AN INEQUALITY FOR A SLICE OF ANY COMPACT SET

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Abstract. In this paper we will prove that if a compact A in R^n belongs to the unit ball in R^n, then A has a slice of measure greater than a calculable constant times the measure of A. Our result is sharp.

1. Introduction

In this paper we prove a very general theorem. Often these kind of questions are studied in the convex special case [1]. Our main theorem is the following.

Theorem 1.1. *Let $A \subset B_n(0,1) \subset \mathbf{R^n}$ be compact. Let γ be such that*

$$\mu_n(A) = \int_{B(0,1)} \gamma^n \mu.$$

Then

$$\max_{e \in \mathbf{R}^n} \mu_{n-1}(K \cap H^e) \geq \gamma^n \mu_{n-1}(B_{n-1}(0,1)).$$

A sharp case is the unit ball obviously. We can state our result for compact sets $A \subset B_n(0,1)$ contained in the unit ball as

$$\begin{aligned}\max_{e \in \mathbf{R}^n} \mu_{n-1}(A \cap H^e) &\geq \gamma^n \mu_{n-1}(B_{n-1}(0,1)) \\ &= \frac{\mu_{n-1}(B_{n-1}(0,1))}{\mu_n(B_n(0,1))} \mu_n(B_n(0,\gamma)) \\ &= \frac{\mu_{n-1}(B_{n-1}(0,1))}{\mu_n(B_n(0,1))} \mu_n(A).\end{aligned} \tag{1.1}$$

If $A \subset B(0,R)$, then we have

$$\begin{aligned}\max_{e \in \mathbf{R}^n} \mu_{n-1}(A \cap H^e) &\geq \frac{\mu_{n-1}(B_{n-1}(0,R))}{\mu_n(B_n(0,R))} \mu_n(A) \\ &= \frac{1}{R} \frac{\mu_{n-1}(B_{n-1}(0,1))}{\mu_n(B_n(0,1))} \mu_n(A)\end{aligned} \tag{1.2}$$

2. Notation

The term $B_n(0,R)$ means the origin centered ball of radius R in $\mathbf{R}^n$. We use the term $B_{n-1}(0,R)$ for the origin centered ball of radius R in $\mathbf{R}^{n-1}$. The term $\mathbf{S}^{n-1}$ means the unit sphere. We use H^e to mean $n-1$-dimensional subspace orthogonal to $e \in \mathbf{S}^{n-1}$. For any set B B_δ means the δ-neighbourhood of B: It is the set of points in $\mathbf{R}^n$ such that the euclidean distance to B is strictly less than δ. We use

2010 *Mathematics Subject Classification.* 28A75,52A23.
Key words and phrases. Geometry,Bourgain‘s Slicing Problem, Hyperplane Conjecture, Asymptotic Convex Geometry.

μ_n for the n-dimensional Lebesque-measure and μ_{n-1} for the $n-1$-dimensional Lebesque-measure.

3. The Proof

Now, for a small $\delta > 0$ there is the smallest number M such that

$$\sum_{i=1}^{M-1} \mu_n(B_n(0,1) \cap H_\delta^{e_i}) \leq \mu_n(B(0,1)) \leq \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H_\delta^{e_i}). \tag{3.1}$$

Of course, the sets $B_n(0,1) \cap H_\delta^{e_i}$ do not necessarily cover $B_n(0,1)$. Thus, M is the suitable natural number such that

$$\mu_n(A) = \gamma_\delta^n \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H_\delta^{e_i}) \tag{3.2}$$

for some density γ_δ^n. Now, there is a density γ^n such that

$$\mu_n(A) = \int_{B(0,1)} \gamma^n \mu, \tag{3.3}$$

and then $\gamma \geq \gamma_\delta$ from above and from (3.1). It follows from above and from (3.2) that

$$\gamma^n \mu_n(B(0,1)) = \mu_n(A) = \gamma_\delta^n \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H_\delta^{e_i}). \tag{3.4}$$

So in principle γ_δ can be solved from

$$\mu_n(A) = \gamma_\delta^n \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H_\delta^{e_i}). \tag{3.5}$$

