Using Classical Doppler Shift to Correct Michelson-Morley's Fringe Shift Calculation

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Abstract

Using classical Doppler shift and wave mechanics, we derive a corrected equation for the fringe shift of the Michelson-Morley interferometer when rotated at an arbitrary angle in the presence of an aether wind, and demonstrate that the 1887 Michelson-Morley interferometer experiment can be interpreted as a positive detection of an aether wind of 235 km/s.

Every suspected cause of disturbance having been eliminated, and an adequate method of procedure having been developed, it is presumed that the persistently observed effects, which though small are systematic, are due to a real ether-drift. The observed displacement of the interference fringes, for some unexplained reason, corresponds to only a fraction of the velocity of the earth in space.

Dayton C. Miller, *Reviews of Modern Physics (Vol. 5)*, 1933

1 Introduction

Light is a wave. It is well-understood that the velocity of a wave is independent of the velocity of its source (for example, consider a stone skipping across a pond; the rate at which each ripple spreads is independent of the speed of the stone). Christian Huygens is generally credited with developing the first wave theory of light in his 1690 *Treatise of Light*.

The property of source-independent propagation is the essential observation in favor of the wave emission model for light, because there is no other plausible explanation for this observation. This property of light was first measured centuries ago—in

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1677 by Ole Roemer's timings of the eclipses of Jupiter's moon Io, in 1728 by James Bradley's observations of stellar aberration, and subsequently by a variety of other optical experiments and observations throughout the 18th, 19th, and 20th centuries (with a notable experiment by Quirino Majorana deserving mention).

Isaac Newton, for example, described in his 1704 book *Opticks* many optical experiments demonstrating refraction and diffraction, and studied interference patterns (for example, Newton's rings) created by light. These effects are all classic examples of wave behavior. Although Newton's observations regarding the polarization of light led him to subscribe to a particle (crepuscular) model of light, it was later recognized that polarization could be explained by a wave model of light with transverse waves traveling through a light-carrying medium. This led physicists in the 19th century (e.g. Stokes, Green, MacCullagh) to develop a variety of increasingly sophisticated theories of an elastic-solid aether that satisfied the known observations of light behavior.

Toward the end of the 19th century, with the formulation of Maxwell's equations (as well as the equations of Ampere, Weber, and Gauss) and the discovery of electromagnetic waves by Heinrich Hertz, the unification of optics and electromagnetism became evident, and in the beginning of the 20th century, the theory of spectral lines took on an important new association with the structure of matter. A theory of the aether, therefore, stood to unify all known physical phenomena, and so understanding the nature of the aether was perhaps the most fundamental task in physics until the acceptance of relativity in the early 20th century; soon after, the theory of the aether was almost universally abandoned along with its program for the unification of physics.

The theory of relativity was a radical departure from classical physics; this theory proposed a new emission model for light that had no analogue in any medium. A radical new theory of physics could only be justified by some experiment which had no conceivable classical explanation. The 1887 Michelson-Morley interferometer experiment, which apparently failed to detect any absolute motion of the Earth due to a luminiferous aether, was this catalyst for change.

In this paper we will show that the Michelson-Morley experiment can be reinterpreted (without introducing any non-classical theory) as a positive detection of an aether wind of 235 km/s.

2 Classical Doppler Shift at a General Angle

For an emission source S and a receiver R moving at the same velocity in the same direction, we claim that the frequency observed in the case when both the source and receiver are moving in tandem is the same as when they are both stationary. This principle of frequency invariance holds in the case of a Michelson-Morley interferometer experiment, for example, because the interferometer is a rigid object in which the sources and receivers (e.g., the beam splitter and mirrors) all move together.

We can deduce the principle of frequency invariance from figure 1. In this figure we can see that the outermost wavefront emitted from S coincides with the outermost wavefront emitted from S', because at $t = 0$, S and S' were in the same position. This means that the number of wavefronts that pass the receiver as it moves from R to R' in time t is the same in the moving case as the stationary case. Thus, $v_S = v_R$ implies $f' = f$. This is a simple concept, but has significant implications for the Michelson-Morley experiment because it means that the wavelength and frequency of

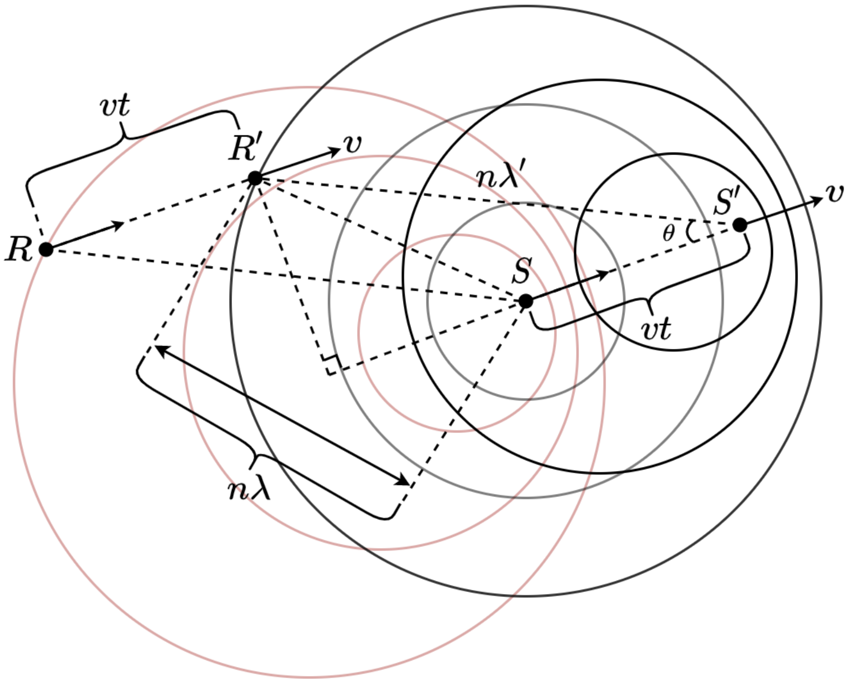

Figure 1: An emission source S and a receiver R moving at the same velocity in the same direction.

light cannot be treated interchangeably. While frequency is constant, wavelength (as can be seen in figure 1) is not, so instead of $c = \lambda f$, we have $c' = \lambda' f$ (where c' is the observed speed of the light wave and λ' is the observed wavelength).

