A Simpler Method of Proving the Irrationality of Various Infinite Series, and Consequential Theorems

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Abstract

Purpose - This paper aims to derive a new method to prove the irrationality of particular cases of infinite series that are less complicated than approaches taken in the past. This could potentially lead to a much more easily teachable method of proving irrationality as well as addressing many open problems.

Design/Methodology/Approach - Using a very simple approach–the limit of the series' partial sums and the behavior of the series as an unsimplified fraction–the proof can be completed using nothing more than the rules of divisibility and modular arithmetic. This can be used to show that the series converges to a value that is impossible to represent with a rational expression. From this theorem, there are also a couple other things that can be derived, like whether or not the partial sums of an infinite series can ever be an integer.

Findings - The results reveal that using this method, a whole class of series can be proven irrational. On top of this, it also results in novel, simpler proofs for older results like the irrationality of e and π, as well as addressing some relevant open problems.

Originality/Value - This method offers a much easier approach to a topic relevant in many domains of math–particularly number theory and analysis–that is simple enough to be taught to high school math students.

Keywords: irrationality; proof; Euler-Mascheroni, Riemann-Zeta; pi; e

1 Introduction

In this paper, a proof for this method will be unveiled. In more detail, it will start off with a fairly well known infinite series that is known to be irrational and from there a step-by-step proof of that series' will be underway. When that is done, a generalized case of this will be explored, completing the proof, as well as some other minor (consequential) theorems that result from this theorem. That being theorems that could be considered less "relevant" but still pertinent to some problems in math. After this, some open problems will be addressed, and discoveries made from this theorem that are not necessarily open questions will be addressed as well.

2 Irrationality of π

It is well known that π is irrational. This is a result that has been proven in many different ways, though the simplest of these proofs, a proof written by Ivan Niven [6], still has to utilize calculus in order to make its point. While Niven's proof is elegant, it still is fairly complicated. Now, a proposition of a new proof:

Consider the Madhava-Leibniz [3] series for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Which of course, in summation notation can be represented like so:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

This proof will be proving the irrationality of $\frac{\pi}{4}$ which will consequently prove the irrationality of π. Consider the following:

$$\sum_{n=0}^{a} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + .. + \frac{1}{2a+1} = \frac{(1\cdot 3\cdot 5\cdot .. \cdot 2a+1) - (1\cdot 5\cdot 7\cdot .. \cdot 2a+1) + (1\cdot 3\cdot 5\cdot 9\cdot .. \cdot 2a+1)..}{(1\cdot 3\cdot 5\cdot 7\cdot ... \cdot 2a+1)} \tag{1}$$

Now, the proof essentially hinges on the nature of how many of the terms in the fraction "cancel out" or how low the denominator and numerator get upon simplification. Observe the cases when a = 3, and then 5.

When a = 3, you get $\frac{76}{105}$. When a = 5 you get $\frac{263}{315}$. You can confirm it for yourself by testing out higher and higher partial sums, but essentially the denominator and numerator both increase as you go higher and higher. Of course heuristic evidence isn't enough to make the case for the proof, so here is the actual logic behind it.

By basic rules of divisibility, if you have $ka + b$ where b is not a multiple of a, then $ka + b$ is not divisible by a. Extending this logic, if you have $k(ab) + k(ac) + k(cb)$ such that they are all relatively prime, then $(a, b, c) \nmid k(ab) + k(ac) + k(cb)$. This is essentially the same as the simpler $ka + b$ situation from above, but with more terms.

The sizes of the simplified integers in the numerator and denominator of the fraction are dependent on how many factors the numerator and denominator share. The ones that they don't share don't cancel out, and result in a bigger denominator or numerator than if they weren't there. Now back to the partial sums.

Take this expression from above for the ath partial sum:

$$\frac{(1\cdot 3\cdot 5\cdot .. \cdot 2a+1) - (1\cdot 5\cdot 7\cdot .. \cdot 2a+1) + (1\cdot 3\cdot 5\cdot 9\cdot .. \cdot 2a+1)..}{(1\cdot 3\cdot 5\cdot 7\cdot ... \cdot 2a+1)} \tag{2}$$

Just by looking at it without doing the arithmetic, and using the $ka + b$ logic, you can deduce that whatever factors appear in every term in the numerator but once are not shared with the denominator. This is because the denominator has factors $1, 3, 5, 7..., 2a + 1$, and if the numerator has any one of those terms everywhere but once, then the numerator must not be a multiple of that term. Still, you can't be certain that the numerator and demoninator always increase until looking deeper.

