triple are called projection maps. They are named so in analogy with the familiar products and their projection maps to the constituent objects. If P' is an object in the category of sets along with two maps, which are the projection maps to E_1 and E_2, then there is a map from P to the product defined above. Exchanging P and the product defined above there is a map from $E_1 \times E_2$ to P. This is depicted in the following commutative diagram:

$$\begin{tikzcd}
 & & E_1 \\
P' \arrow[r, dotted, "\exists!"] \arrow[urr, bend left, "\pi'_1"] \arrow[drr, bend right, "\pi'_2"'] & P \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & E_2
\end{tikzcd}$$

For example, the triple consisting of the cartesian product of two sets and the projection maps is a product, so products exist in the category of sets **Set**.

These two maps are inverses of each other and the isomorphism between them is unique. This is the universal property of the product in the category of magmas. A product object exists in a category if such conditions are true in that category. In fact this construction exists in the category of sets as well. (y, x) is an example of a different product P. There is an obvious isomorphism between P to $E_1 \times E_2$ such that $(y, x) \mapsto (x, y)$.

3.2 Extension

Let $\mathcal{C}$ be a category in which objects called Extensions exist(to be defined now) in which monomorphisms are injective and epimorphisms are surjectives. Let the category have zero objects as well. Objects have properties of atleast groups.

Let $G, F \in \text{Obj } \mathcal{C}$ such that $F \longrightarrow G$. The triple $\mathcal{E} = (E, i, \pi)$ is an extension of G by F if there exists an object E which satisfies a universal property as shown in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
0 \arrow[r] & F \arrow[r, hook, "i"] \arrow[dr, hook, "i_1"'] & E \arrow[r, two heads, "\pi"] \arrow[d, "f"] & G \arrow[r] \arrow[ur] & 0 \\
& & E_1 \arrow[ur, "\pi_1"'] & &
\end{tikzcd} .$$

The following proposition shows that the category of groups has extensions.

Proposition 1. *In the category of groups, let $\mathcal{E}, \mathcal{E}_\infty$ be two extensions of G by F. If $u : E \to E_1$ be a morphism then u is an isomorphism.*

Proof.
$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[dr, "i_1"'] & E \arrow[r, "\pi"] \arrow[d, "u"] & G \\
& E_1 \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$
Let $u(x) = e_1$. So $\pi(x) = \pi_1 \circ u(x) = \pi_1(e_1) = e_G$. Thus $x \in \text{Ker } \pi = \text{Im } i$. So, $x = i(x')$. $i_1(x') = u \circ i(x') = u(x) = e_1 \implies x' = e_f$ since i_1 is injective. Thus $x = e$. So, $\text{Ker } u$ is trivial and thus u is injective.

$\pi = \pi \circ u$ is surjective and π is surjective so u is surjective.

$\square$

A map $s : G \to E$ is called a section of the extension if $\pi \circ s = \text{Id}_G$. A map $r : E \to F$ is called a retraction of the extension if $r \circ i = \text{Id}_F$. For example, direct products in the category of groups. In the category of groups, if E is isomorphic to $F \times G$ then it is called a trivial extension. Following are some equivalent ways of saying that $\mathcal{E}$ is a trivial extension of G by F.

Theorem 2. *$\mathcal{E}$ extension of G by F.[1] TFAE*

1. *$\mathcal{E}$ is a trivial extension.*
2. *$\mathcal{E}$ has a retraction r.*
3. *$\mathcal{E}$ has a section s such that $s(G)$ is contained in the centralizer of $i(F)$.*

Proof. $E \sim F \times G$. So, (i) $\implies$ (ii).
$$\begin{tikzcd} & & F \\ E \arrow[r, hook, two heads, "f"] \arrow[urr, bend left, "r=\pi_1 \circ f"] \arrow[drr, bend right, "\pi"'] & F \times G \arrow[ur, "\pi_1"] \arrow[dr, "\pi"] & \\ & & G \end{tikzcd}$$

(i) implies (iii)
$$\begin{tikzcd} & F \arrow[d, "i"] \\ F \times G \arrow[r, hook, two heads, "f"] & E \\ & G \arrow[u, "s"] \arrow[ul, "i_1"] \end{tikzcd}$$
So, the section is $s = f \circ i_1$. Let $y_1 \in i_1(G), y \in f \circ i(F)$. So, $y_1 = (e, x_1)$ and $y = (x, e)$ and thus $y_1 y = y y_1$. $s(G)$ commutes with every element of $i(F)$, since $F \times G$ is isomorphic to E.

*(*ii implies i)

$(r, \pi) : E \to F \times G$ is a morphism of extensions so it is an isomorphism by the above proposition and thus $\mathcal{E}$ is a trivial extension.

*(*iii implies i)

$(i, s) : F \times G \to E$ is a morphism of extensions so it is an isomorphism.

