Area of any Quadrilateral from Side Lengths

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Abstract. In this paper we show that area of any quadrilateral can be estimated from the four lengths sides. With the Triangle Inequality Theorem and a novel provided diagonal's formula, the boundaries of quadrilateral diagonals are found. Finally Bretschneider's formula can be applied to find a set of possible areas.

1 Introduction

So far, no formulas have been published in the mathematical literature that allow for calculating the area of any quadrilateral based solely on the lengths of its sides. Recent work[1] provides a formula that represent each diagonal of any quadrilateral in terms of its sides and the other diagonal. Theorem 1 provides an equivalent but shorter formula in terms of trigonometric functions. Theorem 2 provides solution set for diagonals of any convex quadrilateral given its side lengths. Finally, is provided a formula to obtain solution set area for any concave quadrilateral as well.

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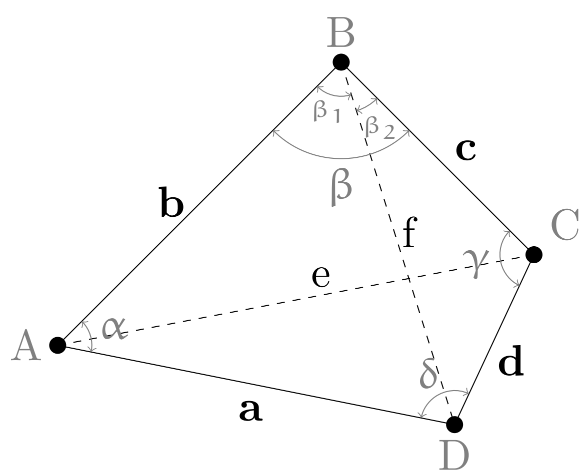
Theorem 1 (Diagonal of any quadrilateral in terms of its sides and the other diagonal). *Let* $ABCD$ *be a general convex quadrilateral, where* a, b, c, d *are lengths of the sides,* e, f *lengths of the diagonals, respectively, then:*

$$e = \sqrt{b^2 + c^2 + 2bc\cos\left(\arcsin\left(\frac{-a^2 + b^2 + f^2}{2bf}\right) + \arcsin\left(\frac{c^2 - d^2 + f^2}{2cf}\right)\right)} \tag{1}$$

$$f = \sqrt{c^2 + d^2 + 2cd\cos\left(\arcsin\left(\frac{-b^2 + c^2 + e^2}{2ce}\right) + \arcsin\left(\frac{d^2 - a^2 + e^2}{2de}\right)\right)} \tag{2}$$

Proof. For a general convex quadrilateral $ABCD$, a, b, c, d side lengths, e, f its diagonals and, $\alpha, \beta, \gamma, \delta$ respectively angles, as following figure shows:



Let area of $\square ABCD$, $\triangle ABC$ and $\triangle CDA$, be A, A_1 and A_2 respectively. From the second axiom of area:

$$A = A_1 + A_2 \tag{3}$$

Let denote f the diagonal segment $\overline{BD}$, then can be written in 2 ways using the Law of Cosines[2]:

$$f^2 = a^2 + b^2 - 2ab\cos(\alpha) \tag{4}$$

$$f^2 = c^2 + d^2 - 2cd\cos(\gamma) \tag{5}$$

So:

$$\gamma = \arccos\left(-\frac{f^2 - c^2 - d^2}{2cd}\right) \tag{6}$$

Replacing f^2 of equation 6 with right side equation 4:

$$\gamma = \arccos\left(\frac{-a^2 - b^2 + c^2 + d^2 + 2ab\cos(\alpha)}{2cd}\right) \tag{7}$$

From Angle Addition Postulate:

$$\beta = \beta_1 + \beta_2 \tag{8}$$

Applying the Law of Cosines, we obtain β_1 as follows:

$$a^2 = b^2 + f^2 - 2bf\cos(\beta_1) \tag{9}$$

Replacing f with equation 4:

$$a^2 = b^2 + \left(a^2 + b^2 - 2ab\cos(\alpha)\right) - 2b\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}\cos(\beta_1) \tag{10}$$

