Resolving Dark Energy and The Cosmological Constant: A Conjecture for Homogeneous Infinitesimals

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Abstract: The discovery in 1998 that the universe is paradoxically accelerating its expansion has led some cosmologists to question the correctness of the non-Euclidean geometric theory of gravity, General Relativity. Physically assigning the term Dark Energy to the Cosmological Constant, sometimes viewed as a constant of integration, as the source of this acceleration has only produced even more questions. In the 17th century, there was also a great paradox between two views for the geometric constituents of a line, heterogeneous (made of points) versus homogeneous (made of infinitesimal segments). Evangelista Torricelli elucidated his logical reasoning on why lines must be made of infinitesimal segments instead of points and created one particular fundamental example among many. In this paper, using primitive notions called homogeneous infinitesimals and a new choice axiom, I produce unknown corollaries to Torricelli's argument. With these primitive notions I can correct Leibniz's notation in order to falsify the relationship between infinitesimals and the Archimedean axiom, scale factors/metrics, redefine the Fundamental Theorem of Calculus, differential forms, n-spheres, Gaussian curvature as well as redefine the relationship between real numbers and infinitesimals. I hypothesize that the voluminal elements of the Ricci tensor are a logically flawed view of homogeneous infinitesimals and metrics are an imperfect measuring paradigm. This allows the conjecture that the intractability of Dark Energy is due to the points of coordinate systems within General Relativity actually being a logically flawed heterogeneous interpretation of homogeneous geometry. I propose that Euclidean and non-Euclidean geometry, and the physics equations based upon them (such as relativity and quantum mechanics), should be rewritten from the perspective of homogeneous infinitesimals. I introduce the logical resolutions to geometrical paradoxes in this paper in order to pave the way for the physical logic.

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1 Introduction

The Dark Energy Task Force[1], a committee of scientists tasked with advising the DOE, NASA and NSF on Dark Energy, has stated, "The acceleration of the Universe is, along with dark matter, the observed phenomenon which most directly demonstrates that our fundamental theories of particles and gravity are either incorrect or incomplete." The theoretical value for the Cosmological Constant (CC) is well known by now as the worst prediction ever made in physics for good reason:

> An alternative explanation of the accelerating expansion of the Universe is that general relativity or the standard cosmological model is incorrect. Iare driven to consider this prospect by potentially deep problems with the other options. A cosmological constant leaves unresolved one of the great mysteries of quantum gravity and particle physics: If the cosmological constant is not zero, it would be expected to be 10^{120} times larger than is observed.

If these problems are fundamental enough for the Task Force to advise that General Relativity (GR) itself could be incomplete or incorrect then it also begs the question: *How* could it be either? I propose an answer: GR could be incorrect if our concept of infinitesimals has always been incomplete.

2 Background

The meaning behind dx (although the notation was invented by Leibniz[1] for the Calculus it has also become ubiquitous in GR)[2] can be traced back to concepts from over 2500 years ago and more rigidly to Bonaventura Cavalieri in 1635[3]. One of the great arguments during his time was whether lines were made of infinitesimal one-dimensional segments of lines (homogeneous)[4] or made of non-dimensional points (heterogeneous)[5]. In the same vein, it was also debated whether area would be composed of infinitesimally thin slices of area versus one dimensional stacked lines and

[1] See Katz[12] for a discussion on who invented the term "infinitesimal".

[2] It would seem to me it is taken for granted. Often the Einstein field equation in compact form doesn't even bother to include the infinitesimal notation $dx_\mu dx_\nu$ with the metric notation $g_{\mu\nu} dx_\mu dx_\nu$ such as $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = k T_{\mu\nu}$.

[3] See [2] p303 for timeline.

[4] See [11] p. 4 for discussion.

[5] Note that this in contrast to the philosophical view that a line exists and a point "lies on" that line.

whether volume was made of infinitesimally thin sheets of volume versus stacked two-dimensional planes. Evangelista Torricelli, a brilliant scientist and inventor in his own right and well known to Galileo, is also known in these debates for his talent at taking a difficult concept and explaining it in many different ways. This has been said to have enabled the transfer of fundamental concepts more so than the voluminous writings of Cavalieri. Torricelli's analysis of the heterogeneous/homogeneous debate [6] landed him firmly on the infinitesimal segment side as recent authors have pointed out[6].

> All indivisibles seem equal to one another, that is, points are equal to points, lines are equal in thickness to lines, and surfaces are equal in depth to surfaces is an opinion that in my judgment is not only difficult to prove, but false.

By this he meant, for the simplest case, that it would seem I should be using infinitesimal segments[7] instead of non-dimensional points. Whereas points can't be distinguished from each other, the segments can have infinitesimal length and that length isn't necessarily the same from one line to another thus distinguishing them (as would be his similar argument for area and volume).

One example in particular that he used to demonstrate his reasoning, prior to his early death at the age of 39, has been called by Francois De Gandt the “condensed" “fundamental example" for Torricelli's view on the heterogeneous/homogeneous paradox[8]. While I have come to very much agree with the sentiment that this is a “condensed paradox", my examination of Torricelli's example has also revealed startling unknown similarities with the chain of logic that was used to create the Calculus, non-Euclidean geometry and ultimately General Relativity. It is concerning that despite the longevity of these paradoxes, infinitesimals themselves appear to have been “banished"[13] from modern mathematics which has been effectively pushed out by real analysis. This “purge"[9] can be seen in that the only arena for which infinitesimals are still studied is in the appropriately named “non-standard analysis[9]" (NSA)[18]. However, NSA seems primarily concerned with incorporating infinitesimals into the concept of the real number $\mathbb{R}$ line whereas a homogeneous infinitesimal line is simply the sum of primitive notion infinitesimals and real numbers are mapped onto their sums (as they

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6 See [2] and [11] page 125.

7 There was a philosophical distinction between indivisibles and infinitesimals. I do not expand upon the indivisible concept as I view this a geometrical and philosophical red herring. See page 24[14].

8 See [6] page 164.

9 as in “real" analysis has become so accepted that analysis of infinitesimals is now “non-standard"

can also be proven to possess properties that real numbers do not). This paper will help elucidate the distinction.

3 Flatness, Curvature and HIs

Allow me to attempt to develop a non-rigorous but common starting paradigm. Imagine that you could have a single one-dimensional finite line segment and it is itself composed solely of “adjacent” “infinitesimal” line segments *SEG* (which I will define as an axiomatic primitive notion, see Section 5) and that the sum of their magnitudes $|SEG|$ is defined as the “length" of that finite line segment. I define that the magnitude of the infinitesimals of this line is in agreement with Eudoxus’ theory of proportions in that their *magnitudes are only measurable relative to another infinitesimal* (see equations 3 and 5) and is the geometrical basis for “relativity". Thus their sums, line lengths, are also in agreement with Bernhard Riemann’s definition [17] that the length of every line is “measurable by every other line” (although his definition does not include any mention of the infinitesimals of which it is composed),

$$\sum |SEG| \equiv \textbf{line length}. \tag{1}$$

In the interest of utilizing a graphical teaching method, I will co-opt the NSA transfer principle and state that I can use graphical segments of length to represent infinitesimal lengths (see Fig.1) since their magnitude and number have the same properties as the infinitesimals.

Figure 1: Graphical segments of length representing magnitudes of infinitesimals of length

Suppose that the magnitude $|SEG^n|$ of a segment n could either be of equal relative magnitude (as in Eqn.2) to another segment $n-1$ or could have a different value (as in Eqn.4) *even within the line itself.* A simple to state introductory hypothesis is whether Euclidean geometry can be derived from

$$|SEG^n| - |SEG^{n-1}| = 0 : \text{intrinsically flat} \tag{2}$$

which I will call *intrinsically flat*. Based upon Eudoxus’ theory of proportions, I could also just use a quotient of the magnitude of any infinitesimal relative to any other infinitesimal. The equation then for intrinsically flat would be

$$\frac{|SEG^n|}{|SEG^{relative}|} = 1 : \text{intrinsically flat}. \tag{3}$$

Allow me to define a “point" as simply an infinitesimal *SEG* that is of null magnitude in the direction along the line so that I can also understand that Euclid’s definition of a straight line (Euclid’s Elements, Book I, Definition 4) is one that “lies evenly with the points upon itself” and in this case both terms are equal or even (with a point between the two segments and at their respective ends). I could then propose that non-Euclidean geometry could be derived from

(4) $$|SEG^n| - |SEG^{n-1}| \neq 0 : \text{intrinsically curved}$$

and

(5) $$\frac{|SEG^n|}{|SEG^{relative}|} \neq 1$$

so that the points would no longer be equally spaced and which I will call *intrinsically curved* such as represented by Figure 2.