However, we managed to state our theorem in terms of γ. Now, let us suppose towards a contradiction that for compact $A \subset B(0,1)$ it holds that

$$\mu_n(A \cap H_\delta^e) < \gamma_\delta^n \mu_n(B_n(0,1) \cap H_\delta^e) \tag{3.6}$$

uniformly in $e \in \mathbf{S}^n$. Now, there is a smallest N such that

$$\mu_n(A) = \mu_n(\bigcup_{e_i \in \mathbf{S}^{n-1}} B_n(0,1) \cap H^{e_i}) \leq \sum_{j=1}^{N} \mu_n(B_n(0,1) \cap H_\delta^{e_j}). \tag{3.7}$$

So, the above implies that $N \leq M$, because of $A \subset B_n(0,1)$. Now, from assumption (3.6), from from $N \leq M$, and from (3.5) we have that

$$\mu_n(A) \leq \sum_{j=1}^{N} \mu_n(A \cap H_\delta^{e_j}) < \sum_{j=1}^{N} \gamma_\delta^n \mu_n(A \cap H_\delta^{e_j}) \leq \sum_{j=1}^{M} \gamma_\delta^n \mu_n(A \cap H_\delta^{e_i}) = \mu_n(A), \tag{3.8}$$

which is a contradiction. So we have for some $e \in \mathbf{S}^n$ that

$$\mu_n(A \cap H_\delta^e) \geq \gamma_\delta^n \mu_n(B_n(0,1) \cap H_\delta^e). \tag{3.9}$$

Because A is compact we can take

$$\max_{e \in \mathbf{S}^n} \mu_n(A \cap H_\delta^e) = \mu_n(A \cap H_\delta^e).$$

We have

$$\mu_n(A \cap H^e_\delta) = \int_{-\delta}^{\delta} \mu_{n-1}(A \cap x + H^e)dx \leq 2\delta \max_{-\delta \leq x \leq \delta} \mu_{n-1}(A \cap x + H^e).$$

So for a small δ we have

$$\begin{aligned} 2\delta \max_{-\delta \leq x \leq \delta} \mu_{n-1}(A \cap x + H^e)) &\geq \mu_n(A \cap H^e_\delta) \\ &\geq \gamma^n_\delta \mu_n(B_n(0,1) \cap H^e_\delta) \\ &\geq 2\delta\gamma_\delta \mu_{n-1}(B_n(0, \gamma_\delta - c\delta) \cap H^e), \end{aligned}$$

where the second to last inequality follows from (3.9) and the last from the geometry of the ball. Because of the slight curvature of the ball we must include the $c\delta$ term. Thus,

$$\mu_{n-1}(A \cap H^e) \geq \liminf_{\delta \to 0} \max_{-\delta \leq x \leq \delta} \mu_{n-1}(A \cap x + H^e)) \geq \limsup_{\delta \to 0} \gamma_\delta \mu_{n-1}(B_n(0, \gamma_\delta - c\delta) \cap H^e).$$

From above it follows that

$$\mu_{n-1}(A \cap H^e)) \geq \limsup_{\delta \to 0} \gamma^n_\delta \mu_{n-1}(B_n(0, 1 - \frac{c\delta}{\gamma_\delta}) \cap H^e). \tag{3.10}$$

Next, from (3.1) and (3.5) we have that

$$\gamma^n \sum_{i=1}^{M-1} \mu_n B_n(0,1) \cap H^{e_i}_\delta) \leq \gamma^n \mu_n(B(0,1)) = \gamma^n_\delta \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H^{e_i}_\delta). \tag{3.11}$$

So that

$$\gamma^n \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H^{e_i}_\delta) \leq \gamma^n_\delta \sum_{i=1}^{M} \mu_n(B_n(0,1) \cap H^{e_i}_\delta) + \mu_n(B_n(0,1) \cap H^{e_M}_\delta).$$

Because

$$\lim_{\delta \to 0} \mu_n(B_n(0,1) \cap H^{e_M}_\delta) = 0,$$

and because $\gamma_\delta \leq \gamma$, we have that

$$\lim_{\delta \to 0} \gamma_\delta = \gamma.$$

So the theorem 1.1 follows from (3.10).

References

[1] J.Bourgain, *On High Dimensional Maximal Functions Associated to Convex Bodies*, Amer.J.Math,Vol 108, No 6, (1986), 1467 1476.