MOND as a transformation between non-inertial reference frames via Sciama's interpretation of Mach's Principle

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Abstract

Milgrom's Modified Newtonian Dynamics (MOND) correction to Newtonian gravity or inertia is shown to be equivalent to a more fundamental transformation between a non-inertial local reference frame and the fixed background of the observable universe, complying with Mach's principle. Both Newton's gravitational constant and Milgrom's MOND acceleration parameter or scale constant are substituted for two varying, measurable, physical, and cosmological parameters under the justification of Sciama's interpretation of Mach's principle: causally connected mass and size of the universe. This Machian interpretation, scale-invariant and free from fundamental constants and free parameters with the exception of the speed of light as the speed of causality and gravity, is based on relative field intensities of the small and large scale of the universe. It respects the Machian effect from Sciama's model, by which, in absence of a background, rotational speed is undefined up to the speed of light. The Machian MOND approximation is a necessary feature of a phenomenological nonlinear theory of modified inertia or nonlinear theory of modified gravity which incorporates Mach's principle in agreement with galaxy rotation curves.

Keywords: Machs principle; dark matter; galaxy rotation curves; MOND; modified Newtonian dynamics; modified gravity; modified inertia; gravitational constant; inertia

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1 Introduction

1.1 Modified Newtonian Dynamics

The solution to the problem of the perihelion precession of Mercury first involved various failed attempts proposing the unobserved existence of a new small planet Vulcan, an asteroid belt, or a gas cloud orbiting the Sun. However, the resolution to the problem came as a modification to the laws of gravity by General Relativity (GR). Similarly, the observation of rotational velocities in disks of galaxies higher than predicted by Newtonian gravity, as depicted in galaxy rotation curves, has driven further research into modifications to the laws of gravity. The discrepancies between observed luminance to gravitational mass ratio in galaxies and their missing Keplerian falloff in their velocity curves, together with faster radial velocities of galaxies in galaxy clusters than Newtonian predictions, among other evidence, are referred to as the "dark matter" problem [1–3]. Modifying gravity instead of proposing the existence of physical cold dark matter is motivated by the Renzo's rule, the Tully-Fisher law, and the cuspy halo problem. These recent modifications to the classical laws of gravity are called Modified (Orbital) Newtonian Dynamics (MOND) hypotheses (or MilgrOmiaN Dynamics), the first and main MOND theory being Milgrom's MOND [4–6], which can be thought of as a modification to inertia through Newton's second law of motion, or to the inverse square law of gravity.

The reason for the anomalous motion of Mercury was that Newtonian gravity is not a good enough approximation for strong gravitational fields, such as near the Sun, where nonlinear GR effects become significant. Milgrom's MOND explains the observed rotation curves of galaxies by imposing a correction based on two observational relationships: The asymptotic velocities in rotation curves are proportional to the fourth root of the mass (Tully-Fisher law) and the mass discrepancies are always observed below a particular acceleration scale. The simplest correction that satisfies these observational conditions results in a true acceleration $g = GM/\mu\,(g/a_0)\,r^2$, achieving on purpose a constant asymptotic rotational speed $v = \sqrt[4]{GMa_0}$. The connection between the classical Newtonian regime and Milgrom's MOND low acceleration regime (at which the inverse-square law of gravity changes to a linear law) is determined by an "interpolating function" $\mu\,(g/a_0) = 1/(1+(a_0/g)^n)^{(1/n)}$ (with $n = 1$ for the simple and $n = 2$ for the standard interpolating function) based on the "true acceleration" g and a constant of acceleration a_0 (or gravitational field intensity, also referred to as the Hubble acceleration or the acceleration scale constant), which is a free parameter adjusted by data fitting and is in principle considered a fundamental and universal constant in Milgrom's MOND. The interpolating function must satisfy the conditions $\mu(x) \to 1$ for $x \ll 1$ and $\mu(x) \to x$ for $x \gg 1$. A fixed-distance or fixed-mass (independent of the system under study) modification alone cannot account for the observed rotation velocities due to the Tully-Fisher law, i.e., it is not possible to simply set a constant of distance or a constant of mass to modify the laws of gravity to fit all rotation curves of galaxies (because some galaxies exhibit the effect of dark matter at small radius while others at large radius, and same happens with different galaxy masses). But by combining both mass and distance in Newtonian acceleration or gravitational field intensity, Milgrom's MOND achieves this limit where the effect of dark matter in most galaxies becomes relevant. Still, Milgrom's MOND, being insufficient to explain all dark matter related phenomena such as galaxy cluster dynamics, [7, 8], being non-relativistic, and not satisfying conservation laws, is an effective theory or approximation to a more fundamental modification to the theory of gravity.