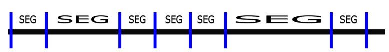

Figure 2: Graphical segments of unequal length representing unequal infinitesimal magnitudes of an intrinsically curved one-dimensional line

Now also suppose that I can examine the number of segments in one line versus another and I called this number the *relative cardinality* (n) so that I can write (in the simplifying case that the line is intrinsically flat) the equation

(6) $$\sum |SEG| = n * |SEG| \equiv \textbf{line length}.$$

This number n will be shown to have similar properties to Cantor’s transfinite number[10], in that n is an “infinite" number, but that value can be greater than, equal to or less than the value of another n. In the case of homogeneous infinitesimals, it will also be shown that they are countable. Let’s rename $|SEG|$ to dx,

(7) $$dx \equiv |SEG|$$

so that Ihave

(8) $$\sum_{1}^{n} |SEG| = n * dx \equiv \textbf{line length}.$$

There is an immediate historical problem with equation 8 in that it appears to run counter to the historical view that infinitesimals are “non-Archimedean"[4] as is understood by

(9) $$ndx \leq 1$$

where dx is an infinitesimal and n is a finite number. However, I am not aware of any research which has considered the ramifications of letting n be transfinite. By this I mean the following equations can be proven to be logically true for mapping real numbers onto sums of homogeneous infinitesimals. This means I can write

$$n_a dx_a \leq n_b dx_b \tag{10}$$

such that

$$5 = 5 \tag{11}$$

or

$$3 < 9 \tag{12}$$

or

$$.2 < e \tag{13}$$

or

$$10 < 23.657\overline{3}. \tag{14}$$

It is important to understand that if I instead wrote equation 10 as

$$A = n_a dx_a, \tag{15}$$

$$B = n_b dx_b, \tag{16}$$

$$A \leq B, \tag{17}$$

then this would mask the relativity of their constituent infinitesimals which I will call the *Axiom of N-M Choice* (ANMC). In words, what this axiom states is that there is an inherent choice made when two sums are compared and that is the relative choice of the Number n of elements and the Magnitude M of those elements. Note that I state *elements* since there is not only the primitive notion (see Section 5) of “infinitesimal length" but also “infinitesimal area", “infinitesimal volume" and so forth[10]. What is commonly considered to be a line[11] is defined as the sum of infinitesimal elements of length, area is defined as the sum of infinitesimal elements of area, volume is defined as the sum of infinitesimal elements of volume, etc. The sums (length, area, volume)

[10] thus it would appear that this is what Riemann unknowingly meant by the term “n-ply" extended magnitudes[17], 1-ply is an element of length, 2-ply is an element of area, 3-ply is an element of volume. It appears he was just missing the concept of relative cardinality.

[11] I state “commonly" because I will demonstrate that there are “lineal" lines, “areal" lines, “voluminal" lines, etc..

are homogeneous in that their composition is strictly made of elements of the same "dimension". Area cannot be composed of elements of length as will be logically proven from the primitive notions and postulates. This is what defines a sum as being "homogeneous". Thus I have

$$n_a dx_a \leq n_b dx_b \tag{18}$$

for sums of elements of length (there are n_a elements of length on the left and n_b elements of length on the right),

$$n_a(dx_{1a}dx_{2a}) \leq n_b(dx_{1b}dx_{2b}) \tag{19}$$

for sums of elements of area and

$$n_a(dx_{1a}dx_{2a}dx_{3a}) \leq n_b(dx_{1b}dx_{2b}dx_{3b}). \tag{20}$$

for sums of volumes (there are n_a elements of volume on the left and n_b elements of volume on the right). If the elements are all of the same magnitude, then the sum with the most elements has the longer length, larger area, larger volume etc. I will limit ourselves to three terms for now (due to the nature of the problem at hand), but by no means is this limited to only three. Note that in each case, n_a represents the total number of elements and not the constituent elemental directional magnitudes dx. In other words, for a square I could write

$$n_a(dx_{1a}dx_{2a}) = (n_{1a}dx_{1a})(n_{2a}dx_{2a}) = (n_{1a2a}dx_a)^2 \tag{21}$$

since

$$n_a = (n_{1a})(n_{2a}) = (n_{1a2a})^2. \tag{22}$$

The two terms $(n_{1a2a}dx_a)$ would represent the length of the sides of the square since

$$(n_{1a}dx_{1a}) = (n_{2a}dx_{2a}). \tag{23}$$

I have named this overall concept and the resulting philosophy the *Calculus, Philosophy and Notation of Axiomatic Homogeneous Infinitesimals* (CPNAHI). Although it is a mouthful, CPNAHI seems to adequately cover the breadth of the concept. Also note that I do not use standard mathematical notation such as R^n for "real space" since there is a conceptual distinction between CPNAHI and real analysis.

3.1 Definition of Lines and CPNAHI n-forms

Allow me to briefly define concepts called lineal lines, areal lines, voluminal lines etc. A lineal line is defined as a path of adjacent infinitesimal elements of length and a lineal line point is just a null infinitesimal. A CPNAHI 0-form is a lineal line. An areal line is defined as a path of adjacent infinitesimal elements of area. An areal line point is defined as an element of area that is null perpendicular to the path and is a CPNAHI 1-form. A voluminal line is defined as a path of adjacent infinitesimal elements of volume. A voluminal line point is an element that is null perpendicular to the path and is a CPNAHI 2-form (and so forth for CPNAHI n-forms). With this understanding and in order to quickly give a reference from which to start, let's briefly examine Evangelista Torricelli's parallelogram paradox[2][11].

4 Torricelli's Parallelogram Paradox

4.1 Historical Analysis of Torricelli's Parallelogram

I could initially recreate the historical explanation for Torricelli's parallelogram involving area and lines of infinitesimal width but let's first consider the simplest observation.

Instead of a line having points on it, assume that a line is made of points and that the number of points in a line determine the length. Two lines that are of the same length have the same number of points. A shorter line has less points and a longer line has more points.

Now take a parallelogram with the four corner points labeled A,B,C, and D as shown in Figure 3. Draw a line BD down the diagonal of it. Let us make a point E on the diagonal line BD. Now draw perpendicular lines from E to a point F on AD, and a second line to a point G on CD. Move these two lines point by point simultaneously so that E moves toward D until they meet, keeping the lines EF and EG always parallel to AB and BC respectively. When I move the lines EF and EG, I am moving their ends simultaneously from point to point on AD, CD and BD.

Since line AD is shorter than the line CD, the number of points that the line AD contains is less than the number of points that line CD contains. However, this creates a paradox. Since I moved the lines point by point and with both points F and G ending up together at point D then this shows that lines AD and CD must also have the same number of points as shown in the equations in Figure 4.

The simplest interpretation of Torricelli's meaning is that the lines AD and CD must be made up of infinitesimal segments (and not dimensionless points) and that these segments must consist of the same number in each line even if they are not of the same magnitude.

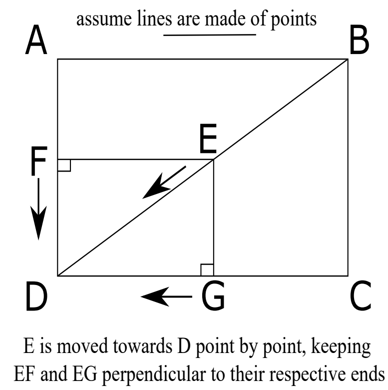

Figure 3: Torricelli: moving perpendicular lines point by point

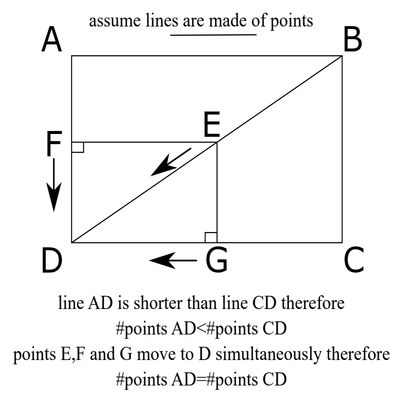

Figure 4: Torricelli: number of points equal or not?

The current most advanced explanation for this paradox is the "understanding that the amount of elements of an infinite set, that is its cardinality, is different from its measure."[12]. While I very much agree with these properties, let us introduce the ANMC to gain more insight.

From Torricelli's observation, I can use the ANMC to write the equality

$$n_{AD} = n_{CD} \tag{24}$$

meaning that lines *AD* and *CD* have the same cardinality in Torricelli's example (same number of one-dimensional infinitesimal segments). Expressing that the magnitudes of these segments cannot be equal I write

[12] See [11] page 125

$$dx_{AD} \neq dx_{CD}. \tag{25}$$

Geometers of Torricelli's era seemed to have been tempted by summing segments to creates lines but were led astray by trying to sum thick lines to create area and thick planes to create volume instead of just fundamental primitive notions of infinitesimal elements of length, area, volume etc. For now, just accept the postulate that a sum of HIs can represent any shape.[13] Despite the shape objections, this can and has been said to have lead to the development of the Calculus. Let us continue to dig further.

4.2 Observing the Axiom of N-M Choice

4.2.1 Intrinsically Flat Lineal Lines Curved With Respect To Another Line

Using the ANMC I can assign equations as a description of Torricelli's parallelogram. Since Torricelli's example states that lines BD, CD and AD are each traversed one segment at a time, then the number of segments in each can be thought of as equal on a one to one basis and I can write

$$n_{BD} = n_{CD} = n_{AD}. \tag{26}$$

Since the segments in any line are all equal, then all adjacent segments in each line are equal and I can write

$$dx^{n}_{BD} - dx^{n-1}_{BD} = 0 \tag{27}$$

$$dx^{n}_{CD} - dx^{n-1}_{CD} = 0 \tag{28}$$

$$dx^{n}_{AD} - dx^{n-1}_{AD} = 0. \tag{29}$$

Since the length of line BD is greater than lines CD which is greater than line AD I can observe that the magnitudes of the infinitesimal segments of each line in relation to each other can be written as

$$dx_{BD} > dx_{CD} > dx_{AD}. \tag{30}$$

Thus the segments of line BD have a magnitude that is *relatively* larger than CD and similarly for AD.