Many "replications" of the Michelson-Morley experiment have used laser cavity resonators to search for changes in frequency, however, these experiments are *not* equivalent to the Michelson-Morley interferometer experiment, in which the frequency remains constant but the wavelength changes. The Michelson-Morley interferometer registers changes in wavelength as fringe shifts, since these produce interference even when frequency remains constant, but cavity resonance detectors only measure frequency and thus will not measure any differences when the wavelengths change. Therefore, the results obtained by laser cavity resonators cannot be considered to weigh as evidence in favor of relativity, because classical wave mechanics also predicts a null result for these experiments.

Returning to figure 1, we can apply the law of cosines to the $SS'R'$ triangle to determine the observed wavelength shift:

$$(n\lambda)^2 = (vt)^2 + (n\lambda')^2 - 2(vt)(n\lambda')\cos\theta \tag{1}$$

Since $n\lambda = ct$, we can substitute $t = \dfrac{n\lambda}{c}$ and divide both sides by n^2 to obtain:

$$
\begin{aligned}
\lambda^2 &= \lambda'^2 + \left(\frac{v\lambda}{c}\right)^2 - 2\lambda\lambda'\left(\frac{v\cos\theta}{c}\right) \\
&= \lambda'^2 - 2\lambda\lambda'\left(\frac{v\cos\theta}{c}\right) + \left(\frac{v\lambda}{c}\right)^2 \\
&= \left(\lambda' - \left(\frac{v\lambda\cos\theta}{c}\right)\right)^2 + \left(\frac{v\lambda}{c}\right)^2 - \left(\frac{v\lambda\cos\theta}{c}\right)^2 \\
&= \left(\lambda' - \left(\frac{v\lambda\cos\theta}{c}\right)\right)^2 + \left(\frac{v\lambda}{c}\right)^2 (1-\cos^2\theta) \\
&= \left(\lambda' - \left(\frac{v\lambda\cos\theta}{c}\right)\right)^2 + \left(\frac{v\lambda\sin\theta}{c}\right)^2
\end{aligned} \tag{2}
$$

Thus,

$$
\begin{aligned}
\left(\lambda' - \left(\frac{v\lambda\cos\theta}{c}\right)\right)^2 &= \lambda^2 - \left(\frac{v\lambda\sin\theta}{c}\right)^2 \\
&= \lambda^2\left(1 - \left(\frac{v\sin\theta}{c}\right)^2\right)
\end{aligned} \tag{3}
$$

. Taking the square root of both sides,

$$
\lambda' - \left(\frac{v\lambda\cos\theta}{c}\right) = \lambda\sqrt{1 - \left(\frac{v\sin\theta}{c}\right)^2} \tag{4}
$$

and factoring out λ we have the result

$$
\lambda' = \lambda\left(\frac{v\cos\theta}{c} + \sqrt{1 - \left(\frac{v\sin\theta}{c}\right)^2}\right) \tag{5}
$$

or alternatively,

$$
\lambda' = \lambda\left(\beta\cos\theta + \sqrt{1 - (\beta\sin\theta)^2}\right) \tag{6}
$$

where $\beta = \frac{v}{c}$. This gives us a classical formula for Doppler shift at an arbitrary angle. Note that for $\theta = 0$ and $\theta = \pi$, equation 6 reduces to the familiar form for longitudinal Doppler shift,

$$
\lambda' = \lambda(1 \pm \beta) \tag{7}
$$

and for $\theta = \frac{\pi}{2}$, equation 6 becomes

$$
\lambda' = \lambda\sqrt{1-\beta^2} = \frac{\lambda}{\gamma} \tag{8}
$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the familiar Lorentz factor.

Here we have derived transverse Doppler shift as a classical wave phenomenon. The previous equation stands in contradiction to the commonly accepted formula for Doppler shift at an angle:

$$\lambda' = \lambda\,(1 \pm \beta\cos\theta) \tag{9}$$

. Equation 9 is clearly wrong because it predicts no wavelength contraction at 90 degrees.

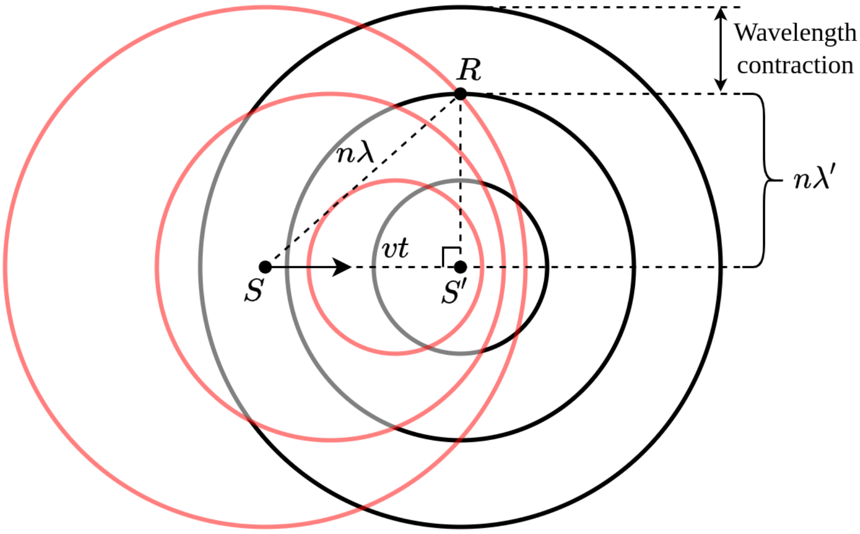

Figure 2: Transverse Doppler shift for a moving source.