Solving for β_1 and simplifying:

$$\begin{aligned}
\cos(\beta_1) &= \frac{\cancel{-a^2} + b^2 \cancel{+a^2} + b^2 - 2ab\cos(\alpha)}{2b\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}} \\
&= \frac{2b^2 - 2ab\cos(\alpha)}{2b\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}} \\
&= \frac{\cancel{2b}(b - a\cos(\alpha))}{\cancel{2b}\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}} \qquad \text{(common factor)} \\
\beta_1 &= \arccos\left(\frac{b - a\cos(\alpha)}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}}\right)
\end{aligned} \tag{11}$$

Also using Law of Cosines we can find β_2 as follows:

$$d^2 = c^2 + f^2 - 2cf\cos(\beta_2) \tag{12}$$

Substituting f with equation 4:

$$d^2 = c^2 + \left(a^2 + b^2 - 2ab\cos(\alpha)\right) - 2c\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}\cos(\beta_2) \tag{13}$$

Solving for β_2:

$$\begin{aligned}\cos(\beta_2) &= \frac{-d^2 + c^2 + a^2 + b^2 - 2ab\cos(\alpha)}{2c\sqrt{a^2 + b^2 - 2ac\cos(\alpha)}} \\ \beta_2 &= \arccos\left(\frac{a^2 + b^2 + c^2 - d^2 - 2ab\cos(\alpha)}{2c\sqrt{a^2 + b^2 - 2ac\cos(\alpha)}}\right)\end{aligned} \tag{14}$$

As stated in equation 8, substituting β_1 and β_2:

$$\beta = \arccos\left(\frac{b - a\cos(\alpha)}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}}\right) + \arccos\left(\frac{a^2 + b^2 + c^2 - d^2 - 2ab\cos(\alpha)}{2c\sqrt{a^2 + b^2 - 2ab\cos(\alpha)}}\right) \tag{15}$$

By the Law of Cosines:

$$e = \sqrt{b^2 + c^2 - 2bc\cos(\beta)} \tag{16}$$

Also from equation 4:

$$\alpha = \arccos\left(\frac{-(f^2 - a^2 - b^2)}{2ab}\right) \tag{17}$$

Substituting equation 17 into 15, followed by further simplification:

$$\begin{aligned}\beta &= \arccos\left(\frac{b - a\left(\frac{-(f^2 - a^2 - b^2)}{2ab}\right)}{\sqrt{a^2 + b^2 - \cancel{2ab}\left(\frac{-(f^2 - a^2 - b^2)}{\cancel{2ab}}\right)}}\right) + \arccos\left(\frac{a^2 + b^2 + c^2 - d^2 - \cancel{2ab}\left(\frac{-(f^2 - a^2 - b^2)}{\cancel{2ab}}\right)}{2c\sqrt{a^2 + b^2 - \cancel{2ab}\left(\frac{-(f^2 - a^2 - b^2)}{\cancel{2ab}}\right)}}\right) \\ &= \arccos\left(\frac{\left(\frac{f^2 - a^2 + b^2}{2b}\right)}{\sqrt{\cancel{a^2 + b^2} - (-(f^2 \cancel{- a^2 - b^2}))}}\right) + \arccos\left(\frac{\cancel{a^2 + b^2} + c^2 - d^2 - (-(f^2 \cancel{- a^2 - b^2}))}{2c\sqrt{\cancel{a^2 + b^2} - (-(f^2 \cancel{- a^2 - b^2}))}}\right) \\ &= \arccos\left(\frac{-a^2 + b^2 + f^2}{2bf}\right) + \arccos\left(\frac{c^2 - d^2 + f^2}{2cf}\right)\end{aligned} \tag{18}$$