[13] No "protruding parts" to paraphrase Descartes. See [11] page 169 for discussion.

Figures 5 is a visual aid for understanding the previous equations. If by the property of congruence I can lay the lines BD, CD and AD next to each other and they are of unequal length, let us then imagine that I can use the vertical dividing lines to help denote the infinitesimal segments within each line. Torricelli's example is represented so that I can understand that the magnitudes of the segments within BD, CD and AD are all the same (intrinsically flat) within their respective lines. However, the magnitudes of the segments within BD must not be the same as CD, nor AD. Again, this is what Torricelli meant when he said that points are indistinguishable whereas segments can differ by their magnitude. They are intrinsically curved relative to each other. I can then also understand that the cardinality within AD must be the same as BD as well as for CD which was chosen when I opted to move the perpendicular lines "point by point".

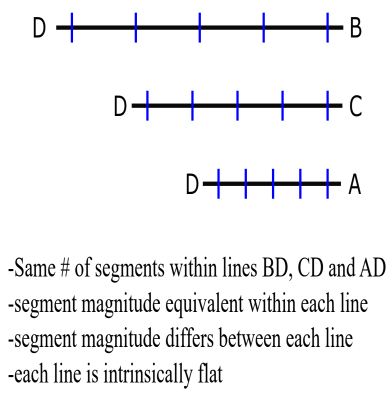

Figure 5: Intrinsically Flat Lineal Lines With Equal Cardinality And Relative Curvature

4.2.2 Intrinsically Flat Lineal Lines With Relative Flatness

Imagine now that I have again taken line BD, CD and AD and laid them next to each. I again traverse the segments on the line one for one but this time I do so in equal magnitude increments as shown in Figure 6. Line BD is still longer but it is longer since the relative cardinality (number of segments) is the greatest as in Figure 6.

I can then write

$$n_{BD} > n_{CD} > n_{AD} \tag{31}$$

$$dx_{BD}^{n} - dx_{BD}^{n-1} = 0 \tag{32}$$

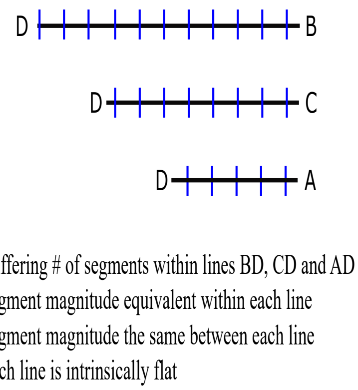

Figure 6: Intrinsically Flat Lines With Differing Cardinality And Relative Flatness

$$dx_{CD}^{n} - dx_{CD}^{n-1} = 0 \quad (33)$$

$$dx_{AD}^{n} - dx_{AD}^{n-1} = 0 \quad (34)$$

$$dx_{BD} = dx_{CD} = dx_{AD} \quad (35)$$

4.2.3 Relative Curvature and Flatness via the ANMC

From these two simple examples I can observe the axiom in that I have the choice to make the inequality

$$n_{AD} * dx_{AD} < n_{CD} * dx_{CD} < n_{BD} * dx_{BD} \quad (36)$$

true by either make n all equal and vary dx or vice versa.

4.3 The Defining of Scale Factors: Sum of Cardinality vs Sums of HIs

Notation provides economy of thought but that notation can be of a bad value if it mischaracterizes the underlying geometry it represents. Let's see what that means by examining sums of infinitesimals and defining scale factors.

4.3.1 Sums of Infinitesimals

From Figure 5 I could view line BD as the summing of two lines AD where the magnitudes of each element in one AD was summed with the magnitude of a corresponding element in the other AD. From Figure 6 I could view line BD as the summing of two lines AD where the cardinality of one line AD was summed with the cardinality of the other line AD creating a longer line, BD, with larger cardinality.

4.3.2 Euclidean Scale Factor

Let's use the common geometric conception that if $A = 4$ is written as the representation of the length of a line, then it is said that a scale factor $m = 3$ would give

$$mA = 12. \tag{37}$$

One could say that if I have another line $B = 12$ then m could be defined as

$$m = \frac{B}{A} \tag{38}$$

as can be seen in Figure 7. However, this hides something geometrical that can be

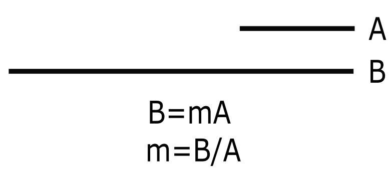

Figure 7: Common understanding of scale factor for simple one-dimensional line

fleshed out with the ANMC.

4.3.3 Scale Factor Defined As A Quotient of Relative Cardinalities

Let us take the equation 18 and relabel it so that I am comparing the length of a line to a "reference" line,

$$n_a dx_a \leq n_{ref} dx_{ref}. \tag{39}$$

If

$$n_a dx_a = 3 = A \tag{40}$$

and

$$n_{ref}dx_{ref} = 12 = B \tag{41}$$

and

$$dx_a = dx_{ref}. \tag{42}$$

then Ican define a *relative cardinality* scale factor m as

$$m_{RC} = \frac{n_{ref}}{n_a} = 3. \tag{43}$$

Thus, writing

$$(m_{RC}n_a)dx_a = 12 = B \tag{44}$$

means that line A has $\frac{1}{3}$ the cardinal number of elements of magnitude as the referenced line B as shown in Figure 8. Line B is scaled 3 times the length of line A. *I specifically note here the similarities between a relative cardinality scale factor and scale factors of Euclidean geometry.*

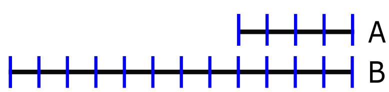

Figure 8: Scaling of relative cardinality

4.3.4 Scale Factor Defined As A Quotient of Relative Magnitudes

The opposing case allowed by the ANMC here is scaling of magnitude dx. Again, if I have the equation 18 and relabel it so that I am comparing the length of a line to a “reference" line,

$$n_a dx_a \leq n_{ref}dx_{ref}. \tag{45}$$

If

$$n_a dx_a = 3 = A \tag{46}$$

and

$$n_{ref}dx_{ref} = 12 = B \tag{47}$$

but instead this time

$$n_a = n_{ref}. \tag{48}$$

then I can define a *relative magnitude* scale factor m_{RM} as

$$m_{RM} = \frac{dx_{ref}}{dx_a} = 3. \tag{49}$$

Thus, writing

$$n_a(m_{RM} dx_a) = 12 = B \tag{50}$$

means that line A has infinitesimal elements that $\frac{1}{3}$ the magnitude as the referenced line B as shown in Figure 9. Again, line B is scaled 3 times the length of line A. *I*

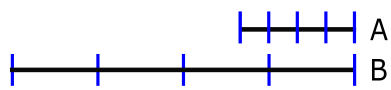

Figure 9: Scaling of relative cardinality

specifically note here the similarities between a relative magnitude scale factor and scale factors h that define the equation $\mathcal{E}_i = h_i \boldsymbol{e}_i$ (2.115 in [3]).

> The $\mathcal{E}_i$ are related to the unit vectors $\mathbf{e}_i$ by the scale factors h_i of Section 2.2. The $\mathbf{e}_i$ have no dimensions; the $\mathcal{E}_i$ have the dimensions of h_i.

There would be a conceptual danger in defining dx as a basis or unit vector. By this I mean

$$m_{RM} dx_a = h\mathbf{e} \tag{51}$$

imposes conceptual locks on infinitesimal magnitudes since m_{RM} does not possess the claimed dimensions of h_i. *I specifically note here that the metric $g_{\mu\nu}$ has been defined in terms of a scale factor*[14] *such that*

$$g_{\mu\nu} dx_\mu dx_\nu = h_\mu dx_\mu h_\nu dx_\nu \tag{52}$$

and the similarities between a relative magnitude scale factor $m_{RM} dx$ and hdx.

4.3.5 ANMC and Area

Before I analyze Torricelli's parallelogram further, let us understand the ramifications of scale factors with area. As with Figures 5 and 6 hopefully it is obvious I can sum together numbers of elements of length to create a longer line or sum together the magnitudes of elements of length to create a longer line.

[14] see Equation 2.8 in [3]

In the same vein there are two methods for summing together two areas, either by their cardinality of elements or by the magnitude of the elements (or some combination of both) as shown in Figure 10 (note that figures are not exact cardinality wise as this is an inherent flaw in using graphical proofs for homogeneous infinitesimals). For the

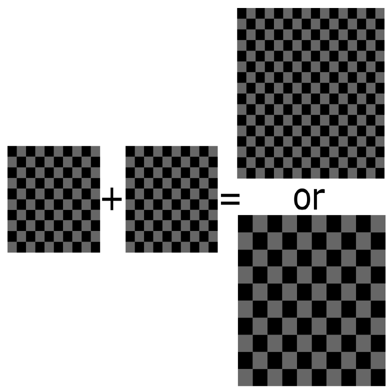

Figure 10: Summation of areas either via cardinality or magnitude

top sum of area this is a sum of cardinalities and I could write

$$n_a(dx_1dx_2) + n_b(dx_1dx_2) = (n_a + n_b)(dx_1dx_2). \tag{53}$$

The bottom sum of area is a sum of magnitudes and could be written as

$$n(dx_{a1}dx_{a2}) + n(dx_{b1}dx_{b2}) = n(dx_{a1}dx_{a2} + dx_{b1}dx_{b2}). \tag{54}$$

Unfortunately the notation can get tedious fast but this is necessary in order to flesh out a difference with Leibniz's notation further in the paper.

