Fast Convergence

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ABSTRACT: We give a sequence that converges to π quickly.

I. Introduction: Rate of Convergence

Definition. If a sequence $x_1, x_2, \ldots, x_n$ converges to a value s and if there exist real numbers $\lambda > 0$ and $\alpha \geq 1$ such that

$$\lim_{n \longrightarrow \infty} \frac{|x_{n+1} - s|}{|x_n - s|^{\alpha}} = \lambda \tag{1}$$

then we say that α is the rate of convergence of the sequence.

When $\alpha = 1$ we say the sequence converges linearly and when $\alpha = 2$ we say the sequence converges quadratically. If $1 < \alpha < 2$ then the sequence exhibits superlinear convergence.

Remark: $\alpha = 3 \Longrightarrow$ cubic convergence, $\alpha = 4 \Longrightarrow$ quartic convergence, and so on...

II. Fast Sequence for Pi

Define the sequence x_n as

$$x_{n+1} = f(x_n) \quad , \; x_1 = 3 \, , \; n = 1, 2, 3, \; \ldots \tag{2}$$

where

$$f(x) = x + \sin(x) + 4\left(1 - \frac{2}{1 + \cot\left(\frac{x+\sin(x)}{4}\right)}\right) \tag{3}$$

we have

$$f(\pi) = \pi \tag{4}$$

$$x_n \longrightarrow \pi \; , \; n \longrightarrow \infty \tag{5}$$

$$f^{(1)}(\pi) = f^{(2)}(\pi) =. .. = f^{(8)}(\pi) = 0 \tag{6}$$

$$f^{(9)}(\pi) = -35 \tag{7}$$

$$|x_{n+1} - \pi| \sim \frac{35}{9!} \, |x_n - \pi|^9 \tag{8}$$

$$|x_{n+1} - \pi| \sim \left(\frac{1}{10\,368}\right)^{\frac{3^{2n}-1}{8}} (\pi - 3)^{3^{2n}} \tag{9}$$

Remark 1: $f^{(k)}(x)$ is the kth derivative of $f(x)$.

Remark 2: The number Pi is defined by

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right) \tag{10}$$

III. Numerical Tests

$$x_{n+1} = f(x_n) \quad , \quad x_1 = 3 \ , \quad n = 1, 2, 3, \ \dots \tag{11}$$

n	$\lvert x_n - \pi \rvert$
1	$1.4159 \dots \times 10^{-1}$
2	$2.1997 \dots \times 10^{-12}$
3	$1.1630 \dots \times 10^{-109}$
4	$3.7553 \dots \times 10^{-985}$

IV. Endnote

CONTRACTIONS: $F : D \longrightarrow \mathbb{R}$ is a contraction if $\exists\ \lambda \in (0,\ 1)$ s.t. $\lvert F(x) - F(y) \rvert \leq \lambda\ \lvert x - y \rvert\ \forall\ x,\ y \in D$.

CONTRACTIVE MAPPING THEOREM: Let $U \subseteq \mathbb{R}$ be closed. If $F : U \longrightarrow U$ is a contraction, then $x_{n+1} = F(x_n)$ has a unique fixed point $x^* \in U$.

BROUWER'S FIXED POINT THEOREM: Let D be a nonempty, compact, convex subset of Euclidean space $\mathbb{R}^n$, and let $f : D \longrightarrow D$ be a continuous function. Then, f has at least one fixed point in D.

SCHAUDER FIXED POINT THEOREM: Let K be a nonempty, compact, convex subset of a Banach space X, and let $T : K \longrightarrow K$ be a continuous mapping. Then, T has a fixed point in K .

V. References

1. Agarwal, P., Jleli, M., Samet, B.: Banach Contraction Principie and Applications, Fixed Point Theory in Metric Spaces. Singapore: Springer, 2018.
2. Amdeberhan, T., Medina, L. A., Moll, V. H.(Editors): Gems in Experimental Mathematics. AMS Special Session Experimental Mathematics, Washington, DC, 2009.
3. Borwein, J., Bailey, D.: Mathematics by experiment, A K Peters Ltd., Natick, MA, 2004.
4. Borwein, J., Bailey, D., Girgensohn, R.: Experimentation in mathematics: Computational paths to discovery, A K Peters Ltd., Natick, MA, 2004.
5. Kincaid, D., Cheney, W.: Numerical Analysis: Mathematics of Scientific Computing. Brook Cole, Pacific Grove, CA, 3rd Edition, 2002.
6. Kirk, W. A., Sims, B.: Handbook of Metric Fixed Point Theory, Springer, 2001.
7. Kress, R.: Numerical Analysis. Springer-Verlag, New York, NY, 1998.
8. Lefschetz, S.: On the fixed point formula, Annals of Mathematics, 38(4), 1937.