□

4 Monomorphisms and Epimorphisms

Definition 3 (Monomorphism)**.** *An monomorphism in a category $\mathcal{C}$ is a morphism $f : U \to V$ that satisfies left cancellation: for any C and any morphisms $g, h : W \rightrightarrows U$, whenever $f \circ g = f \circ h$, then $g = h$.*

Proposition 4. *Composition of monomorphisms are monomorphisms.*

Proof. $h_1, h_2 : U \rightrightarrows V$ morphisms.
$f : V \to W$, $g : W \to X$ monomorphisms.
If $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ then using associativity, $g \circ (f \circ h_1) = g \circ (f \circ h_2) \implies f \circ h_1 = f \circ h_2 \implies h_1 = h_2$. So, composition of monomorphisms is a monomorphism. □

Proposition 5. *In the category of sets and groups, a monomorphism and injective map are same.*

Proof. Let $f : V \to W$ be an injective map in the category of sets(groups). Let $h_1, h_2 : U \rightrightarrows V$ be set(group) morphisms such that $f \circ h_1 = f \circ h_2$. So, $f(h_1(x)) = f(h_2)(x) \implies h_1(x) = h_2(x)$ for all $x \in U$. So, $h_1 = h_2$ and thus f is a monomorphism.

Let $f : V \to W$ be a monomorphism in the category of sets(groups). Let $h_1, h_2 : U \rightrightarrows V$ be set(group) morphisms. Let U be a singleton set consisting of the element u. Let $h_1(u) = x$ and $h_2(u) = y$. Let $f(x) = f(y)$, so $f(h_1(u)) = f(h_2(u))$ and thus $f \circ h_1 = f \circ h_2$ since it $f \circ h_1(u) = f \circ h_2(u)$ for all elements $u \in U$. Since f is a monomorphism, $h_1 = h_2$ and thus $x = y$. So, f is an injective map in the category of sets and groups.

□

Definition 6 (Epimorphism). *An epimorphism in a category $\mathcal{C}$ is a morphism $f : A \to B$ that satisfies right cancellation: for any C and any morphisms $g, h : B \rightrightarrows C$, whenever $g \circ f = h \circ f$, then $g = h$.*

Proposition 7. *Composition of epimorphisms are epimorphisms.*

Proposition 8. *In the category of sets and groups, a epimorphism and surjective map are same.*

The result will be proved only in the category of sets. Readers are invited to prove it in the category of groups.

Proof. Let $f : A \to B$ be an epimorphism in the category of sets. Let $g, h : B \rightrightarrows C$ be set morphisms. Let $C = 0, 1$. Let$g(b) = 1$ if $b \in f(A)$ and $g(b) = 0$ otherwise. Let $h(b) = 1$ for all $b \in B$. $g \circ f(a) = h \circ f(a) = 1$ for all $a \in A$. So, $g \circ f = h \circ f$ and thus $g = h$ since f is an epimorphism. This impels $B = f(A)$ and thus f is surjective.
Let $f : A \to B$ be a surjective map. Let $g, h : B \rightrightarrows C$ be set morphisms such that $h_1 \circ f = h_2 \circ g$. Since f is surjective, $B = f(A)$ and thus for all $b \in B$ there is an element $a \in A$ such that $b = f(a)$. $h_1(b) = h_1(f(a)) = h_2(f(a)) = h_2(b)$ for all $b \in A$. So, $h_1 = h_2$ and thus f is an epimorphism. □

Example 9. *An injective map is an epimorphism in the category of rings. Given the morphism $i : \mathbb{Z} \to \mathbb{Q}$, let $h_1, h_2 : \mathbb{Q} \to R$ where R is another ring such that $h_1 \circ i = h_2 \circ i$. $1_R = h_1(1) = h_1(q/q) = h_1(q)h_1(1/q)$ where $q \in \mathbb{Z} - 0$. So, $h_1(1/q) = 1/h_1(q) = h_1(q)^{-1}$. Similarly, $h_2(1/q) = h_2(q)^{-1}$. $h_1(p/q) = h_1(p)h_1^{-1}(q) = h_1(i(p))(h_1(i(q)))^{-1} = h_2(i(p))(h_2(i(q)))^{-1} = h_2(p/q)$ where $p \in \mathbb{Z}$. So, $h_1 = h_2$ and thus i is an epimorphism.*

References

[1] N. Bourbaki. *Algebra I: Chapters 1-3*. Actualités scientifiques et industrielles. Springer, 1998.

[2] Eilenberg S. Cartan, H. *Homological Algebra*. Princeton University Press, Princeton, 1956.

[3] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58(2):231–294, 1945.

[4] Peter J Freyd. *Abelian categories*, volume 1964. Harper & Row New York, 1964.

[5] Daniel M Kan. Adjoint functors. *Transactions of the American Mathematical Society*, 87(2):294–329, 1958.

[6] M. Kashiwara and P. Schapira. *Categories and Sheaves*, volume 332 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, 2005.