The acceleration constant in Milgrom's MOND $a_0 \approx 1.2 \cdot 10^{-10}\ m/s^2$, was already related to the cosmological scale through the Hubble constant H_0 (or the cosmological constant, respectively) $a_0 \sim cH_0 \sim c^2\sqrt{\Lambda}$ originally by Milgrom through the Hubble parameter and then through the cosmological constant [9], stating that this coincidence could point to a basic theory underlying MOND's phenomenology. Milgrom stated that "*an attractive possibility is that MOND results as a non-relativistic, small-scale expression of a fundamental theory by which inertia is a vestige of the interaction of a body with "the rest of the Universe", in the spirit of Mach's principle.*" [10], although he did not further develop his theory in relation to Mach's principle.

Progress in fundamental physics has been made by reducing the number of fundamental constants (such as the gravitational acceleration of the Earth and other planets for a universal gravitational constant, the unification of the speed of light with vacuum permittivity and permeability, and atomic constants for the Planck constant). Therefore, it is of considerable interest to explore modifications of the laws of gravity without the need for more fundamental constants.

Various theories of modified gravity are based on Milgrom's MOND acceleration constant: John Moffat's MOG scalar–tensor–vector gravity approach to galaxy rotation curves is based on two parameters, which are set to fit galactic Milgrom's MOND's acceleration constant [11], Erik's Verlinde's entropic gravity attempts to propose an underlying framework for Milgrom's MOND's acceleration constant [12], and Deur's self-interacting gravitons model correction to Newtonian gravity contains a physical constant counterpart to Milgrom's MOND's acceleration constant in the form of $a = \Lambda/a_0 = \sqrt{\Lambda}/c^2$ [13]. Others, such as quantized inertia, rely on another acceleration parameter in the form of $a = c^2/R_u$ where R_u is the radius of the cosmological comoving horizon [14].

1.2 Mach's Principle

Newtonian gravity and classical mechanics are founded on the principles of absolute space, absolute time, and absolute motion (including acceleration). These were originally opposed by Leibniz (and others, such as Berkeley in his 1721 *De Motu*, and Huygens) arguing that as observers, we can only epistemically access relative notions of space, time, and motion. Newton's justification of absolute space and motion is portraited in his rotating bucket of water experiment at the beginning of his 1687 masterpiece *Philosophiæ naturalis principia mathematica*, which can be summarized as follows: A bucket of water is set spinning around its axis and, at first, the walls of the bucket rotate relative to the stationary water while the surface of the water remains flat as prior to the spinning. After the water starts to rotate as well, it rises towards the walls and its shape is no longer flat when the spinning bucket and the water are at rest relative to each other. Interactions between water and walls (due to their relative velocities) cannot explain the shape of the water, and Newton assumed that this experiment could be used to measure rotation with respect to an absolute space, "*without relation to anything external, which remains always similar and immovable*".

The definition of absolute space, time, and motion always suffers from circular reasoning. In Newtonian mechanics, absolute space is defined as the frame absent of inertial forces (also referred to as fictitious forces, which come from our choice of reference frame that is rotating in absolute space), but the absence of inertial forces is used to prove and identify the existence of absolute space in Newton's rotating bucket of water experiment. According to Newton's first law of motion, an object moves inertially if it is free from outside influences, but the fact that it is free from outside influences is inferred only by observing that it moves inertially. In Special Relativity, light travels at the same speed in all inertial frames and absolute acceleration is defined relative to inertial frames, but inertial frames are defined as frames absent of acceleration (accelerometers always measure acceleration with respect to a reference frame of calibration, which could be non-inertial). If a body is measured to have acceleration in one inertial frame, all other inertial frames will agree that it is accelerating.