We can also derive the transverse Doppler shift more directly as shown in figure 2. At time zero, a source at S traveling at speed v emits a wavefront (shown in red) traveling with speed c, and at time t this wavefront reaches a stationary receiver at R. During this time, the source emits n wavefronts and moves from S to S'. Since S resides at the center of the outermost red wavefront, the distance from S to R is equal to n stationary wavelengths, $n\lambda$, and since these wavefronts travel at speed c, the travel time can be expressed as $t = \dfrac{n\lambda}{c}$. The distance from S' to R is equal to n observed wavelengths, $n\lambda'$. Using the Pythagorean theorem, we can express the relation betwen distances as

$$\begin{aligned} (n\lambda)^2 &= (n\lambda')^2 + (vt)^2 \\ &= (n\lambda')^2 + \left(v\left(\frac{n\lambda}{c}\right)\right)^2 \end{aligned} \tag{10}$$

. Canceling the n's and rearranging, we have

$$\begin{aligned} \lambda'^2 &= \lambda^2 - \left(\frac{v\lambda}{c}\right)^2 \\ &= \lambda^2\left(1 - \left(\frac{v}{c}\right)^2\right) \end{aligned} \tag{11}$$

and taking the square root of both sides, we again obtain:

$$\lambda' = \lambda\sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{\lambda}{\gamma} \tag{12}$$

.

3 Optical Paths

An optical path difference is the difference in the phase of light between two different paths. The phase of a traveling wave (in cycles) is

$$\phi = ft - \frac{x}{\lambda} \tag{13}$$

. To determine the phase of a path through a particular medium with an index of refraction $\eta = \frac{c}{c'}$, first, let's change the variables λ and t to λ' and t' to make clear that these are the observed wavelength and travel times. Then, we can apply the following trick:

$$\begin{aligned}
\phi &= ft' - \frac{x}{\lambda'} \\
&= ft' - \frac{\lambda f x}{\lambda f \lambda'} \\
&= ft' - \frac{cx}{\lambda c'} \\
&= ft' - \frac{\eta x}{\lambda}
\end{aligned} \tag{14}$$

. For a scenario in which light travels along two different geometrical paths, the phase shift is

$$\begin{aligned}
\Delta\phi_n &= \phi_2 - \phi_1 \\
&= \left(ft'_2 - \frac{\eta x_2}{\lambda}\right) - \left(ft'_2 - \frac{\eta x_2}{\lambda}\right) \\
&= f(t'_2 - t'_1) + \frac{\eta x_1 - \eta x_2}{\lambda} \\
&= \frac{\eta \Delta x}{\lambda} - f\Delta t'
\end{aligned} \tag{15}$$

. However, the observed fringe shift Δ_{FS} in an interferometer (ignoring the index of refraction since $\eta \approx 1$ for an interferometer in air) is given by

$$\Delta_{FS} = \frac{\Delta x}{\lambda} \tag{16}$$

. This equation is empirically verified; if you hold the interferometer still and adjust one path to be longer or shorter, you can verify that the number of fringe shifts counted corresponds to the wavelength of light being used. Note that the time-dependent terms in this phase shift calculation have dropped out, because we are not measuring the phase of light at a particular time—rather, we are measuring over a change in distance as light flows continuously. Therefore, we can disregard the motion of light in the interferometer altogether and simply consider it to be filled with standing waves.

A careful reader might notice that we have expressed this fringe shift formula somewhat vaguely. We could have written the formula as

$$\Delta_{FS} = \frac{\Delta x_1}{\lambda} \tag{17}$$

or as

$$\Delta_{FS} = \frac{x_1 - x_2}{\lambda} \tag{18}$$

, where Δx_1 is the difference in the first path length before and after adjustment, and $x_1 - x_2$ is the difference between the two paths after adjustment (with the assumed difference being zero before adjustment). Both fringe shift formulas yield the same result, but which one is correct?

A little thought might convince you that the first equation must be the correct version; only the difference in the adjusted path matters—the path that was not adjusted could have been any length at all to begin with (for example, x_2 could have been ten times longer than x_1), and it would not have affected the result.

Let us consider another hypothetical scenario. Suppose we have two paths of different lengths so that $x_1 \neq x_2$, and imagine we slowly triple the length of each path at the same rate simultaneously. What happens? Well, using our model of standing waves, it is evident that nothing should happen when both paths are scaled by the same factor; otherwise, in a one-way interferometer with two parallel paths directed into an aether wind one could produce several hundred thousand fringe shifts by rotating the apparatus 180 degrees (mathematically, scaling the wavelengths by some factor should be equivalent to scaling the distances).

However, equation 17 predicts that the fringe shift will be $\frac{2x_1}{\lambda}$ and equation 18 predicts $\frac{2(x_1 - x_2)}{\lambda}$. Even if we attempted to modify equation 17, for example, so that

$$\Delta_{FS} = \frac{\Delta x_1}{\lambda} - \frac{\Delta x_2}{\lambda} \tag{19}$$

, this would still predict $\frac{2(x_1 - x_2)}{\lambda}$ instead of 0. Clearly, the situation becomes more complex when both paths change simultaneously, particularly if they are not changing by the same factor, which is exactly the situation we are confronted with regarding the rotation of the Michelson-Morley interferometer.