I could also view this as scaling of either the cardinality of elements of area or of their magnitudes such that I could write

$$mn(dx_1dx_2) = (n_1 + n_2 + n_m)(dx_1dx_2) \tag{55}$$

using m to scale the cardinal number of the elements or

$$n(m(dx_{a1}dx_{a2})) = n((dx_{a1}dx_{a2})_1 + (dx_{b1}dx_{b2})_2 + (dx_{b1}dx_{b2})_m) \tag{56}$$

to scale the magnitudes of each of the elements of area.

4.4 Analysis of the Area of Torricelli's Parallelogram Using ANMC

Now that I have an understanding of area, let's take Torricelli's parallelogram, remove some of the notation and split up the opposing triangles as in Figure 11. I have left point

E so that I still have some visual reference to the diagonal line in the parallelogram. Without proof, assume that $Area1 = Area2$ and that $Area3 = Area4$.

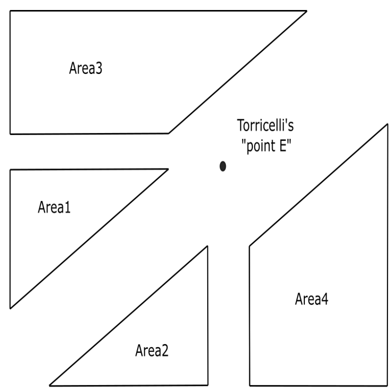

Figure 11: Torricelli's parallelogram divided up with areas labeled

I might mentally understand that as E “moves" toward the bottom left, that the area in $Area3$ and $Area4$ increases while the area in $Area1$ and $Area2$ decreases. I might also understand that the previous equalities still hold in that $Area1 = Area2$ and that $Area3 = Area4$. Let's add a concept from ANMC to our graphic to help us understand why this is.

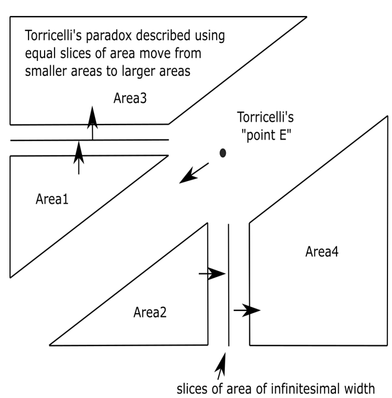

Figure 12: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via slices of area

In Figure 12 I can obtain the concept that infinitesimal slices of area are being moved from $Area1$ to $Area3$ and from $Area2$ to $Area4$. Since the magnitude relationships between the top and bottom areas are constant, then these slices of area in the top and bottom must be equal also.

Let's now rotate the top areas as a triangle so that the slices of area for both are vertical in our graphic as in Figure 13. If I view these slice of area instead as columns of

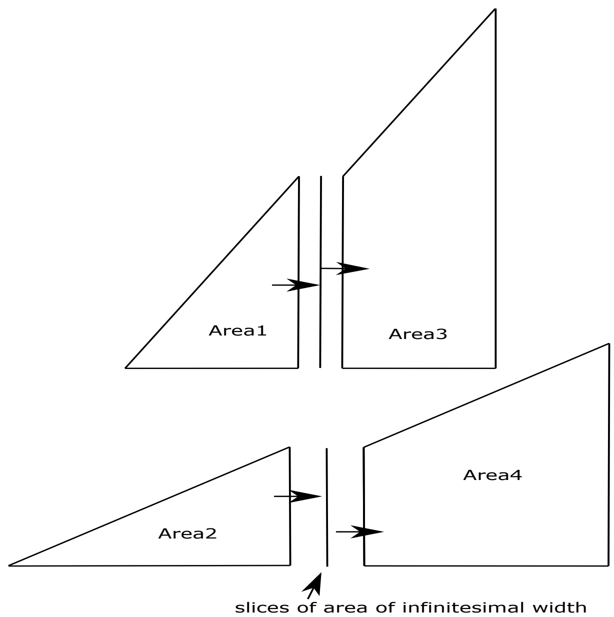

Figure 13: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via vertical slices of area

elements of area, then I can draw Figure 14.

Let me point out in the graphic there are three elements of area shown in the column (and not six elements of length) to conceptually represent the cardinality. I can also see that the elements of area in the top triangle have the same "height" as the "width" of the elements in the bottom triangle and vice versa. I can also see that the top and the bottoms have the same cardinal number of slices (and I can assume number of elements). If I am not averse to counting the number of elements along the bottom of each triangle, I can see that they too have the same cardinal number.

It will be helpful to represent these columns of area with Figure 15 where I have chosen the real number values to represent the lengths of the sides and bottoms of the top and bottom triangle. Note that I can write that each column of area in the top is equal to each column of area in the bottom,

$$n_{ytop} dy_{top} dx_{top} = n_{ybot} dy_{bot} dx_{bot}. \tag{57}$$

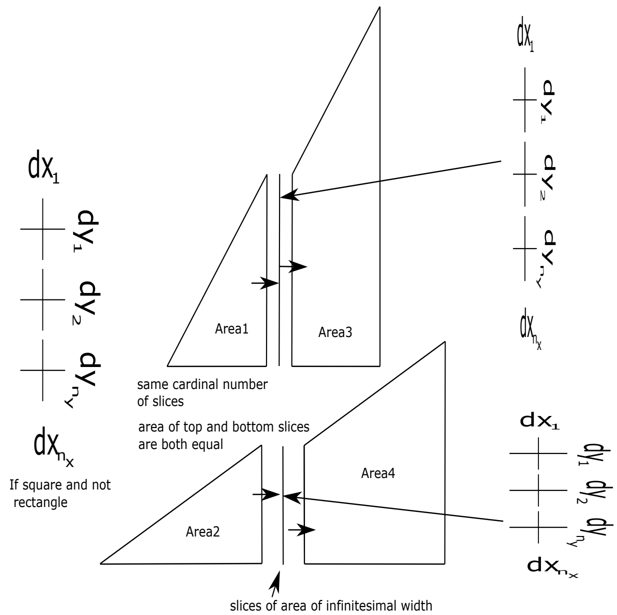

Figure 14: Torricelli's parallelogram with area being removed from Area1 to Area3 and from Area2 to Area4 via columns of vertical infinitesimal elements of equal area

The height of the labeled top column is

$$n_{ytop} dy_{top} = 1 \tag{58}$$

and the height of the labeled bottom column is

$$n_{ybot} dy_{bot} = \frac{1}{2}. \tag{59}$$

The bottom column is twice as "thick" as the top column and the bottom column is half the height of the top column since they possess equal infinitesimal areas. *It is important to note that I am measuring the height of a column of elements of area and not the length of a one-dimensional line! It is also important to note that each element is intrinsically curved within itself and not flat (the dx and the dy are not of equal magnitude).*

4.4.1 Comparison of Torricelli ANMC Solution With Leibniz Area Solution

One particular example of a proof provided by Leibniz [15] is particularly famous for a term he purposefully omits in his solution. This is useful to me since his proof is for

[15] see pg.255 [7]

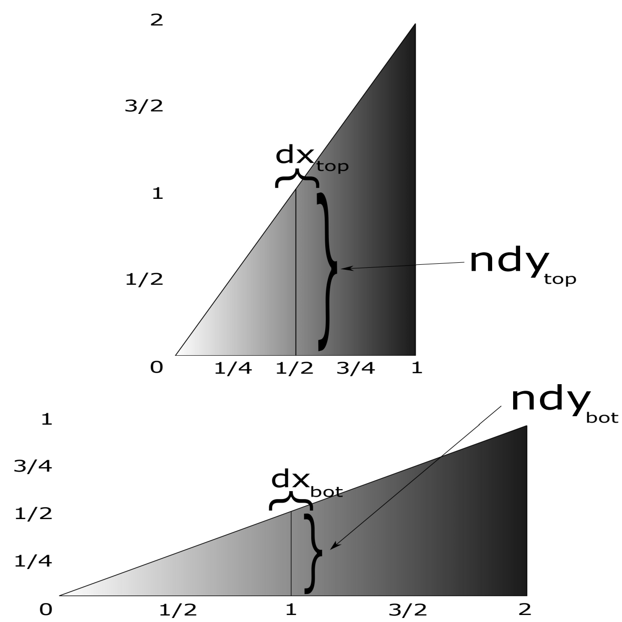

Figure 15: Torricelli's parallelogram composed of columns of elements of area

determining the equation for a change in area and the ANMC solution to Torricelli's parallelogram provides an interesting conceptual counter argument for the concept of relative cardinality scaling.