Leibniz's relativistic space, time and motion ideas were further developed by Ernst Mach in his 1883 *The Science of Mechanics* (which inspired Einstein to develop GR, who originally coined the term "Mach's principle"), in which he criticizes Newton's conclusion of his bucket of water experiment stating that "*No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick.*"[15]. According to Mach, the relative rotation of the water with respect to the bucket produces no noticeable centrifugal forces, and such forces are instead produced by its relative rotation with respect to Earth and the other celestial bodies (in this way, the relative motion between the several leagues thick bucket and the water could produce noticeable centrifugal forces in the water, and perhaps their non-relative motion could reduce these forces when both rotate with respect to the rest of the universe). In contrast to Newton, who attempted to explain the physical effects of inertia through a sort of resistance to motion within an unobservable absolute space with no physical properties, Mach conceived inertia as an interaction that required other external bodies to manifest. Newton did not suspect that the change in the water's shape could be due to rotation relative to the rest of the universe because his action at a distance gravitational force from his infinite and homogeneous universe acting on the water cancels out according to Newton's shell theorem and it is independent of velocity or acceleration. But no local phenomena can ever be isolated from the rest of the universe, which certainly reaches the water even if the sum of these forces is zero. According to Mach, and in opposition to Newton's conclusions, a spinning bucket of water in an empty universe would not change the water's shape (one could not detect relative motion in an empty universe, it would be undefinable), and if the rest of the universe was set spinning while the bucket was at rest, the water surface would curve, as both the spinning of the bucket or the spinning of the universe are indistinguishable systems without an absolute space.

Moreover, there is an exact coincidence between the local measurement of the angular velocity of the Earth through Foucault's pendulum and the cosmological measurement through the apparent movement of distant stars and galaxies, which Newtonian gravity and mechanics cannot explain because it does not causally connect both measurements (and consider the determination of inertial frames by the fixed stars and Foucault's pendulum a coincidence), but constitutes the basic idea of Mach's principle: "*The universe, as represented by the average motion of distant galaxies, does not appear to rotate relative to local inertial frames.*"[16].

According to Mach, and known as Mach's principle, the inertia of a body is not an independent and intrinsic property of matter (unlike in Newtonian mechanics and GR, where a particle in an empty universe has inertial properties), but rather the result of the action of the universe as a whole. Mach suggested that the fixed background distribution of matter in the universe must exert inertial forces on a local accelerating body. In this way, the reference system with respect to which the universe is at rest or in uniform and rectilinear motion is a true inertial reference system. In Ernst Mach words, "*I have remained to the present day the only one who insists upon referring the law of inertia to the Earth, and in the case of motions of great spatial and temporal extent to the fixed stars.*" [15]. Hence, inertial frames should be defined with respect to this rest frame, and local physical laws must be determined by the large-scale structure of the universe.

A frame linearly accelerated relative to an inertial frame in the Minkowskian spacetime of Special Relativity is locally identical to a frame at rest in a gravitational field. Einstein's equivalence principle assumes that inertial and gravitational masses are equal, and the metric tensor in GR determines the inertial mass of a body. Gravity is the only long distance force that cannot be screened. Moreover, there always exists a reference frame in which inertial forces vanish, just as for the case of gravitational forces in free fall. These equivalences indicate that both inertial and gravitational forces are of the same nature. Following Mach, in any two-body interaction, the influence of all other matter inside their causal spacetime should be taken into account. Inertia is thus a form of gravitational induction, appearing when a body is accelerated with respect to the rest of the universe, subjected to retarded action. It seems that to describe, for example, the Earth's motion around the Sun, only local masses and distances are required, but the Machian perspective implies that the universe's action is already taken into account in the Newtonian laws. The only way this can be true is through the Newtonian gravitational constant, which, together with the choice of an inertial absolute space frame, both constitute the two arbitrarily choices of Newton.