3.1 Michelson-Morley's Approach

Michelson and Morley derived their fringe shift formula by considering light in the frame of the stationary aether (this is the first subtle difference between their formula and the empirically verified one). We can imagine they began with the empirically verified equation in mind:

$$\Delta_{FS} = \frac{\Delta x}{\lambda} = \frac{x_1 - x_2}{\lambda} \tag{20}$$

. Note that they are working with the second version of the fringe shift equation, which we previously determined to be incorrect.

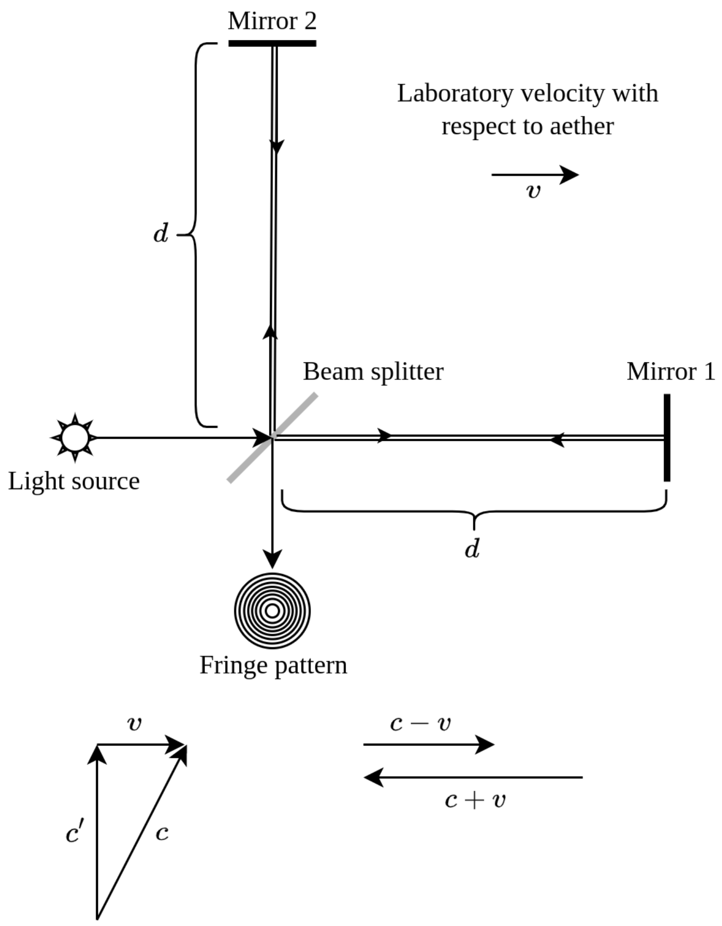

Figure 3: Experimental setup for the Michelson-Morley interferometer.

Then they computed the lengths x_1 and x_2 in the aether frame. The lengths were computed by working out the travel times for each path. For the horizontal path,

$$\begin{aligned} t_1' &= \frac{L}{c-v} + \frac{L}{c+v} \\ &= \frac{2Lc}{c^2 - v^2} \\ &= \frac{2L}{c} \cdot \frac{1}{1 - \frac{v^2}{c^2}} \\ &= \frac{2L}{c} \cdot \frac{1}{1 - \beta^2} \end{aligned} \tag{21}$$

, thus

$$x_1 = ct_1' = \frac{2L}{1-\beta^2} \approx 2L\left(1 + \beta^2\right) \tag{22}$$

. For the vertical path, from figure 1 we have $c'^2 + v^2 = c^2$, so

$$
\begin{aligned}
t_2' &= \frac{2L}{\sqrt{c^2 - v^2}} \\
&= \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
&= \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - \beta^2}}
\end{aligned}
\tag{23}
$$

thus

$$
x_2 = ct_2' = \frac{2L}{\sqrt{1-\beta^2}} \approx \frac{2L}{1 - \frac{1}{2}\beta^2} \approx 2L\left(1 + \frac{1}{2}\beta^2\right) \tag{24}
$$

, where $\beta = \frac{v}{c}$. The difference between these two distances is

$$
x_1 - x_2 = L\beta^2 \tag{25}
$$

. Substituting this into the fringe shift equation above, we have

$$
\Delta_{FS} \approx \frac{L}{\lambda}\beta^2 \tag{26}
$$

. Michelson and Morley also believed (incorrectly) that by rotating the interferometer 90 degrees, they would double their observed fringe shift displacement, so their expectation was a fringe shift of $\approx \frac{2L}{\lambda}\beta^2$.

To better understand the problem with Michelson-Morley's logic, disregard the time dependency (as we have already done with the empirically verified fringe shift formula) and imagine the interferometer is filled with standing waves. In the empirically verified formula, the number of standing waves in the interferometer *decreases* by $\frac{\Delta x_1}{\lambda}$ during measurement. Yet, in the Michelson-Morley experiment, the number of standing waves in the interferometer is *unchanged* after (and nearly unchanged during) a rotation of ninety degrees.

To understand why this difference is critical, consider the interferometer in the lab frame without any aether wind, and imagine that instead of just changing one path length, we increase one path length while simultaneously decreasing the other path length by the same amount. Would there be any observable fringe shift? No! The fringes are an interference pattern, and interference is caused by the sum of the waves. If we have two waves moving in opposite directions, for example $\sin(kx - \omega t)$ and $\sin(kx + \omega t)$, the sum of these two waves $\sin(kx - \omega t) + \sin(kx + \omega t)$ does not move in either direction, so in this scenario there would be no observed fringe displacement. (If you are still not convinced, ask yourself, in which direction would the fringes move? Since the fringes are an interference pattern, increasing one path and decreasing another should not be distinguishable from decreasing one path and increasing the other.)