Essentially, let

(60) $$xy = \text{area}$$

so that

(61) $$d(xy) = \text{change in area.}$$

Leibniz adds infinitesimals dx and dy to this notation for x and y so that

(62) $$d(xy) = (x+dx)(y+dy) - xy$$

denotes a change in area of xy. From treating his terms algebraically he ends up with

(63) $$d(xy) = xy + xdy + ydx + dxdy - xy$$

which should give

(64) $$d(xy) = xdy + ydx + dxdy$$

but instead it equals

(65) $$d(xy) = xdy + ydx.$$

He states

> the omission of the quantity $dxdy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small, will leave $xdy + ydx$.

This is an important conceptual distinction from CPNAHI in that it does not seem that he views $dxdy$ as an element of area and thus can be discarded. ANMC keeps track of the relative number of elements and their relative magnitudes. Without these two sides of the same coin of measurement, both the number of non-dimensional points in a line or the magnitudes of infinitesimal elements is paradoxical.

It may also be helpful to consider the distinction between Leibniz's argument for a "change in area" and the concept of a change in area caused by scaling. These "ghosts of departed quantities"[5] will be analyzed further in depth in a future paper since Equation 65 is essentially his viewpoint of our two columns in Figure 15.

4.4.2 The Fundamental Theorem of Calculus of Axiomatic Homogeneous Infinitesimals: The Integral

With the postulate that two areas are equal if the sums of their homogeneous infinitesimal elements of area are equal I can also use the ANMC and the column solution for Torricelli's parallelogram to create a process for summing these elements of area. In simple terms, I can find the total area by adding up columns. Let's take a look at Figure 16.

In this figure I could represent the column of area two ways, one as a double bounded strip of area or as a column of plus signs that represent the individual elements. This illustrates the difficulty in using graphical means to teach homogeneous infinitesimals. The area in each of these columns can be represented by

$$(n_y dy)dx = \text{area of column} \tag{66}$$

where $n_y dy$ is the height of the column and $1 * dx$ is the width since they are one infinitesimal wide. If I were to sum column by column then I could write

$$\sum_{1}^{n_x}(n_y dy)dx = n_{total}(dxdy) = \text{total area of all } n_x \text{ columns} \tag{67}$$

since I am summing up all n_{total} elements of area $dxdy$ column by column.

Since our elements of area are flat, then I could define a cardinal relationship between the elements of area. If the height of a column is dependent on the length of a row

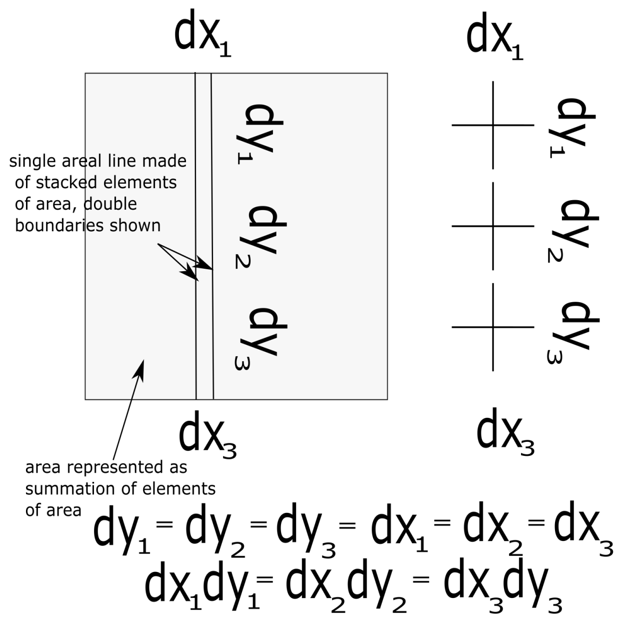

Figure 16: Summation of columns of intrinsically flat elements of area as strips of area or columns of plus signs

of elements, then I could say that $n_y dy$ is a function of $n_x dx$. Since $dy = dx$ for all elements, then n_y can be said to be a function of n_x. Let us call this a cardinal function and essentially it means that the cardinal number of elements in a column is dependent on the cardinal number of elements in a row. This can be defined as a function f where

$$n_y dy = f(n_x dx) \tag{68}$$

or

$$n_y = f(n_x). \tag{69}$$

In simple terms I could ask: What is the number of y elements in a column for a particular number of x elements in a row?

Note the similarity of Equations 67 and 68 with Leibniz summation notation for an integral,

$$\int f(x)dx \tag{70}$$

and thus of a one-form.

4.4.3 The Fundamental Theorem of Calculus of Axiomatic Homogeneous Infinitesimals: The Derivative

Let us suppose that I want to measure the change in area of the columns as I move from left to right. Since the elements of area are flat, I could measure the change in the height of the columns by measuring the change in their cardinality as I move column by column. I could then write

$$\frac{(n_y dy)}{dx_n} - \frac{(n-1)_y dy}{dx_{n-1}} \tag{71}$$

which simplifies to

$$\frac{(n_y - (n-1)_y)dy}{dx} \tag{72}$$

which can be written as the change in the number of dy per dx,

$$\frac{(\Delta n_y)dy}{dx}. \tag{73}$$

If $\Delta n_y = 0$ then there is no change in the height of the columns and the derivative is zero. If $\Delta n > 0$ then the column height is increasing and the derivative is positive and vice versa for less than zero.

Note here that if I attempt to define this as equal to Leibniz's notation of

$$\frac{(\Delta n_y)dy}{dx}^{CPNAHI} \equiv \frac{dy}{dx}^{Leibniz} \tag{74}$$

it exposes the flaw in his since in CPNAHI the quotient

$$\frac{dy}{dx}^{CPNAHI} \tag{75}$$

is a ratio of infinitesimal magnitudes and here always equals 1 since our elements of area are flat.

Therefore, integration is just summing columns of infinitesimal elements of area to find the total area, and differentiation is just measuring the change in the number of elements in those columns as I move along the area.

4.4.4 CPNAHI epsilon-delta limit equivalence

To define the epsilon-delta limit in terms of CPNAHI I again need to make sure that I understand the concept and distinction of lineal lines, areal lines, voluminal lines etc. A lineal line is defined as a path of adjacent infinitesimal elements of length. An areal line is defined as a path of adjacent infinitesimal elements of area. A voluminal line is defined as a path of adjacent infinitesimal elements of volume and so forth. It is important to understand that infinitesimals are a language and are not limited to descriptions of physical space and time.

If I have an area that I have decided is composed of flat elements of area, then I define that only areal lines can exist however I can map lineal lines onto this area and postulate that the elements of length have the same properties as the background flat geometry on which it is mapped (The same goes for lineal and area lines mapped onto voluminal elements). By this I mean that all textbooks which show “lines" drawn on a page are, in CPNAHI conceptual terms, a mapping of lineal lines onto flat elements of area such as Figure 17. I have added in the elements of area on Keisler’s drawing [13]. Note that this graphical teaching method hides the nature that Δx will become defined as the change over a single infinitesimal wide and Δy as a change in the number of y infinitesimals in height. In other words, “rise over run" is the change in number of elements in height per width of each single element.

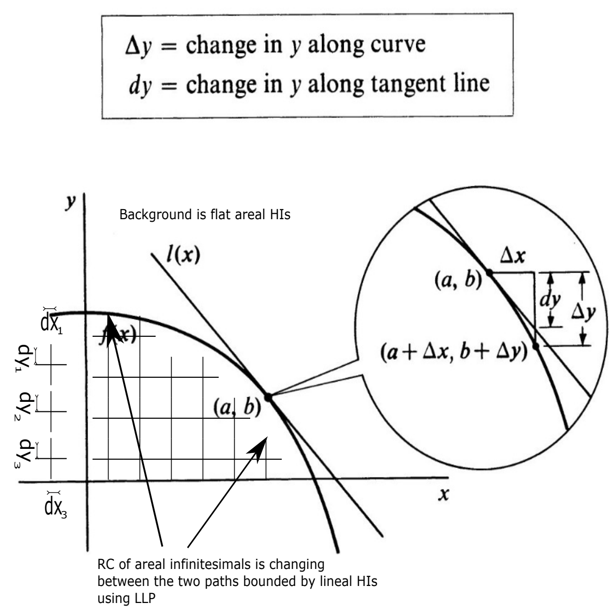

Figure 17: Keisler’s graphical definition of a derivative using non-CPNAHI concept of infinitesimals: modified as per author’s public license

The concept of the $\epsilon - \delta$(epsilon-delta) limit comes from the concept of functional relationships. However, in CPNAHI this is a cardinal relationship (being distinct from a magnitude relationship which I have not yet introduced). If I keep the cardinal function simple (i.e. non-CPNAHI smooth continuous function) then I can view

$$\delta \equiv dx \tag{76}$$

and

$$y = n_y dy = L \tag{77}$$

with

$$(n_y + 1)dy = \epsilon^{pos} \tag{78}$$

and

$$(n_y - 1)dy = \epsilon^{neg}. \tag{79}$$

This should bring in a needed conversation concerning the distinction of the definition of what *change* means between Leibniz's dy and his dx. It appears that making ϵ as small as I want does not mean minimizing the magnitude of dy since that would be an endless circular argument since all $dy = dx$ and instead is a cardinality argument meaning that I can go down to one infinitesimal above and below L. I leave this to a future paper.

