ON THE NICOLAS CRITERION FOR RIEMANN HYPOTHESIS

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Abstract. A criterion given by Jean-Louis Nicolas is used to offer a proof for the Riemann Hypothesis in a straightforward way.

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There is a vivid interest in the Riemann Hypothesis proposed by Bernhard Riemann in 1859. While there are no reasons to doubt the validity of the Riemann Hypothesis [1], many colleagues consider it the most important unsolved problem in pure mathematics [2]. The Riemann Hypothesis is of great interest in number theory because it implies results about the distribution of prime numbers. In this short note, we offer a proof of the Riemann hypothesis via the Nicolas criterion.

Nicolas has shown [3] that if

$$G(k) = \frac{N_k}{\varphi(N_k)} - \mathrm{e}^{\gamma} \ln\ln N_k > 0, \tag{1}$$

the Riemann Hypothesis is true. The primordial of order k is given by

$$N_k = \prod_{i=1}^{k} p_i, \tag{2}$$

$\gamma \approx 0.577216$ is the Euler–Mascheroni constant, and $\varphi(N)$ is Euler's totient function, i.e., the number of integers less than N that are coprime to N. Euler's product formula for the totient formula reads

$$\varphi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right), \tag{3}$$

where $p|N$ are the primes p that divide the integer N.

If N is the primordial of order k, one has

$$\varphi(N_k) = N_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \tag{4}$$

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Dividing by the product and using the Taylor series, one obtains

$$G_0(k)=\frac{N_k}{\varphi(N_k)}=\prod_{i=1}^k\frac{p_i}{p_i-1}=\prod_{i=1}^k\sum_{\alpha=0}^\infty\frac{1}{p_i^\alpha}. \tag{5}$$

Since the Taylor series for the function $f(x)=1/(1-x)=1+x+O(x^2)$ is convergent for $x=1/p_i<1$, this Taylor series development is valid for any prime p_i. On the other hand, one has

$$\prod_{i=1}^k\sum_{\alpha=0}^h\frac{1}{p_i^\alpha}=\prod_{i=1}^k\frac{1+p_i+\ldots+p_i^h}{p_i^h}=\frac{\sigma((N_k)^h)}{(N_k)^h}, \tag{6}$$

where $\sigma(N)$ is sum of divisors function. Written in the form

$$G(k)=\lim_{h\to\infty}\frac{\sigma((N_k)^h)}{(N_k)^h}-\mathrm{e}^\gamma\ln\ln N_k>0, \tag{7}$$

the Nicolas criterion is closely related to Robin's theorem [4], stating that if

$$\frac{\sigma(N)}{N}-\mathrm{e}^\gamma\ln\ln N<0 \tag{8}$$

for $N>5040$, the Riemann hypothesis is valid (cf. also Ref. [5]). While Robin's theorem provides a lower limit for a general integer N, found both in the first and the second term, for the Nicolas criterion the primordial number in the first term is taken to an infinite power. Intuition suggests that this first term $G_0(k)$ is large, certainly larger than the second term in $G(k)$, and $G(k)>0$. This will be confirmed in the following.

We use finite difference methods [6] for the first term to calculate

$$\begin{aligned}\Delta G_0(k) &= G_0(k)-G_0(k-1)\\ &= \prod_{i=1}^k\frac{p_i}{p_i-1}-\prod_{i=1}^{k-1}\frac{p_i}{p_i-1} = \frac{G_0(k)}{p_k}\end{aligned} \tag{9}$$

and $\Delta N_k=N_k(p_k-1)/p_k$. Applied to the second term, one has

$$\Delta\mathrm{e}^\gamma\ln\ln N_k=\frac{\mathrm{e}^\gamma\Delta\ln N_k}{\ln N_k}=\frac{\mathrm{e}^\gamma\Delta N_k}{N_k\ln N_k}=\frac{\mathrm{e}^\gamma(p_k-1)}{p_k\ln N_k}. \tag{10}$$

Because of

$$\Delta G(k)=\frac{G_0(k)}{p_k}-\frac{\mathrm{e}^\gamma(p_k-1)}{p_k\ln N_k}, \tag{11}$$

a local extremum exists if

$$G_0(k) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \to \frac{\mathrm{e}^{\gamma}\,(p_k - 1)}{p_k} \to \mathrm{e}^{\gamma} \approx 1.781 < 2, \tag{12}$$

where we used $N_k/\exp(p_k) = 1$ for $p_k \to \infty$ [7]. However,

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \gg 2. \tag{13}$$

Therefore, there are no local extrema for $G(k)$, which means that this function has its maximum at $k = 1$. Indeed, one obtains the first values $G(1) = 2.653$, $G(2) = 1.961$, $G(3) = 1.5697$, $G(4) = 1.3889$. Therefore, as $G(k) > 0$ for $k \to \infty$, the Nicolas criterion is satisfied and the Riemann Hypothesis is true.

The final part of the proof is simple. In Ref. [3], Nicolas has found that if the criterion fails, $G(k)$ has both infinitely many positive and negative values. However, as we have shown that $G(k)$ does not have local extrema, such a possibility is excluded. This means that the Riemann Hypothesis is true because Nicolas's criterion does not fail.

References

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