In Newtonian gravity, the gravitational constant G is the only fundamental constant of the theory, and it is known with far less precision than any other fundamental dimensional constant in physics. The uncertainty in its measurement, together with its problematic in Quantum Field Theory in Planck's energy (which is used as a cutoff for the energy density of the vacuum and the theoretical quantum corrections estimation of the Higg's mass, leading to the vacuum catastrophe and hierarchy problems) has led to questioning its constant nature repeatedly over the last century, see [17] for reference. GR carries this constant (a varying G would imply a violation of the strong equivalence principle) with the addition of the speed of light, which can be derived from vacuum permittivity and permeability constants. Therefore, it is reasonable to question whether the gravitational constant is also a derived parameter.

The first historically suggestion of gravity arising necessarily as a consequence of the relativity of inertia came from Hans Reissner [18]. Schrödinger soon identified in 1925 [19] that GR did not fully implement Mach's principle, and proposed a relationship between the gravitational potential of distant masses and the speed of light: "*This remarkable relationship states that the (negative) potential of all masses at the point of observation, calculated with the gravitational constant valid at that observation point, must be equal to half the square of the speed of light.*". The conclusion was reached by imposing that the kinetic energy had an origin in a potential-like interaction (such as all other forms of energy, which have an origin in an interaction) and was relational, so that $K = mv^2/2$ based on absolute mass and velocity can be expressed through the gravitational Newtonian potential Gm/r so that $K = (Gm_i/r_{ij})(m_j v_{ij}^2/2c^2)$ with the necessity of introducing c^2. The term $Gm_i/r_{ij}c^2 = GM_u/c^2R_u \sim 1$ is considered missing in the original formulation, with M_u being the sum of all masses and R_u the distance between the moving particle m_j and M_u. This relationship was later credited to Reissner by Schrödinger himself.

Schrödinger's formulation appeared repeatedly in several works after him [20–24], but it was popularized by Dennis Sciama, who developed a modified cosmological vector potential theory of inertia in which the inertial law arises as a side effect of gravity [25]. He proposed that local inertial forces result from the gravitational induction of the universe, so that the dynamics of a rotating body would be affected by the large-scale distribution of mass of the universe. The induced inertia decreased with $1/r$ with r being the distance to the inertial body, so that the action of global matter dominates over the action of local matter, and it is not significantly modified by the acceleration of local matter, leading to the illusion that inertia depends only on the body itself. He postulated that "*In the rest-frame of any body the total gravitational field at the body arising from all the other matter in the universe is zero*". The gravitational constant at any point was determined by the total gravitational potential of the distribution of matter of the universe, taking the form of $G \sim c^2R_u/M_u = c^2/\Phi_u$ with M_u and R_u the mass and radius of the observable universe, causally connected by the speed of light c in the past light-cone of the point considered to evaluate G, for the equivalence of inertial and gravitational masses to hold. In this way, local phenomena were strongly coupled to global properties of the universe. Sciama's model predicted a non-infinite and expanding universe, since the potential of an infinite non-expanding eternal universe would also be infinite. Sciama's relationship was also derived through an alternative approach based on the velocity of waves in a medium of a certain pressure and mass density, in which the rest energy of a given gravitational mass is its gravitational potential energy due to the distribution of masses of the universe [26].

By pure dimensional analysis, Reissner's and Schrödinger's equation (henceforth Sciama's relationship) is the only possible derivation of the units of measurement of the gravitational constant through a mass, a distance, and the speed of light. If Sciama's relationship is derived from a more fundamental theory, it can be expected to also contain integers and mathematical constants, but these will be omitted for simplicity, since it does not significantly affect the resulting orders of magnitude of the comparison between M_u and R_u.