Consider any ath partial sum where 2a + 1 is a prime. Since it is a prime, there are no numbers below it that are factors of it, and up until the (2a + 1)th partial sum, 2a + 1 will appear everywhere but once in the numerator, guaranteeing that it is a factor unshared between the numerator and denominator. So for any ath partial sum, every prime $p \leq 2a + 1$ such that $2a + 1 < 2p$ is guaranteed to be an unshared factor. The new question then becomes if whether or not these primes compound fast enough between p and $2p$ such that these unshared factors reach infinity. This is a question that is quite easy to answer with the Prime Number Theorem [4], and one that has been addressed in Bertrand's Postulate [1].

$$\lim_{p\to\infty} \pi(2p) - \pi(p) = \infty \tag{3}$$

Computing this limit is fairly straightforward.

$$\lim_{p\to\infty} \pi(2p) - \pi(p) = \infty \tag{4}$$

$$\pi(2p) \approx \frac{2p}{\ln 2 + \ln p} \text{ and } \pi(p) \approx \frac{p}{\ln p} \tag{5}$$

Because this is a limit,

$$\lim_{p\to\infty} \pi(2p) - \pi(p) = \lim_{p\to\infty} \frac{2p}{\ln 2 + \ln p} - \frac{p}{\ln p} \tag{6}$$

After some elementary limit techniques,

$$\lim_{p\to\infty} \frac{2p - p}{\ln p} = \infty \tag{7}$$

Therefore the unshared factors approach infinity, and the numerator and denominator approach infinity.

This logic is justifiable based on the fact that there is no constant modulus that decreases as the series gets bigger. As an example of how important the statement of the compounding prime factors is, take a geometric series:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} \cdots \tag{8}$$

You cannot use the same logic here because the result simplifies in a very convenient way. When expanding the denominator, you are left with this:

$$\sum_{n=0}^{k} \frac{1}{2^n} = \frac{1+2+4+8\cdots}{2^k} = \frac{1}{2^k} + \frac{2+4\cdots}{2^k} \tag{9}$$

Evidently the denominator and numerator will not go to infinity because they are perfectly divisible by each other apart from the one residual $\frac{1}{2^k}$ which will quickly go to 0 and leave you with the remaining fraction, leaving you with the fact that the series simplifies to 2, a rational number. If the numerator is N and the denominator D,

$$N \equiv 1 \pmod{D}. \tag{10}$$

Since for every partial sum, the denominator gets higher, whatever the modulus is, be it 1 or 1,000,000, it will eventually go to 0 if it is constant.

In the case of a series with infinitely many prime denominators, if you look at any term in the denominator of the kth partial sum, the numerator will not constantly have the same modulus relative to the denominator. This can be demonstrated:

$$\sum_{n=0}^{a} \frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+..+\frac{1}{2a+1} = \frac{(1\cdot3\cdot5\cdot..\cdot2a+1)-(1\cdot5\cdot7\cdot..\cdot2a+1)+(1\cdot3\cdot5\cdot7\cdot..\cdot2a-1)..}{(1\cdot3\cdot5\cdot7\cdot...\cdot2a+1)} \tag{11}$$

Let the denominator be denoted as D and the numerator as N. To prove that the modulus changes for at least one term, let's focus on the ath partial sum, and contrast it with the $a+1$th.

$$N \equiv (1\cdot3\cdot5\cdot7\cdot...\cdot2a-1) \pmod{D} \equiv k \pmod{2a+1}, k < 2a+1 \tag{12}$$

If you were to then take the $a+1$th partial sum:

$$\sum_{n=0}^{a} \frac{(-1)^n}{2n+1} = \frac{N}{D}, N \equiv (k2a+3k) \pmod{2a+1}, D \equiv 0 \pmod{2a+1} \tag{13}$$

$$(k2a+3k) = k(2a+1)+2k \equiv (2k) \pmod{2a+1} \not\equiv k \pmod{2a+1} \tag{14}$$

And by induction, this is true for every term and any partial sum you take. It is also true that this likely applies for any series that is non geometric but can be only be explicitly mathematically demonstrated with series consisting of many prime denominator terms. Mathematically, if you were to "churn" this out, you would find that there is no point in which any one of the terms in the numerator or any of their products is divisible by the denominator, when you are just left with the primes. Now, to go back and reference the main argument of this section, that being the irrationality of π:

For a number to be a positive rational number, it needs to equal $\frac{a}{b}$ where $a, b \in \mathbb{N}$.