Now consider that the Michelson-Morley experiment was very similar to the imagined scenario described above: There were two optical path lengths, one longer than the other, and during a rotation of 90 degrees, these optical paths were swapped so that the shorter path became the longer path and vice versa. The situation was therefore very similar to the case of two wave-trains moving in opposite directions, so Michelson-Morley's fringe shift expectation was wrong.

3.2 Correcting the Approach

The task of predicting fringe shifts is in fact more complex than Michelson-Morley supposed.

The optical path difference between the two branches when the interferometer has not been rotated is:

$$\delta_{n0} = \frac{x}{\lambda_{H0}} - \frac{x}{\lambda_{V0}} \tag{27}$$

, and similarly, after rotation it is

$$\delta_{n\theta} = \frac{x}{\lambda_{H\theta}} - \frac{x}{\lambda_{V\theta}} \tag{28}$$

, where λ_{H0} is a harmonic average of the observed wavelengths in the horizontal branch (keeping in mind they are different lengths on the forward and return trips) and λ_{V0} is the harmonic average of the wavelengths in the vertical branch. Likewise $\lambda_{H\theta}$ and $\lambda_{V\theta}$ are the average wavelengths when the interferometer is tilted at angle θ.

A note about computing wavelengths: Keeping in mind $c' = \lambda' f$ from our principle of frequency invariance, we can compute an average wavelength by starting with

$$t' = \frac{L}{c'_{\uparrow}} + \frac{L}{c'_{\downarrow}} \tag{29}$$

, where $c_{\uparrow}$ represents the observed speed during the forward trip and $c_{\downarrow}$ represents observed speed during the return trip.

Since $c' = \dfrac{2L}{t'}$ (where c' is an average of the observed forward and return speeds), this implies

$$\frac{1}{c'} = \frac{1}{2}\left(\frac{1}{c'_{\uparrow}} + \frac{1}{c'_{\downarrow}}\right) \tag{30}$$

. Multiplying both sides by f, we have

$$\frac{1}{\lambda'} = \frac{1}{2}\left(\frac{1}{\lambda'_{\uparrow}} + \frac{1}{\lambda'_{\downarrow}}\right) \tag{31}$$

, so

$$\lambda' = \frac{2}{\dfrac{1}{\lambda'_{\uparrow}} + \dfrac{1}{\lambda'_{\downarrow}}} \tag{32}$$

, which is a harmonic average.

One might hypothesize that the predicted fringe shift Δ_{FS} should be the difference between the optical path differences $\delta_{n\theta}$ and δ_{n0}, so that

$$\Delta_{FS} = \delta_{n\theta} - \delta_{n0} \tag{33}$$

. However, this predicts that if the interferometer is rotated, and then the number of wavelengths for each branch is scaled by some factor $g > 1$, the fringe shift will be $\Delta_{FS} = g\delta_{n\theta} - \delta_{n0}$ instead of $\Delta_{FS} = \delta_{n\theta} - \delta_{n0}$, which is incorrect, as previously discussed. A scalar transformation of both paths simultaneously should not cause an observable fringe shift.

4 Corrected Fringe Shift Calculation

We will attack this problem first by computing the fringe shift for a general angle, then we will examine the specific case where $\theta = \frac{\pi}{2}$. In contrast to Michelson-Morley, we will carry out our derivation in the lab frame. Without loss of generality, we will assume the aether wind is directed along the $-x$-axis and that when the apparatus is rotated, the horizontal path is at angle θ to the x-axis.

First, recall that the formula for Doppler shift at a general angle is

$$\begin{aligned}\lambda' &= \lambda\left(\frac{v}{c}\cos\theta + \sqrt{1 - \left(\frac{v}{c}\sin\theta\right)^2}\right)\\ &= \lambda\left(\beta\cos\theta + \sqrt{1 - (\beta\sin\theta)^2}\right)\end{aligned} \tag{34}$$

, in which λ' is the observed wavelength, λ is the stationary wavelength, and $\beta = \dfrac{v}{c}$.

Since $\cos(\theta + \pi) = -\cos(\theta)$, the number of waves along the horizontal roundtrip path when tilted at some angle θ (yes, we're still referring to this path as "horizontal" even when tilted at an angle) is given by

$$\begin{aligned}N_{H\theta} &= N_{H\uparrow} + N_{H\downarrow}\\ &= \frac{L}{\lambda_{H\uparrow}} + \frac{L}{\lambda_{H\downarrow}}\\ &= \frac{L}{\lambda}\left(\frac{1}{\beta\cos\theta + \sqrt{1-(\beta\sin\theta)^2}} + \frac{1}{-\beta\cos\theta + \sqrt{1-(\beta\sin\theta)^2}}\right)\end{aligned} \tag{35}$$

. Likewise, since $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$, the number of waves along the vertical roundtrip path is given by

$$\begin{aligned}N_{V\theta} &= N_{V\uparrow} + N_{V\downarrow}\\ &= \frac{L}{\lambda_{V\uparrow}} + \frac{L}{\lambda_{V\downarrow}}\\ &= \frac{L}{\lambda}\left(\frac{1}{-\beta\sin\theta + \sqrt{1-(\beta\cos\theta)^2}} + \frac{1}{\beta\sin\theta + \sqrt{1-(\beta\cos\theta)^2}}\right)\end{aligned} \tag{36}$$

. Returning to our expression for N_H, we can simplify the first fraction as follows:

$$
\begin{aligned}
N_{H\uparrow} &= \frac{L}{\lambda}\left(\frac{1}{\beta\cos\theta+\sqrt{1-(\beta\sin\theta)^2}}\right)\left(\frac{\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}{\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}\right)\\
&= \frac{L}{\lambda}\left(\frac{\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}{\beta^2-1}\right)
\end{aligned}
\tag{37}
$$

, in which we have used $(\beta\cos\theta)^2+(\beta\sin\theta)^2=\beta^2(\cos^2\theta+\sin^2\theta)=\beta^2$ to simplify the denominator in the final step.