4.4.5 CPNAHI Chain Rule

As a quick note of the usefulness of CPNAHI, lets briefly consider the Chain Rule in CPNAHI. If the number of elements of y is a function of the number of elements x, but the number of elements of z is a function of the number of elements of y, then the chain rule just allows us to find the relationship where the number of elements of z is a function of of the number of elements of x, with

$$n_y dy = f(n_x dx) \tag{80}$$

and

$$n_z dz = f(n_y dy) \tag{81}$$

then Ican write

$$\frac{(\Delta n_y)dy}{1 * dx} * \frac{(\Delta n_z)dz}{1 * dy} = \frac{(\Delta n_z)dz}{1 * dx}. \tag{82}$$

5 Primitive Notion and Postulates

CPNAHI Primitive Notion Let a homogeneous infinitesimal (HI) be a primitive notion.

CPNAHI Postulates

(1) Postulate of Homogeneity: Homogeneous Infinitesimals (HIs) can have the property of direction with magnitude which gives length for one direction, area for two, volume for three etc. Only HIs of length can sum to create lines. Only HIs of area can sum to create area. Only HIs of volume can sum to create volume. etc..[16].

(2) HIs conform to the boundaries of any shape.

(3) HIs can be adjacent or non-adjacent to other HIs.

(4) A set of HIs can be a closed set.

(5) A lineal line is defined as a closed set of adjacent HIs (path) with the property of length. These HIs have one direction.

(6) An areal line is defined as a closed set of adjacent HIs (path) with the property of area. These HIs possess two orthogonal directions.

(7) A voluminal line is defined as a closed set of adjacent HIs (path) with the property of volume. These HIs possess three orthogonal directions.

(8) Higher directional lines possess higher orthogonal directions.

(9) The cardinality of these sets is infinite.

(10) The cardinality of these sets can be relatively less than, equal to or greater than the cardinality of another set and is called *relative cardinality*(n or RC).

(11) Postulate of HI proportionality: RC, HI magnitude and sums each follow Eudoxus' theory of proportion.

(12) The magnitudes of a HI can be relatively less than, equal to or the same as another HI.

(13) The magnitude of a HI can be null.

[16] This is also in accordance with Eudoxus' theory of proportions which I view as equivalent to not being possible to sum heterogeneous infinitesimals. In simpler words, "stacked" two-dimensional planes cannot integrate into a volume.

(14) If the HI within a line is of the same magnitude as the corresponding adjacent HI, then that HI is intrinsically flat relative to the corresponding HI.

(15) If the HI within a line is of a magnitude other than equal to or null as the corresponding adjacent HI, then that HI is intrinsically curved relative to the corresponding HI.

(16) A HI that is of null magnitude in the same direction as a path is defined as a point. A lineal HI point has the property of 0 dimensions. An areal HI point has the property of length orthogonal to the path. A voluminal HI point has the property of area orthogonal to the path.

(17) Adjacent points within adjacent areal lines are said to create an arc (i.e. the circumference of a circle).

(18) Adjacent points within adjacent voluminal lines are said to create a surface (i.e. the surface of a sphere).

5.1 CPNAHI: Axiom Issues

As one author states[8], axioms should be given without justification. However I see no way to simply launch into a proof using CPNAHI as even Leibnizian notation would seem to be flawed. I know of nothing within the body of knowledge of geometry nor mathematics[17] that already contains analysis of both Torricell's work and non-Euclidean geometry from which to launch my hypotheses. Therefore, I instead will justify my primitive notions and postulates by resolving certain historical paradoxes that presently seem unsolved and/or present a different method of describing them. It is my hope that these similarities and rudimentary proofs will motivate individuals to my view that CPNAHI has compelling features. There are many things that I have not yet attempted (such as Grassmann algebra or even simply deriving contravariance and covariance from CPNAHI) but I fear letting the perfect be the enemy of the good. Some of these postulates will inevitably be found to be derivable from each other but I think the practice of deducing that will be more informative than an absolutely correct set here.

[17] Nor alternate axiom systems. See Chapter 15 [15].

6 Conceptual Comparisons

6.0.1 Background and Foreground Geometry

Let us assume that I can have a sum of areal HIs to create area (but not limited to just areal). From this I can understand the concept of a coordinate system as in Figure 18. “Events" are located on this coordinate system which do not effect the background itself and can be denoted by mapping lineal lines and lineal points onto it. I refer to this as Flat Background Geometry. Since the background is composed of flat areal HIs, then I can represent the lineal line lengths with coordinates using the CPNAHI equation for a flat lineal line. Thus the coordinates are actually just dependent upon the relative cardinality of a lineal line since all $dy = dx$. Coordinate transformations can be defined by changing line length by adding and subtracting relative cardinality.

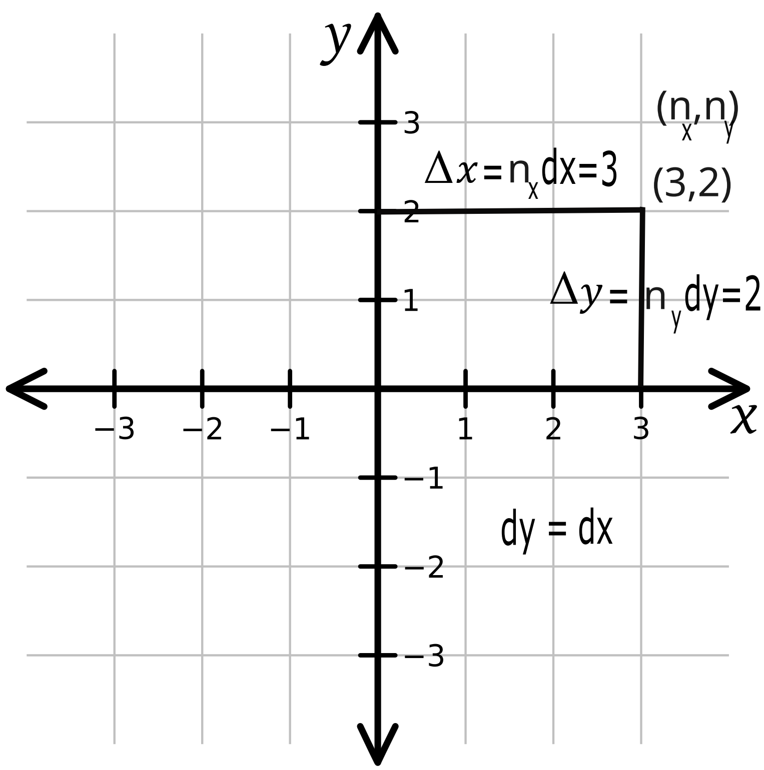

Figure 18: Intrinsically Flat Areal HIs Creating Cartesian Coordinate System

Let us also assume that I can have a different sum of areal HIs. No events take place “on" this geometry. Instead, all events are simply changes in relative cardinality and magnitudes of the HIs themselves. I refer to this as Curved Foreground Geometry. I could also argue that a coordinate transformation would occur if I defined the changing length of a line by adding and subtracting relative magnitudes of infinitesimals.

6.0.2 Anti-derivative

Hopefully CPNAHI makes the concept of anti-derivatives simpler. If an area is defined by the boundaries of lineal lines and a derivative is the measure of the change in that area column by column, then the anti-derivative is simply taking the derivative (change in number of elements of the column) and reversing the process so that I can find the total area that was measured via the changing height of the columns. Note that there is a constant of integration because I am not measuring the total number of elements in a column, only the number that changed within the lineal line envelope. A constant of integration represents the unknown potential number of elements that were not part of the change. I refrain from further explanation here for the sake of brevity and will expand upon it in a future paper.

6.0.3 Pythagorean Theorem

Derivation of the Pythagorean Theorem has been cited as an example of the failure of infinitesimal methods[18]. However in my opinion I believe it can not only be derived from CPNAHI but is actually a fundamental demonstration of the relativity of homogeneous infinitesimals. In future research I will flesh this out, but for now I refer to Figures 19 and 20. The first figure demonstrates that Area C, C^2, changes depending on whether I have chosen to sum the cardinal number of elements or summing the magnitudes as in the second. Thus Ican write

(83)
$$AreaC = n^C(dx_1dx_2)^C = C^2 = AreaA + AreaB = n^A(dx_1dx_2)^A + n^B(dx_1dx_2)^B = A^2 + B^2$$

which I consider will probably be proven true down to the cardinal summation of four elements of area or the summation of the magnitudes of four elements of area. Note that the elements within the bounded triangle are flat for cardinality summation but not necessarily for magnitude summation.