If it is approaching infinity in the numerator and denominator, that would it imply that it is approaching a ratio of infinities.

$$\infty \notin \mathbb{N} \quad \therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \notin \mathbb{Q} \tag{15}$$

3 Main Theorems

Using the reasoning from above, the following can be stated:

if there are infinitely many primes of the form $f(n), n \in \mathbb{Z}$, and $g(n) \in \mathbb{Z}$ for all n, then

$$\sum_{n=a}^{\infty} \left(\frac{g(n)}{f(n)}\right)^k \notin \mathbb{Q}.$$

$$\sum_{n=a}^{b} \left(\frac{g(n)}{f(n)}\right)^k \notin \mathbb{Z}.$$

4 Open Problems

One of the biggest burning questions is whether or not $\zeta(5)$ is rational. Well consider the definition of the zeta function [4]:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Now let's apply the theorem to this situation. Of course, there are infinitely many primes of the form $f(n) = n$, and in this case $s = 5$, meaning that s is a natural number. By the criteria of the above theorem, $\zeta(5) \notin \mathbb{Q}$.

To take another leap, the above theorem just requires s to be a natural number. This means that not only is $\zeta(5) \notin \mathbb{Q}$ but for any $s \in \mathbb{N}$ $\zeta(s) \notin \mathbb{Q}$.

Another open irrationality problem is the rationality of the Euler-Mascheroni Constant γ.

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right)$$

This takes a bit more effort to convert into the $\sum_{n=a}^{\infty} \left(\frac{g(n)}{f(n)}\right)^k$ format.

First, consider the infinite series representation for $ln(1+n)$:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$\lim_{n\to\infty} \ln(n+1) = \ln(n)$$

So in the context of a limit to infinity,

$$\ln(n) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(n+1)^k$$

Therefore you can rewrite γ like so:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}n^k}{k} \right)$$

Which then becomes

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \frac{(-1)^{k-1}n^k}{k} \right)$$

Fortunately the denominator is straightforward. Just k. Of course, there are infinitely many primes of the form $f(k) = k$.

Now to try to apply this theorem to it, let's look at a slightly altered series:

$$\left(\sum_{k=1}^{\infty} \frac{1}{k} - \frac{(-1)^{k-1}n^k}{k} \right)$$

Criteria 1: there are infinitely many primes of the form k. This condition is satisfied.

Criteria 2: $1 - (-1)^{k-1}n^k \in \mathbb{Z}$ for all n. This is also satisfied.

$$\therefore \gamma \notin \mathbb{Q}$$

5 Other Observations

This theorem has a lot of other implications that are not necessarily open problems. Below is a list of some of the more interesting ones:

$$\sum_{n=1}^{a} \frac{1}{n} \notin \mathbb{Z}$$

$$\sum_{n=1}^{a} \frac{1}{T_n^k} \notin \mathbb{Z}, \quad k \in \mathbb{Z} \tag{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{T_n^k} \notin \mathbb{Q}, \quad k \in \mathbb{Z} \tag{17}$$

(The irrationality of the twin prime sum only applies if the Twin Prime Conjecture [5] is true.)

$$\sum_{n=1}^{a} \frac{1}{(cn+b)^k} \quad \notin \mathbb{Z} \quad \text{where } c, b, k \in \mathbb{Z} \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{1}{(cn+b)^k} \quad \notin \mathbb{Q} \quad \text{where } c, b, k \in \mathbb{Z} \tag{19}$$

The above statements are true because of Dirichlet's work on prime arithmetic progressions [2].

$$\sum_{p \in \mathbb{P}} \frac{1}{p^s} \notin \mathbb{Q}, s \in \mathbb{N} \tag{20}$$

The above prime series is the prime zeta function, the zeta function excluding the composites. It is sometimes notated as P(s).

6 Conclusion

This theorem is a much simpler method to accomplish a task that has been historically daunting in the world of mathematics. Not only is it generalized but it is also simple enough to teach in schools and places where people may not have a college level math education, as well as the fact that it can be used on open problems that cannot be practically solved with previously available methods.

References

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$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

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