Substituting equation 18 into 16:

$$e = \sqrt{b^2 + c^2 + 2bc\cos\left(\arcsin\left(\frac{-a^2 + b^2 + f^2}{2bf}\right) + \arcsin\left(\frac{c^2 - d^2 + f^2}{2cf}\right)\right)} \tag{19}$$

Hence the proof follows. ■

Remark 1 (Diagonal of any concave quadrilateral in terms of its sides and obtuse angle). *If α obtuse angle ($\pi/2 < \alpha < \pi$) is provided, then equation 4 holds.*

Theorem 2 (Diagonal boundaries of any quadrilateral in terms of its sides). *Let $ABCD$ be a general convex quadrilateral, where a, b, c, d are lengths of sides, e, f lengths of diagonals, where $e_{min}, e_{max}, f_{min}$ and f_{max} represent the minimun and maximum possible lengths of the diagonals, respectively, then f is constrained such that:*

$$f_{min} < f < f_{max} \tag{20}$$

Where:

$$f_{min} = \sqrt{c^2 + d^2 + 2cd\cos\left(\arcsin\left(\frac{-b^2 + c^2 + e_{max}^2}{2ce_{max}}\right) + \arcsin\left(\frac{d^2 - a^2 + e_{max}^2}{2de_{max}}\right)\right)}$$

$e_{max} = a + d$ *if* $b + c > a + d$ *Or* $e_{max} = b + c$ *if* $a + d > b + c$*, And* $f_{max} = c + d$ *if* $a + b > c + d$ *Or* $f_{max} = a + b$ *if* $c + d > a + b$.

Proof. For a convex quadrilateral $ABCD$, a, b, c, d side lengths, e, f diagonals. Let area of $\square ABCD$, $\triangle ABC$, $\triangle CDA$, $\triangle DAB$ and $\triangle BCD$, be A, A_1, A_2, A_3 and A_4 respectively.

From the Second Axiom of Area:

$$\begin{aligned} A &= A_1 + A_2 \\ A &= A_3 + A_4 \end{aligned} \tag{21}$$

From Triangle Inequality theorem:

$$\begin{aligned} c + d &> f \\ a + b &> f \\ b + c &> e \\ a + d &> e \end{aligned} \tag{22}$$

Let denote f_{max} the maximum length size of f diagonal, because is also the shared side ($\overline{BD}$) of adjacent triangles $\triangle DAB$ and $\triangle BCD$. So if $a+b > c+d$, then $f_{max} = c + d$. Conversely $f_{max} = a + b$ if $c + d > a + b$.

Also let denote e_{max} the maximum length size of e diagonal. Diagonals e, f are inversely proportional by Theorem 1, so the minimum length side of f diagonal is as follows:

$$f_{min} = \sqrt{c^2 + d^2 + 2cd\cos\left(\arcsin\left(\frac{-b^2 + c^2 + e_{max}^2}{2ce_{max}}\right) + \arcsin\left(\frac{d^2 - a^2 + e_{max}^2}{2de_{max}}\right)\right)} \tag{23}$$

Where $e_{max} = a + d$ if $b + c > a + d$, otherwise $e_{max} = b + c$ if $a + d > b + c$.
Hence the proof follows. ■

2 The Area of any Quadrilateral

The area of any convex quadrilateral follows from applying Bretschneider's Formula[3] to f value constricted by Theorem 2, but concave case still not so trivial.

Remark 2 *Using Heron's formula[4], the area of a concave quadrilateral can be calculated by considering the adjacent triangle formed by the obtuse angle α and subtracting its area from that of the overall figure. Specifically, given the side lengths of the quadrilateral and indicating the obtuse angle is α, the area can be determined as follows:*

$$K = \frac{1}{4}\sqrt{4e^2f^2 - (b^2 + d^2 - a^2 - f^2)^2} - 2\sqrt{s(s-a)(s-b)(s-f)} \tag{24}$$

Where $s = \dfrac{a + b + f}{2}$ is the semi-perimeter.

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