[18] see pg.11 of [19]

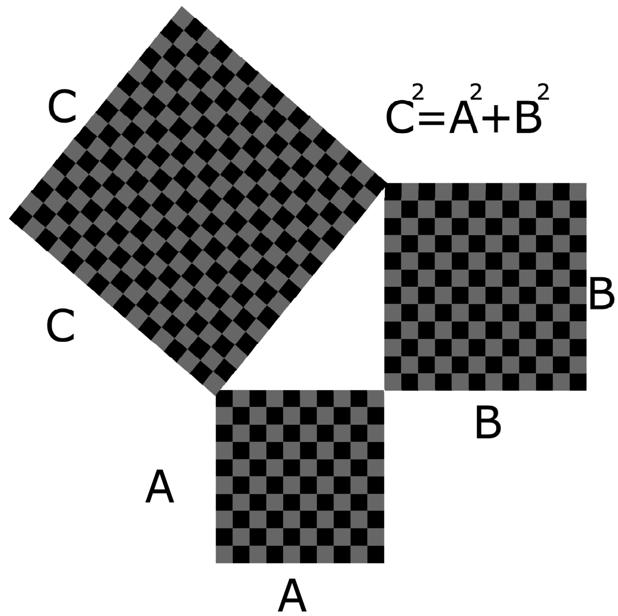

Figure 19: Pythagorean Theorem via Cardinality Summation

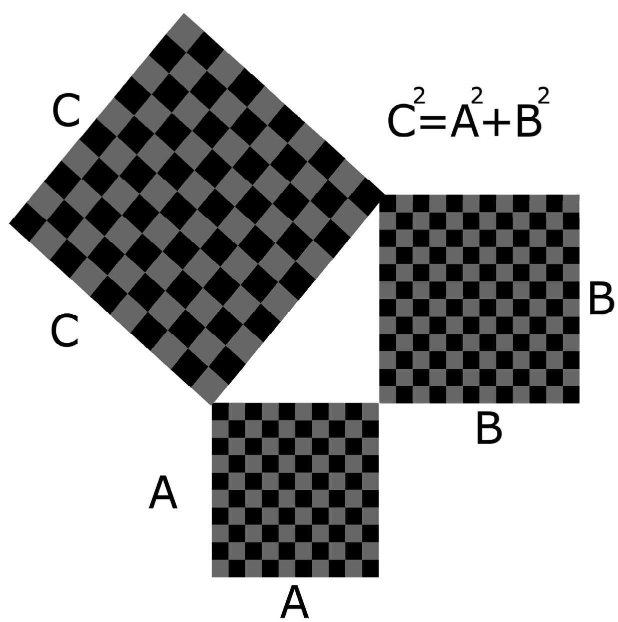

Figure 20: Pythagorean Theorem via Magnitude Summation

6.0.4 Parallelogram Law or Vector Addition

I note here that vector addition can be defined via summing the components of a homogeneous infinitesimal. In other words, on a background of flat areal elements $dxdy$, a vector A can be defined as having the components $(n_x dx, n_y dy) = A$. Summing two vectors $A + B$ is just summing their components,

$$A + B = (n_{xA}dx, n_{yA}dy) + (n_{xB}dx, n_{yB}dy) = ((n_{xA} + n_{xB})dx, (n_{yA}dy + n_{yB}dy) \tag{84}$$

6.0.5 Euclid's Parallel Postulate

If I have an area bounded by two lineal lines, then it should be trivial (but not notationally short) to define Euclid's Parallel Postulate as conservation of the relative cardinality of a column between the two lineal lines. I view Euclid's postulate as one form of geometric N-M conservation. In other words, there are other definitions of "straight" depending on whether the relative cardinality or magnitude is preserved. The Parallel Postulate, like summation of columns of elements of area, is just a specific case. I will expand on this in a future paper.

6.0.6 CPNAHI one-forms: resolution of the circle-line paradox

Torricelli also pointed out a paradox concerning circles [19]. For our purposes, let us rename an areal line as a CPNAHI one-form. In Figure 21 I have detailed a set of CPNAHI one-forms that are arranged radially to form a circle from the dx which I can call a CPNAHI 1-sphere. These one-forms are not flat in that the dx gets bigger as ndy increases. In Figure 22 I have on the left a standard circle with commonly conceived lines drawn from the center. On the right I have a CPNAHI 1-sphere. This comparison allows us to examine the assumptions in these questions: For the circles on the left, can more lines cross the outer circle than the inner circle? Can one-dimensional lines overlap? Do lines possess area?

For the circles on the right, they can be considered as composed of the dx of the elements of area. The elements are not flat meaning that $dy \neq dx$ and the magnitude of the dx grows as ndy increases, which is why the radius gets larger. Same number of elements, just larger. I consider it possible that π may be derivable from CPNAHI circles since the left circle is a lineal line mapped onto a background of flat areal elements and the right one seems more geometrically fundamental.

[19] see pg. 25 [14]

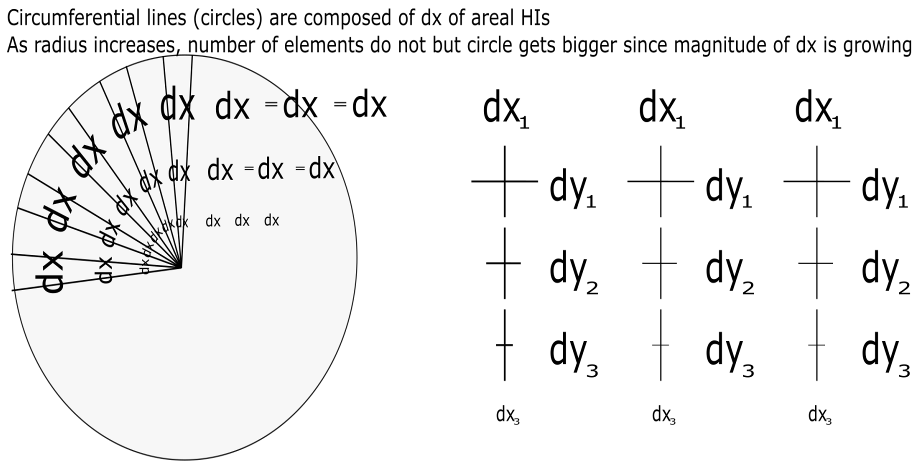

Figure 21: CPNAHI 1-sphere formed from intrinsically curved CPNAHI one-forms

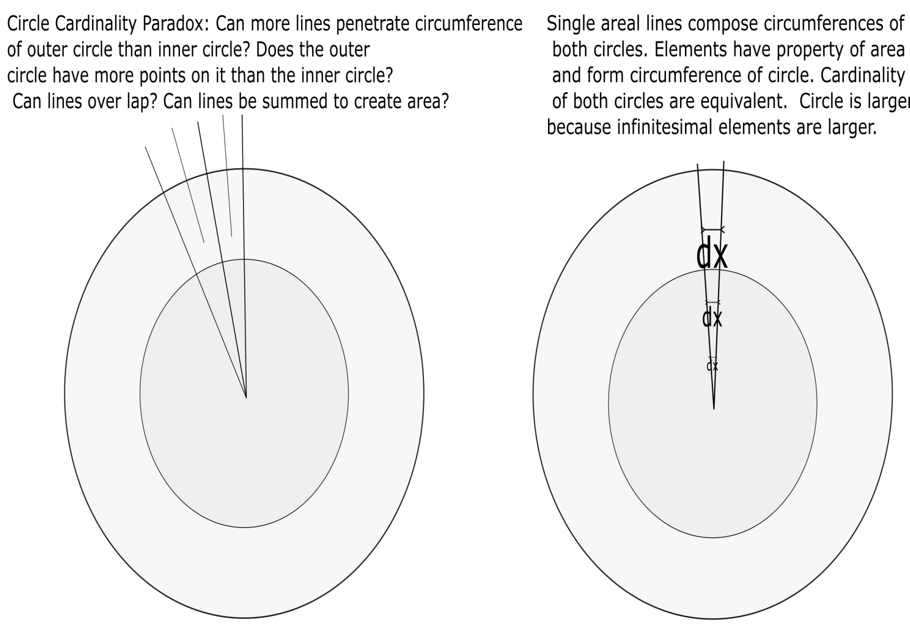

Figure 22: Euclidean circle vs CPNAHI circle

6.0.7 CPNAHI surfaces of voluminal elements: manifolds and Gaussian curvature

Since a volume is defined as the sum of infinitesimal elements of volume, let's define a "surface" as composed of adjacent voluminal lines where their points gives the surface the property of area[20]. In Figure 23 this surface is composed of the shaded squares which represent infinitesimal elements of area, or elements of volume that are null in the z direction. Let us call this surface a CPNAHI manifold. As a mental aid, think of slicing through a volume. The stacks of voluminal elements here define a voluminal line[21]. I can define the "curvature" of this surface by the change in the magnitude of

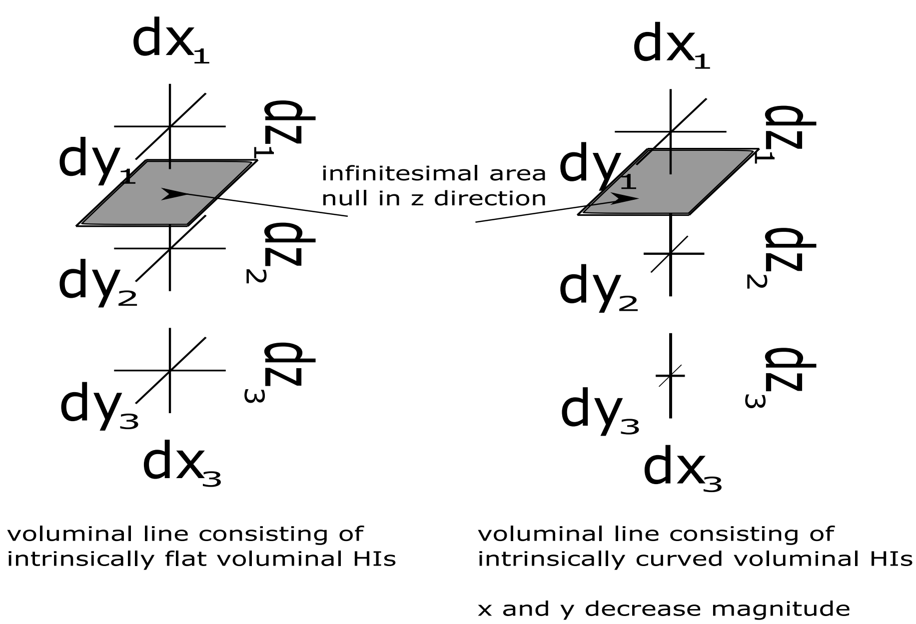

Figure 23: Voluminal lines: intrinsically flat and intrinsically curved

dx and *dy* elements across the surface. If this surface composed a sphere then let us name this a CPNAHI 2-sphere. Let us write

$$dx_1 - dx_2 = \Delta dx \tag{85}$$

[20]The affine connection of rolling a "Euclidean plane" along the surface would seem to be derivable from this definition.