Sciama's relationship with today's measurements of the current cosmological model is satisfied only in terms of orders of magnitude, because the Hubble tension impedes a precise calculation. It yields the orders of magnitude of the observed gravitational constant when considering not the baryonic and dark matter content for M_u, but the total energy content (since all forms of energy must gravitate according to GR), which roughly corresponds to the critical density of the universe as measured to be almost flat. However, the critical density is calculated with the assumption of a constant gravitational constant itself and considering dark matter, which could be eliminated through a correction to the gravitational constant, as will be proposed in the next section. Thus, an honest calculation must be done with direct observations only by the total luminosity of the universe and its average mass-to-light ratio, which yields a value $10^{52}kg < M_u < 10^{53}kg$, and the observable radius of the universe, which can be estimated by direct observations of the Hubble parameter (subjected to the Hubble tension), which yields around $R_u \sim 10^{26}m$.

Few authors have attempted to develop a MOND for galaxy rotation curves based on Mach's principle, or relate Milgrom's MOND to Sciama's relationship and Mach's principle [27–33]. The most significant points in the literature relating these topics are summarized onwards.

Alexander Unzicker explored galaxy dynamics in relation to Mach's Principle and the rotating bucket of water experiment, and proposed a basis for a MOND to solve flat rotation curves without considering Milgrom's MOND [34]. For Unzicker, the rotation of a galaxy is the reason for gravitation, and a modification to the gravitational constant in function of size, mass, and angular momentum of the galaxy could explain flat rotation curves. He first proposes another interpretation of Sciama's formulation substituting c^2 for v^2 so that $v^2 = G \sum m_i/r_i$ and considering v the asymptotic rotation velocity of the galaxy, but lacks an interpolating function through which it can be effectively applied to the solar system. He proposes a thought experiment: According to Mach, the system of two distant masses rotating around their center of mass in equilibrium of gravitational and centrifugal forces in an empty universe, since rotation between them is undefined due to absence of absolute space, must be equivalent to the system of those two distant masses not rotating and without gravity (only radial or relative velocities are measurable, and tangential velocities are impossible to measure instantaneously). The maximum possible angular velocity is limited by $w_{max} = c/r$ so that the maximum tangential speed is $v = c$ for one of the masses rotating around the other one, since the speed of light cannot be surpassed in any case. This luminal rotation in absence of background is a feature of introducing Sciama's relationship in Newton's law of gravity.

Jaume Gine explored a Machian interpretation of the modified second law of motion for the simple interpolating function of Milgrom's MOND based on the accelerated expansion of the universe [30], in which $a_0 = GM_u/{R_u}^2 = c^2/R_u$ through Sciama's relationship is the acceleration that an experimental body feels induced by the rest of the matter of the universe in its inertial frame of reference, and the value of acceleration for which inertial and gravitational masses of a body can differ. Thus, the equivalence principle between inertial and gravitational masses is broken for accelerations smaller than a_0. In a following paper [31], Gine attempted to derive a phenomenological version of Milgrom's MOND modification to Newton's second law. He considers a_0 to be the acceleration at which the edge of the universe at distance R_u goes away from a considered central inertial point due to the expansion of the universe. By relativizing acceleration considering the distant universe at rest, a new interpolating function similar to Milgrom's MOND simple interpolating function is derived.

Even though interesting ideas have been put forward relating MOND to Mach's principle, an exact equivalence will be presented onwards, which should arise as an approximation in any nonlinear theory of modified inertia or gravity which attempts to explain dark matter effects in galaxies. As Milgrom stated in one of his conferences, "*the only system that is strongly general relativistic and in the MOND regime is the Universe at large*".