Likewise, we can simplify the second fraction so that

$$
\begin{aligned}
N_{H\downarrow} &= \frac{L}{\lambda}\left(\frac{1}{-\beta\cos\theta+\sqrt{1-(\beta\sin\theta)^2}}\right)\left(\frac{-\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}{-\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}\right)\\
&= \frac{L}{\lambda}\left(\frac{-\beta\cos\theta-\sqrt{1-(\beta\sin\theta)^2}}{\beta^2-1}\right)
\end{aligned}
\tag{38}
$$

. Adding these together, we have

$$
\begin{aligned}
N_{H\theta} &= N_{H\uparrow}+N_{H\downarrow}\\
&= \frac{L}{\lambda}\left(\frac{-2\sqrt{1-(\beta\sin\theta)^2}}{\beta^2-1}\right)\\
&= \frac{2L}{\lambda}\left(\frac{\sqrt{1-(\beta\sin\theta)^2}}{1-\beta^2}\right)
\end{aligned}
\tag{39}
$$

. Following a similar process for $N_{V\theta}$, we arrive at

$$
N_{V\theta} = \frac{2L}{\lambda}\left(\frac{\sqrt{1-(\beta\cos\theta)^2}}{1-\beta^2}\right)
\tag{40}
$$

. Note that for $\theta=0$, these expressions become

$$
N_{H0} = \frac{2L}{\lambda}\left(\frac{1}{1-\beta^2}\right)
\tag{41}
$$

and

$$
N_{V0} = \frac{2L}{\lambda}\left(\frac{\sqrt{1-\beta^2}}{1-\beta^2}\right)
\tag{42}
$$

respectively.

As the interferometer rotates, the total number of wavelengths in the vertical and horizontal paths will either increase or decrease. Let us define G as the growth factor for the total number of waves in the interferometer at a given angle, so that

$$
\begin{aligned}
G &= \frac{N_{H\theta}+N_{V\theta}}{N_{H0}+N_{V0}}\\
&= \frac{\sqrt{1-(\beta\sin\theta)^2}+\sqrt{1-(\beta\cos\theta)^2}}{1+\sqrt{1-\beta^2}}
\end{aligned}
\tag{43}
$$

, noting that the common factors in the numerator and denominator have been cancelled out. Since we have defined the aether wind to be along the $-x$ axis, $G \geq 1$ for all angles. We can also see that $G = 1$ when $\theta = \frac{n\pi}{2}$ for any integer $n \geq 0$. This means that the total number of wavelengths in the interferometer reaches a minimum at these angles.

If the number of wavelengths in the horizontal path and vertical path both change at the same rate as the interferometer is rotated, there will not be any fringe shift. However, if the number of wavelengths in the horizontal path changes at a different rate than the number of wavelengths in the vertical path, there *will* be an observable fringe shift.

The number of residual wavelengths δ_N that contribute to the observable fringe shift is given by

$$\delta_N = (N_{H\theta} - N_{V\theta}) - G(N_{H0} - N_{V0}) \tag{44}$$

To understand the rationale for the formula for δ_N, note that if the number of wavelengths in the vertical path and horizontal path both change by some constant factor g during rotation, then $\delta_N = 0$ (otherwise, $\delta_N \neq 0$ and there are some "residual" wavelengths). We can prove this in two ways:

Proof 1. Suppose $N_{H\theta} = gN_{H0}$ and $N_{V\theta} = gN_{V0}$, where $g > 0$ is some constant factor. Then

$$\begin{aligned} G &= \frac{N_{H\theta} + N_{V\theta}}{N_{H0} + N_{V0}} \\ &= \frac{gN_{H0} + gN_{V0}}{N_{H0} + N_{V0}} \\ &= \frac{g(N_{H0} + N_{V0})}{N_{H0} + N_{V0}} \\ &= g \end{aligned} \tag{45}$$

and

$$\begin{aligned} \delta_N &= (N_{H\theta} - N_{V\theta}) - G(N_{H0} - N_{V0}) \\ \delta_N &= (N_{H\theta} - N_{V\theta}) - g(N_{H0} - N_{V0}) \\ \delta_N &= (gN_{H0} - gN_{V0}) - g(N_{H0} - N_{V0}) \\ \delta_N &= g(N_{H0} - N_{V0}) - g(N_{H0} - N_{V0}) \\ \delta_N &= 0 \end{aligned} \tag{46}$$

$\blacksquare$.

Proof 2. Suppose $N_{H\theta} = gN_{H0}$ and $N_{V\theta} = gN_{V0}$, where $g > 0$ is some constant factor. Then

$$\begin{aligned} \frac{N_{H\theta} - N_{V\theta}}{N_{H\theta} + N_{V\theta}} &= \frac{gN_{H0} - gN_{V0}}{gN_{H0} + gN_{V0}} \\ &= \frac{N_{H0} - N_{V0}}{N_{H0} + N_{V0}} \end{aligned} \tag{47}$$

. Cross-multiplying, we have

$$\begin{aligned}\frac{N_{H\theta}-N_{V\theta}}{N_{H0}-N_{V0}} &= \frac{N_{H\theta}+N_{V\theta}}{N_{H0}+N_{V0}} \\ &= G\end{aligned} \tag{48}$$

Thus,

$$N_{H\theta}-N_{V\theta} = G(N_{H0}-N_{V0}) \tag{49}$$

so

$$\begin{aligned}0 &= (N_{H\theta}-N_{V\theta}) - G(N_{H0}-N_{V0}) \\ &= \delta_N\end{aligned} \tag{50}$$

$\blacksquare$.

