Understanding when the correlations are causation

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Abstract

In this paper, we will expose for the Gaussian multiple causation a theorem relating the causation to correlations. This theorem is based on another equality which will be also proven.

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1 Introduction

The aim of this paper is to propose a theorem relating the conditions in which the correlations imply causality. The main theorem is based from an intermediate result relating the Gaussian conditional variance (Schur's complement) to the variance applied to the difference between the current variable and its conditional mean.

2 Conditional variance and marginal variance

In what follows, we will make the link between the conditional variance (Schur's complement) $K_{X^2} - K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}$ and the marginal variance $Var(.)$ of the difference between the current variable X and the conditional mean $E[X|\Omega]$:

$$\boxed{\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])}$$

Proof:

$\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])$

$\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = \Sigma_{X^2} - 2.Cov(X, E[X|\Omega]) + Var(E[X|\Omega])$

$2.Cov(X, E[X|\Omega]) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$2.Cov(X, \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\mu_\Omega) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$2.Cov(X, \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega}) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

Using the bilinearity of the covariance we obtain:

$2.\Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$\Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

We will now develop $Var(E[X|\Omega])$:

$Var(E[X|\Omega])$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\mu_\Omega)$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega})$

As we have the following relationship:

$Var(A.(\vec{Y} - \vec{\mu})) = Var(A.\vec{Y} - A.\vec{\mu}) = Var(A.\vec{Y}) = A.Var(Y).A^t$

we obtain:

$Var(E[X|\Omega])$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega})$

$= \Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega^2}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

$= \Sigma_{X\Omega}.I.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

$= \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

We have proven the relationship: $\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])$

3 Context when the Correlations are causation

Theorem:

If $\Omega \equiv \vec{\omega}$ *is a set of causes with* $\#\Omega \geq 2$ *and* X *a variable then there exists a causal relationship between the variables* Ω *and the variable* X*:*

$$X = E[X|\Omega] = \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}\mu_\Omega = \sum_{i=1}^{\#\Omega}\beta_{X\omega_i}.\omega_i + \beta_X$$

If:

$$K_{X\Omega}.K_{\Omega}^{-1}.K_{\Omega X} = 1$$

Where $\Sigma_{(\Omega,X)^2}$ *is the covariance-variance matrix between the variables* (Ω, X) *and* $K_{(\Omega,X)^2} = diag^{-1}(\Sigma_{(\Omega,X)^2}).\Sigma_{(\Omega,X)^2}.diag^{-1}(\Sigma_{(\Omega,X)^2}$ *is the correlation matrix between the variables* (Ω, X)*.*

Proof:

We will consider a symmetric matrix $\Sigma_{(\Omega X)^2}$ strictly positive definite or a symmetric matrix $\Sigma_{(\Omega X)^2}$ having a single negative eigenvalue:

$$\Sigma_{(\Omega X)^2} = \begin{pmatrix} \Sigma_{\Omega^2} & \Sigma_{\Omega X} \\ \Sigma_{X\Omega} & \Sigma_{X^2} \end{pmatrix}$$

We will project this matrix onto the boundary of the cone of symmetric positive semi-definite matrices (see paper [4] page 9).The projection is done by spectral decomposition and by putting the last eigenvalue equal to 0:

$$\Sigma_{(\Omega X)^2}^{+} = P_S(\Sigma_{(\Omega X)^2}) = \begin{pmatrix} \Sigma_{\Omega^2}^{+} & \Sigma_{\Omega X}^{+} \\ \Sigma_{X\Omega}^{+} & \Sigma_{X^2}^{+} \end{pmatrix}$$

The symmetric matrix $\Sigma_{(\Omega X)^2}^{+}$ is singular because it is onto the boundary of the cone of the positive semi-definite matrix, we obtain therefore:

$$det(\Sigma_{(\Omega X)^2}^{+}) = det(\Sigma_{\Omega^2}^{+}).(\Sigma_{X^2}^{+} - \Sigma_{X\Omega}^{+}.(\Sigma_{\Omega^2}^{+})^{-1}.\Sigma_{\Omega X}^{+}) = 0$$

$\Sigma_{\Omega^2}^{+}$ is strictly positive definite, we can therefore deduce:

$$det(\Sigma_{\Omega^2}^{+}) > 0 \Longrightarrow \Sigma_{X^2}^{+} - \Sigma_{X\Omega}^{+}.(\Sigma_{\Omega^2}^{+})^{-1}.\Sigma_{\Omega X}^{+} = 0$$

From the theorem proof in the previous section, we obtain therefore:

$$\Sigma_{X^2}^{+} - \Sigma_{X\Omega}^{+}.(\Sigma_{\Omega^2}^{+})^{-1}.\Sigma_{\Omega X}^{+} = Var(X - E[X|\Omega]) = 0$$

The equality implies that we have the causal relationship:

$$X = E[X|\Omega] = \Sigma^+_{X\Omega}.(\Sigma^{-1}_{\Omega^2})^+.\vec{\omega} + \mu_X - \Sigma^+_{X\Omega}.(\Sigma^{-1}_{\Omega^2})^+\mu_\Omega = \sum_{i=1}^{\#\Omega} \beta_{X\omega_i}.\omega_i + \beta_X$$

We will factorize the variance $\Sigma^+_{X^2}$ into quadratic form $\Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{X,\Omega}$ to show the correlations $K_{X,\Omega}.K^{-1}_{\Omega^2}.K_{\Omega,X}$:

$\Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{X,\Omega}$

$= \Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(diag^{-1}(\Sigma^+_{\Omega^2}))^{\frac{1}{2}}.K^{-1}_{\Omega^2}.(diag^{-1}(\Sigma^+_{\Omega^2}))^{\frac{1}{2}}.\Sigma^+_{\Omega,X}$

$= \Sigma^+_{X^2} - (\Sigma^+_{X^2})^{\frac{1}{2}}.K_{X,\Omega}.K^{-1}_{\Omega^2}.(\Sigma^+_{X^2})^{\frac{1}{2}}.K_{\Omega,X}$

$= \Sigma^+_{X^2}.(1 - K_{X,\Omega}.K^{-1}_{\Omega^2}.K_{\Omega,X})$

The expression $X = E[X|\Omega]$ is valid when we have:

$$\Sigma^+_{X^2} - \Sigma^+_{X\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{\Omega X} = \Sigma^+_{X^2}.(1 - K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X}) = 0$$

This implies that we have the causal relationship when the quadratic form $K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X}$ verifies the following equality:

$$K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X} = 1$$

We have proven the theorem.

4 Conclusion

In this paper, we have proved a theorem giving the conditions under which correlations imply causality. The paper therefore aimed to relate the notion of correlation to that of causality.

[1]Elements of information theory. Author: Thomas M.Cover and Joy A.Thomas. Copyright 1991 John Wiley and sons.

[2]Optimal stastical decisions. Author: Morris H.DeGroot. Copyright 1970-2004 John Wiley and sons.

[3]Matrix Analysis. Author: Roger A.Horn and Charles R.Johnson. Copyright 2012, Cambridge university press.

[4]Computing the nearest correlation Matrix-A problem from finance. Author: Nicholas Higham.copyright 2002, The university of Manchester

[5]Causal effect vector and multiple correlation. Author Nabil Ait-Taleb. copyright 2024