2 Milgrom's MOND from Mach's Principle

Starting from the well-known form of Milgrom's MOND standard interpolating function correction to Newtonian gravity $g_N = GM/r^2$ or to Newton's second law $F = m_i g$, the "true" gravitational acceleration g is

$$g = \frac{G}{\mu\left(\frac{g}{a_0}\right)} \frac{M}{r^2} \quad or \quad F = \mu\left(\frac{g}{a_0}\right) m_i g \tag{1}$$

$$\mu\left(\frac{g}{a_0}\right) = \frac{1}{\sqrt{1 + \left(\frac{a_0}{g}\right)^2}} \quad or \quad \mu\left(\frac{g}{a_0}\right) = \frac{1}{1 + \frac{a_0}{g}} \tag{2}$$

with a_0 being Milgrom's MOND acceleration or gravitational field intensity constant (also known as Milgrom's MOND scale constant), M the active gravitational mass, m_i the inertial mass, and r the distance between the centers of mass of the active and passive gravitational masses, so that for the motion of a test particle in a gravitational field in the low acceleration or deep MOND regime, $\mu\left(g/a_0\right) \to g/a_0$ and $g = \sqrt{a_0\ g_N}$, so that with $g = v^2/r$, $v = \sqrt[4]{GMa_0}$ for achieving flat rotation curves in agreement with the Tully-Fisher law in a remarkable simple way with a single introduced free parameter.

Milgrom's MOND acceleration or gravitational field intensity constant a_0 is usually interpreted as a fundamental constant in standard MOND theory, but possibly related to the cosmological parameter of the cosmological constant $a_0 \sim c^2\sqrt{\Lambda}$ with $\Lambda = 8\pi G\rho_{vac}/c^2$ and ρ_{vac} the energy density of the cosmological constant or dark energy according to GR, which coincides in orders of magnitude (a better match can be done with $a_0 = c^2\sqrt{\Lambda/3}$, although integers and pi are onwards omitted for simplicity). In order to express a_0 in terms of the mass and radius of the universe, ρ_{vac} is substituted for the matter density of the universe $\rho_{vac} \sim \rho_u \sim M_u/{R_u}^3$ with M_u the causally connected mass to the local system of study through the radius R_u of the observable universe, a relationship which also coincides in orders of magnitude. Newton's gravitational constant G is substituted for Sciama's interpretation of Mach's principle $G \sim c^2R_u/M_u$, so that $a_0 \sim c^2/R_u$, and by reverse engineering, $v \sim c\sqrt[4]{M/M_u}$ in the equivalent deep MOND regime and $g \sim c^2\sqrt{M/M_u}/r$. Inserting both g and a_0 into the original interpolating function, it is now scale-invariant in mass and distance, and the speed of light cancels out, resulting in a Machian MOND formulation

$$g = \frac{c^2}{\frac{M_u}{R_u}\ \mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right)} \frac{M}{r^2} \quad or \quad \frac{M}{r^2} = \frac{M_u/R_u}{c^2}\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) g \tag{3}$$

$$\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) = \frac{1}{\sqrt{1 + \left(\frac{M_u/{R_u}^2}{M/r^2}\right)}} \quad or \quad \mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) = \frac{1}{1 + \sqrt{\frac{M_u/{R_u}^2}{M/r^2}}} \tag{4}$$

Thus, a transformation (4) can be identified between the non-inertial reference frame due to gravity of a local system with a mass parameter M and distance r, and a global reference frame of observable mass M_u and size R_u of the universe. The variable changed by this transformation is either the gravitational force (or equivalently, a variable gravitational constant), or the inertial law. Note that in (1), the interpolating function depends on true acceleration, while in Machian MOND depends on Newtonian acceleration.

It is trivial that, since only equivalent substitutions for G and a_0 according to Sciama's interpretation of Mach's principle which yield the same values are done (omitting integers and mathematical constants), and M and r take the same meaning as in Milgrom's MOND, the resulting correction (3) and (4) for rotation curves of galaxies is equivalent to standard Milgrom's MOND. For $\frac{M_u/{R_u}^2}{M/r^2} << 1$, $\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) \to 1$ and Newtonian gravity is restored. For $\frac{M_u/{R_u}^2}{M/r^2} >> 1$, $\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) \to \sqrt{\frac{M/r^2}{M_u/{R_u}^2}}$, $v = c\sqrt[4]{M/M_u}$, and the speed of rotating stars around galaxies far away from their galactic centers is determined by the relative active gravitational masses between the local system and the global system. The resulting velocities in this case do not strongly depend on the values of M_u due to the fourth root, so they are not too sensitive to the uncertainties of the measured mass value.

