Power Series Representation of the Golden Mean and Real Numbers: The Snake Bits in Its Tail

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Abstract

This is a simple number-theoretical exercise for interested pupils presenting a connection between the golden mean power series expansion an real numbers.

Mathematical Exercise

The golden mean or golden ratio is an omnipresent number in nature, found in the architecture of living creatures as well as human buildings, music, finance, medicine, philosophy, and of course in physics and mathematics including quantum computation [1][2][3]. It is the most irrational number known with the simplest continued fraction representation at all [4] and a number-theoretical chameleon with a self-similarity property. All these properties render it to be suitable for quantum computer application. In this contribution the golden mean is denoted by φ. Its inverse number, denote by big Φ, is just $\Phi = \varphi^{-1} = 1 + \varphi$. One yields numerically

$$\varphi = \frac{\sqrt{5}-1}{2} = 0.6180339887\ldots \qquad \varphi^5 = \frac{\sqrt{125}-11}{2} = 0.090169943\ldots$$

$$\Phi = \varphi^{-1} = \frac{\sqrt{5}+1}{2} = 1 + \varphi = 1.6180339887\ldots$$

It is important to notice that in numerous papers, for instance such about topological quantum computation (*TQC*), the golden mean is taken as $\frac{\sqrt{5}+1}{2}$. The infinite continued fraction representation is outlined in reference [4].

The known series representation for φ can be applied more generally with some new didactic insights. It delivers

$$\sum_{n=1}^{\infty} \varphi^n = \sum_{n=1}^{\infty} \left(\frac{1}{\Phi}\right)^n = \varphi + \varphi^2 + \varphi^3 + \cdots = 1 + \varphi = \Phi$$

If we use the reciprocal of the sum we get

$$\left(\sum_{n=1}^{\infty} \varphi^n\right)^{-1} = \varphi$$

Now we generalize this equation for positive real numbers

$$\left(\sum_{n=1}^{\infty} m^{-n}\right)^{-1} = m - 1$$

As an important example we apply the equation for the interesting *Fibonacci* number $m = 13$

$$\left(\sum_{n=1}^{\infty} \left(\frac{1}{13}\right)^n\right)^{-1} = 12$$

Summing only over uneven numbers $n = 1,3,5, \ldots$ we yield

$$\left(\sum_{n=1,3,5\ldots}^{\infty} \left(\frac{1}{13}\right)^n\right)^{-1} = 13 - \frac{1}{13}$$

or general for all positive real numbers

$$\left(\sum_{n=1,3,5\ldots}^{\infty} m^{-n}\right)^{-1} = m - \frac{1}{m}$$

The change from $m - \frac{1}{m}$ to $m + \frac{1}{m}$ can be performed by the relation

$$\left(m + \frac{1}{m}\right)^2 = 4 + \left(m - \frac{1}{m}\right)^2$$

The term $\left(13 + \frac{1}{13}\right)^2$ can be used to approximate coefficients of the icosahedron equation such as 171 or 228 [5] [6]. These coefficients can vice versa be applied to compose the inverse of *Sommerfeld*'s structure constant α^{-1} [2] [3]

$$\left(13 + \frac{1}{13}\right)^2 = 171.0059172$$

$$\frac{4}{3} 171 = 228$$

$$\frac{3}{5} 228 + \varphi^3 = \frac{12}{20} 228 + \varphi^3 = 137.036 = \alpha^{-1}$$

Notice that number 12 is the number of vertices of an icosahedron respectively number 20 is the number of faces of this regular polyhedron. In this way, the most important constant of physics, besides the circle constant, is connected with the icosahedron and the golden mean, and also with the helix of life.
Furthermore, the equation [2]

$$\frac{4}{3}(x+x^{-1})^2 = 228$$

can be recast into the depressed quadric polynomial equation

$$x^4 - 169x^2 + 1 = 0$$

with solutions

$$x_{1,2} = \pm\, 12.99977241 \approx \pm\, 13$$

and $$x_{3,4} = \pm\, {x_{1,2}}^{-1} = \pm\, 0.076924423 = \pm\frac{1}{12.99977241} \approx \pm\frac{1}{13}.$$

Another simple approximation connects the fifth power of the golden mean with reciprocal terms of number 13

$$\frac{\varphi^5}{1+\varphi^5} = 0.0827118 \approx \frac{1}{13} + \frac{1}{13^2} = 0.082840235$$

An own related contribution written years ago may supplement this paper [7]. An extended version will be presented soon.

References

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