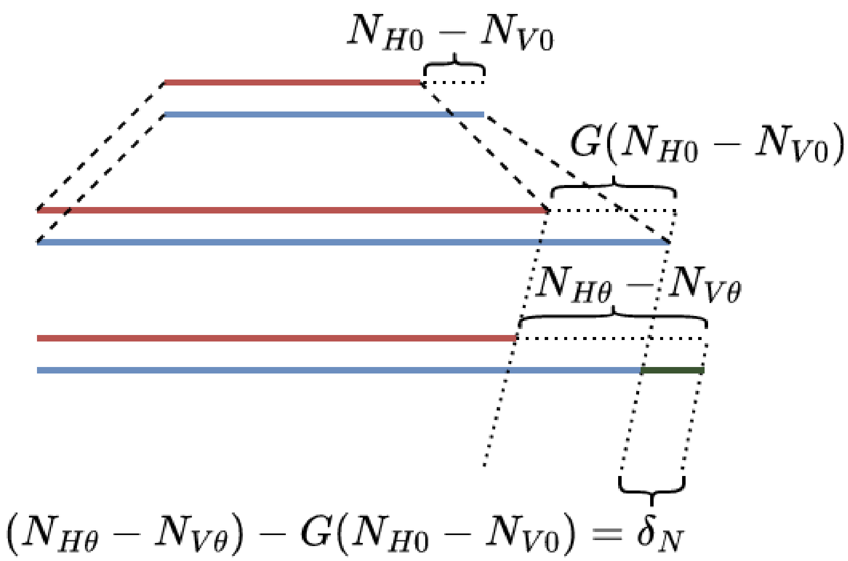

Figure 4: Residual wavelengths for N_H (blue) and N_V (red).

We can visualize the formula for residual wavelengths δ_N in figure 4, where we have illustrated the optical path lengths for N_H and N_V side-by-side. When each path scales by a constant factor g, there is no fringe shift, even though the gap between the two optical paths increases. (Otherwise, one would observe hundreds of thousands of fringe shifts during the rotation of a one-way interferometer that sends one beam through glass and another through air. Increasing or decreasing the number of wavelengths by the same scaling factor does not cause any fringe shift.) Thus, the only effective portion of the optical path that can contribute to fringe shift is δ_N. Therefore, the fringe shift is determined by the expansion or contraction of the wavelengths in δ_N.

Suppose that during rotation, each of the residual wavelengths expands or contracts by a factor Δ_λ, given by

$$
\begin{aligned}
\Delta_\lambda &= \frac{\lambda_\theta - \lambda_0}{\lambda_0} \\
&= \frac{\lambda_\theta}{\lambda_0} - 1 \\
&= \frac{\beta\cos\theta + \sqrt{1-(\beta\sin\theta)^2}}{1+\beta} - 1
\end{aligned} \tag{51}
$$

The fringe shift Δ_{FS} is therefore

$$
\Delta_{FS} = \delta_N \Delta_\lambda \tag{52}
$$

. The difference in the number of wavelengths in each arm of the interferometer at an angle θ is

$$
N_{H\theta} - N_{V\theta} = \frac{\sqrt{1-(\beta\sin\theta)^2} - \sqrt{1-(\beta\cos\theta)^2}}{1-\beta^2} \tag{53}
$$

, and the difference when $\theta = 0$ is

$$
N_{H0} - N_{V0} = \frac{1-\sqrt{1-\beta^2}}{1-\beta^2} \tag{54}
$$

. The fringe shift $\Delta_{FS}(\beta, \theta)$ at a general angle is therefore given by

$$
\begin{aligned}
\Delta_{FS}(\beta,\theta) = &\frac{2L}{\lambda} \\
&\cdot \left(\frac{\sqrt{1-(\beta\sin\theta)^2} - \sqrt{1-(\beta\cos\theta)^2}}{1-\beta^2} - G\left(\frac{1-\sqrt{1-\beta^2}}{1-\beta^2}\right)\right) \\
&\cdot \left(\frac{\beta\cos\theta + \sqrt{1-(\beta\sin\theta)^2}}{1+\beta} - 1 \right)
\end{aligned} \tag{55}
$$

, where

$$
G = \frac{\sqrt{1-(\beta\sin\theta)^2} + \sqrt{1-(\beta\cos\theta)^2}}{1+\sqrt{1-\beta^2}} \tag{56}
$$

. Michelson and Morley conducted measurements by turning the interferometer ninety degrees, so setting $\theta = \frac{\pi}{2}$, we have $G = 1$ and

$$
\Delta_{FS}\left(\beta, \frac{\pi}{2}\right) = \frac{4L}{\lambda}\left(\frac{\sqrt{1-\beta^2}-1}{1-\beta^2}\right)\left(\frac{\sqrt{1-\beta^2}}{1+\beta} - 1\right) \tag{57}
$$

. Using the binomial approximation $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ and the power series approximation $(1-x)^{-1} \approx 1 + x + x^2$, we note that this fringe shift effect is on the order of β^4 rather than β^2 as Michelson and Morley expected. For comparison, the formula Michelson and Morley used to estimate fringe shift was