It is also trivial that, integrating an average mass density of the homogeneous and isotropic universe with uniform matter density at large enough scales over the causally connected spherical volume, the integral results in $NM_u/{R_u}^2$ (with N being a rational factor that can be omitted for simplicity since it does not affect the orders of magnitude of the relationship between M_u and R_u). Thus, under Machian MOND, the numerator $M_u/{R_u}^2$ in (4) is the total field intensity (the gravitational field intensity without the gravitational constant G) of all masses of the universe at the point in which the correction is applied (omitting integers for simplicity), taking the same meaning as M/r^2 in the denominator, which is the field intensity of the masses of the local system at the point in which the correction is applied.

By decreasing the mass of the universe within the observable radius M_u, the correction term (4) decreases the gravitational force. But, considering also Sciama's relationship, the gravitational force increases (having the same observational consequence as considering Sciama's relationship alone, the decrease of inertia), and local velocities increase. Decreasing the mass of the universe until the absence of any background ($M_u \to M, R_u \to r$), the transformation μ (4) reduces to some simple number K which can be set to unity with the appropriate choice of integers in the formulation of the Machian transformation and Sciama's Mach's principle interpretation (a simple way to obtain $K = 1$ is considering Schrödinger's original definition of G and multiplying the term in parenthesis of the transformation by 3), and velocities approach $v \to Kc$ at the edge of the galaxy. This is already a feature of Sciama's relationship, and it is reasonable according to Mach, since without any background, rotation is undefined and tangential speed can take up any value up to the limit of the speed of light. The proposed transformation respects this result and also respects the scale invariance in mass and distance that Sciama's relationship introduces (doubling the mass or size of both the universe and the local system results in the same estimated velocities) to the scale-dependent Newtonian gravity (Milgrom's MOND is only scale-invariant when $a_0 \to \infty$ fixing Ga_0).

3 Discussion

It is shown that Milgrom's MOND correction to Newtonian gravity (or inertia and Newton's second law) can be reformulated based on Mach's principle without dimensional constants or free parameters except for the speed of light, ensuring scale invariance.

Substituting both Newton's gravitational constant G for Sciama's interpretation of Mach's principle $G \sim c^2/(M_u/R_u)$ with M_u and R_u (mass and radius of the observable universe) of global, measurable, variable and physical parameters relationship, and Milgrom's MOND acceleration constant a_0 for $a_0 \sim GM_u/{R_u}^2 \sim c^2/R_u$, yields equivalent results to Milgrom's MOND: in the deep-MOND regime, $v = \sqrt[4]{GMa_0} \equiv c\sqrt[4]{M/M_u}$. The gravitational force (or inertial law) depends on the relative field intensities of local and global mass distributions. In this way, the mass and size of the observable universe are always accounted for when considering non-inertial reference frames, as it is always the case for gravitational systems and non-inertial motion, in agreement with Mach's principle.

The relationship pointed out by Milgrom between his MOND's acceleration constant a_0 and the cosmological constant or the Hubble parameters is interpreted merely as accidental due to the cosmological coincidence by which, at present epoch, the energy densities associated with dark energy (the cosmological constant) and visible matter are of the same orders of magnitude.

The Machian MOND's interpolating function is a transformation $\mu(M_u, R_u, M, r)$ (4) between different scales and non-inertial reference frames which, by using the standard MOND interpolating function, resembles a Lorentz-like transformation factor with a plus sign instead of a minus sign: the relationship between a local frame in spacetime, and that of a global frame which is at rest (the background of the rest of the universe) with respect to the first one, with both being gravitationally accelerated relative to each other. It expresses how gravity changes locally due to the background universe through relative masses and distances in both systems, in accordance with relational mechanics. The speed of light in Sciama's relationship takes the meaning of the speed of gravity and causality justifying the choice of R_u as the radius of the observable universe containing mass M_u, which is the part of the universe causally affecting local dynamics, for Mach's interpretation not to violate locality.