[21]Compare this description with that of a "ray" with a differentiable manifold S^2 in Figure 9.3 of [16]

as an expression of intrinsic curvature. If $\Delta dx = 0$ then the infinitesimals are of equal magnitude and they are flat relative to each other. If $\Delta dx <> 0$ then they are intrinsically curved relative to each other. If I create a chart and list the ways that the these infinitesimals can change, this chart matches the definitions of Gaussian curvature as seen in Figure 24[22].

	Gaussian curvature k	**CPNAHI surface**
elliptic	k_1 and k_2 same sign	Δdx and Δdy same sign
parabolic	k_1 or k_2 equals zero but other does not	Δdx or Δdy equals zero but other does not
hyperbolic	k_1 and k_2 opposite sign	Δdx and Δdy opposite signs
planar	$k_1 = k_2 = 0$	$\Delta dx = \Delta dy = 0$

One obvious section that should be included in this paper is an analysis of prior research by Gauss, Riemann, Bolyai and Lobachevsky etc. during the development of non-Euclidean geometry when they analyzed Torricelli's HIs. Unfortunately, I only have access to commonly published works and not perhaps unpublished notes. I currently find no published evidence that any of them analyzed Torricelli's work. However, absence of evidence is not evidence of absence and thus will have to rely upon the peer review process to enlist mathematical historians.

In light of the breadth of this research versus the readability of an introductory paper, I have decided to follow Torricelli's example and try to err on the side of simplicity. His known simplifications have proven to be more effective for the initial spread of ideas than a dense but unread "Geometria Indivisibilibus".

6.1 Cardinal Functions and Magnitude Functions

Let us call the relationship between cardinal numbers of elements on a flat background, such as $n_y dy = f(n_x dx)$, a *Cardinal Function*. Let us then call the dependency of the magnitude of one infinitesimal upon another a *Magnitude Function* such that Ian write

$$|dx_1| = f(|dx_2|). \tag{86}$$

I have included magnitude brackets to specifically point out that this is a functional relationship between the magnitudes of two infinitesimals. Instead of coordinate transformations, Magnitude Functions can measure offsetting changes in magnitudes of infinitesimals. In future research I propose to examine whether tensors can be replaced via cardinal and magnitude functions.

[22] https://www.grad.hr/itproject_math/Links/sonja/gausseng/ehpp/ehpp.html

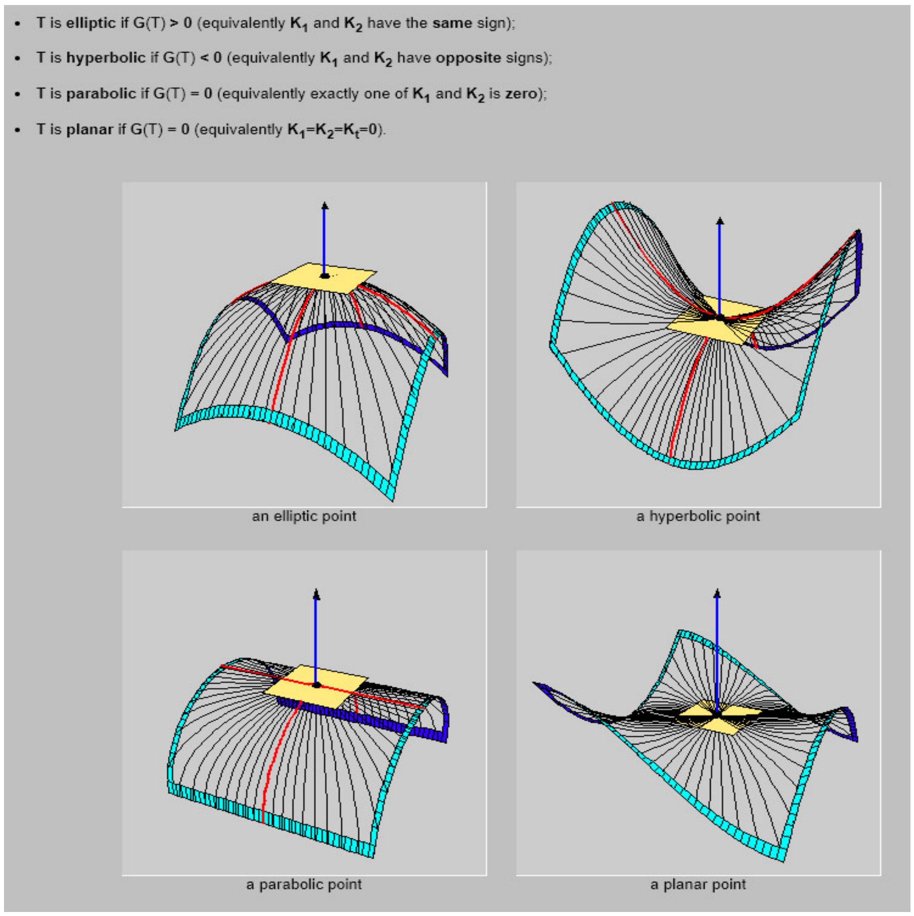

Figure 24: Elements of area on CPNAHI surface as compared to Gaussian curvature

6.2 Constants of Integration: The Cosmological Constant vs homogeneous infinitesimals

As a non-rigorous description, lets consider hypothetically that the metric $g_{\mu\nu}dx_{\mu}dx_{\nu}$ is used to measure how a voluminal homogeneous infinitesimal changes magnitude (i.e. Ricci tensor/ value of Christoffel symbols). In other words, if I have a ball inside a glass case and the ball shrinks inside that case then there is a change in volume of the ball. If the glass case and ball BOTH shrink equivalently then this is the type of volume change in a foreground geometry I am speaking of. Let's say now that these volume elements only change size due to the presence of "energy-momentum". Outside the neighborhood of this "energy-momentum" the elements revert back to the original size, defined as “unit",“1" or basis vectors and is viewed as a coordinate system. In this model, there is no known mechanism that would allow the metric to measure a universal change (coordinate system itself is growing or shrinking non-locally). This would seem to be equivalent to a “scalar" multiple of the metric, $\Lambda g_{\mu\nu} dx_\mu dx_\nu$. In other words, $g_{\mu\nu}$ has been arbitrarily chosen and defined as the magnitude from which to measure change but doesn’t have an intrinsic absolute value. Λ is just an acknowledgment that it is geometrically possible to choose any value from which to measure, and not just “unitary". This is the same concept as derived from flat homogeneous infinitesimals in that there is no absolute value for the magnitude of dx. I can only measure *relative to it*. Similarly, the anti-derivative (Section 6.0.2) to a cardinal function (Section 6.1) also contains a term (constant of integration) which accounts for the similar relationship for cardinality. You can only measure the change in relative cardinality and you can only measure the change in relative magnitude. They both require “constants of integration", a common concept of the Cosmological Constant.

6.3 Elements of Volume: Coordinate Systems vs Elements of Density With Magnitude Functions

It has been said that the Ricci tensor “measures the extent to which the volume of a geodesic ball on the surface differs from the volume of a geodesic ball in Euclidean space [23]. If these geodesic balls in General Relativity represent the curvature of a coordinate system representing space-time but can be proven derivable from CPNAHI voluminal homogeneous infinitesimals, then this should compel a re-evaluation of the logic of the model. Dark Energy models would seem to make it more logical to represent these elements of volume as a measure of the density of an elastic medium (continuum mechanics) and a change in size would represent a change in density and strain of this elastic medium. In line with the issues presented in the Report of the Dark Energy Task Force [1], redefining energy-density ρ as a change in the density of the vacuum $\Delta\rho_{vac} = \rho$, modeled as an elastic-medium-fluid, would seem to fit the conceptual model. This perfect fluid/elastic medium model would seem to make more logical sense for differing expansion and contraction rates in the absence of matter (modeled as waves in this medium), than a universally flat coordinate system when no energy-momentum is present. In other words, the (unit) basis vectors of the quote at 4.3.4 should model the strain and density changes of a perfect fluid rather than a coordinate system. Instead of a stress-energy tensor and the Einstein Field Equation,

[23] see https://pi.math.cornell.edu/files/Research/SeniorTheses/rudeliusThesis.pdf

time dilation (as well as wave-length change) could be written as

$$\Delta|dt| = f(\Delta|dx|^3) \tag{87}$$

to derive a new equation for local gravitational effects. What could be causing this fluid to change density universe wide, and the interpretation of c, will be expanded upon in a future paper.

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