$$
\Delta_{FS} \approx \frac{2L}{\lambda}\beta^2 \tag{58}
$$

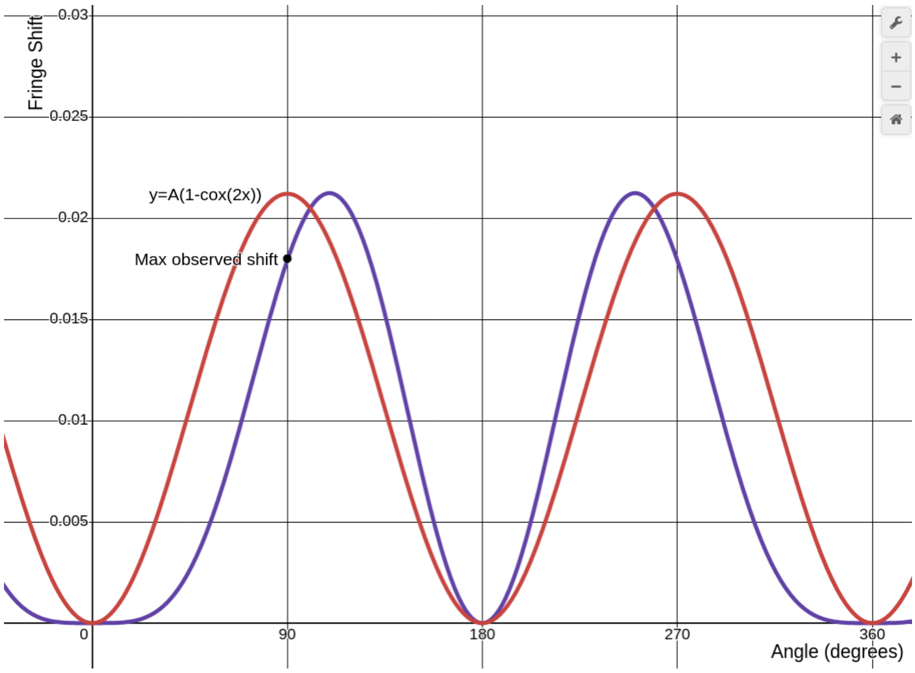

Figure 5: Predicted fringe shift displacements for $\Delta_{FS}(\beta = 0.000784, \theta)$ (blue), with a cosine function (red) for comparison.

. The length of the Michelson-Morley interferometer was 11 meters and the wavelength of their light source was 590 nanometers, so $L = 11$ m, $\lambda = 590 \cdot 10^{-9}$ m, and the maximum fringe displacement they observed was 0.018 fringes. Therefore, the observed fringe shift for their experiment corresponded to $\beta = 0.000784$, indicating an aether wind of 235 km/s rather than Michelson-Morley's estimation of 6.59 km/s. Notably, this is within 3.4 percent of D.C. Miller's estimation of 227 km/s (using a phase shift analysis of Miller's data).

The predicted fringe shift displacements for $\Delta_{FS}(\beta = 0.000784, \theta)$ is plotted in figure 5. Note that this curve is not sinusoidal (the flatter bottom of the curve can be seen around 0 and 360 degrees), and the maximum displacement does not correspond to a rotation of ninety degrees; rather, the rotation angle that produces the greatest displacement is roughly 109 degrees. This may partially account for why D.C. Miller (who made measurements along a larger range of rotations) reported somewhat disproportionately larger displacement amplitudes in his experiments (even after accounting for the $\approx$3x larger size of his interferometer).

The predicted fringe shift curve is also bimodal over a 360-degree rotation, which is expected (since turning the interferometer 180 degrees should not cause any fringe shift) and appears consistent with D.C. Miller's findings, shown in figure 6. "The unit for the scale of ordinates is one-hundredth of a fringe width, while the abscissae correspond to azimuth intervals of 22^o, beginning at the north point and proceeding clockwise around the horizon. A chart of this kind is plotted for each set of observations. These charted "curves" of the actual observations contain not only the second-order, half-period ether-drift effect, but also a first-order, full-period effect [...]. The residual curves are of very small amplitude and are evidence of the fact that the incidental and random errors are small." This demonstrates that the "half-period" displacement during rotation did not escape Miller's notice.

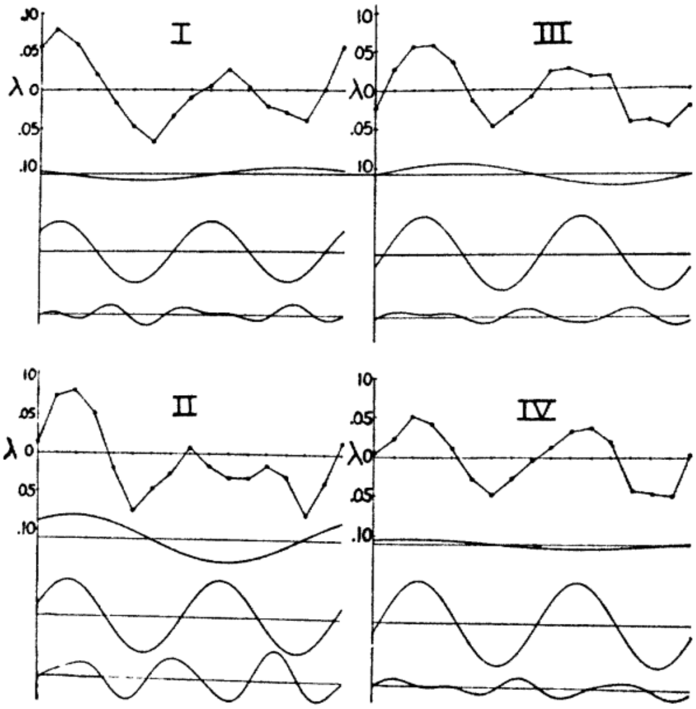

Fig. 21. Harmonic analysis of ether-drift observations.

Figure 6: Figure 21 from D.C. Miller's 1933 report in the *Reviews of Modern Physics* journal (vol 5).

Using Michelson and Morley's fringe shift formula, an aether wind of 235 km/s would have corresponded to a shift of 23 full fringes, whereas they would have considered 0.4 fringe shifts (an aether wind of ≈30 km/s) to be a positive detection. Since 235 km/s is well above the threshold Michelson and Morley expected, we conclude that the Michelson-Morley measurement did in fact constitute a positive detection of the aether wind.

5 Acknowledgments

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