Machian MOND satisfies several definitions of Mach's principle: Newton's gravitational constant G is a dynamical field, local inertial frames are affected by the cosmic distribution of matter, and rotation is undefined up to the speed of light in the absence of the rest of the universe. If the correction applies to inertia instead of just to gravity, then inertial mass would also be affected by the global distribution of matter and an isolated body in otherwise empty space would have no inertia. In agreement with Mach's principle, if inertial mass is defined by the potential of the distribution of mass (following Sciama's argument), it is reasonable to think that it could also be affected by the field intensity of the distribution of mass. Thus, Machian MOND introduces an even stronger (and nonlinear) dependence of local dynamics on the distribution of matter than theories that just consider Sciama's relationship, and can be considered more Machian.

Milgrom himself favors modifying inertia (which is the true aim of Mach's principle), under the justification that Newtonian inertia was already modified by special relativity. The formal limit where $a_0 \to 0$ in which Milgrom's MOND approaches the standard Newtonian action does not hold, since the mass and radius of the universe (for which a_0 is replaced) always exist. Machian MOND, or a similar and equivalent approximate formulation, is considered to be a necessary nonlinear feature of a phenomenological theory of modified inertia or modified gravity which incorporates Mach's principle, since Milgrom's MOND is clearly nonlinear due to its external field effect. This might arise by imposing Mach's principle (for instance, through the relativity of accelerations and inertia, and inertia being of gravitational origin) to a non-Machian nonlinear theory of gravity, such as GR. So far, only linear and non-relativistic full Machian theories of modified inertia with Sciama's relationship have been constructed.

Milgrom's MOND originally violates conservation of linear momentum for N-body systems. This can be fixed through a non-relativistic Lagrangian formulation of MOND as a modification to Newtonian gravity, which satisfies all standard conservation laws [35]. Applying the Gauss law for cases of high symmetry, (1) is recovered. Machian MOND (3) can be derived from an action in the same way. As modified gravity, it does not violate Einstein's equivalence principle (although the strong equivalence principle is violated) because the modification to the gravitational acceleration does not depend directly on the passive gravitational or inertial mass.

It is well known that, according to Bekenstein, a higher value for Milgrom's MOND a_0 could resolve the tension between standard MOND and galaxy cluster's core dynamics (and that a_0 seems to grow with scale), where Newtonian accelerations take around that same value (provided that a_0 stays the same in galaxies for agreement with rotation curves). Machian MOND, although being equivalent to Milgrom's MOND, is effectively a theory of a varying a_0 with time scale and could in principle resolve this issue of Milgrom's MOND. It could also in principle solve the cosmological issues of Milgrom's MOND [36]. As modified gravity, the equivalent variation of the Newtonian G is both temporal due to the expanding universe and dependence on the mass and radius of the observable universe, and spatial since local potentials should also be included in Sciama's relationship and the correction through the interpolating function depends on the local field intensity.

It would be interesting to compare Machian MOND with observational constraints of a varying gravitational constant at the early universe considering a cosmological model without dark matter (assuming that Machian MOND solves partly the need for dark matter), due to its dependence on the mass and radius of the observable universe, which have both varied over time. A similar study has been done for quantized inertia by comparing the prediction of a specific increase in the galaxy rotation anomaly at higher redshifts [37]. The formulation of quantized inertia is similar to the one proposed in this article (without Sciama's relationship), but its interpretation is not based on Mach's principle. In fact, no Planck's constant is used in the quantized inertia formulation, which suggests that its origin is not related to quantum phenomena.

The form of the transformation (4) is not unique and depends on the chosen Milgrom's MOND interpolating function for which the Machian substitutions are made. The transformation between non-inertial reference frames could also be applied directly as a correction to relativistic momentum, mass, and energy. Furthermore, which integers and mathematical constants play a role within the Machian transformation and within Sciama's relationship is left to be discussed for a better agreement with observations.

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Declarations

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