REPRESENTATION OF AN INTEGRAL INVOLVING TRIGONOMETRIC FUNCTIONS BY TRIPLE INTEGRAL

EDIGLES GUEDES

Abstract. In this paper, we present an integral representation involving trigonometric functions and variable transformation techniques to turn it into a triple integral. The proposed integral is initially simplified using trigonometric identities, so we rewrite the original integral in terms of a three-variable integral representation. The main theorem demonstrates the equivalence between the initial integral and the resulting triple integral, illustrating the applicability of trigonometric identities in calculations of complicated integrals.

"But he answered and said, It is written, Man shall not live by bread alone, but by every word that proceedeth out of the mouth of God." Matthew 4:4 (KJV)

1. Introduction

Integral representations are essential tools in mathematical analysis and calculus (S. Woods [1934]). They serve to simplify extremely complicated expressions and solve difficult integrals. In this paper, we explore an elegant integral representation for the following integral

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\sin^2 x+\cos x}\mathrm{d}x.$$

This is rewritten as a triple integral involving polynomials. The use of trigonometric identities and changes of variables plays a crucial role in this process, allowing us to express the original integral in an alternative and beautiful form. This approach not only makes it easier to calculate the integral if necessary, but also illustrates the power of integral transformations.

2. A beautiful integral representation

Theorem 1. *The integral representation is valid*

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\sin^2 x+\cos x}\mathrm{d}x = 4\int_0^1\int_0^1\int_0^1 \frac{\left(1+z^2\right)z^2}{(1+3z^2)(1+x^2z^2)(1+y^2z^2)}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2.1}$$

Proof. Note that the left-hand side of (2.1) can be rewritten as follows

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\sin^2 x+\cos x}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{x^2}{-2\cos^2\left(\frac{x}{2}\right)(\cos x-2)}\mathrm{d}x \\ &= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\sec^2\left(\frac{x}{2}\right)}{\cos x-2}\mathrm{d}x.\end{aligned} \tag{2.2}$$

We know that the following trigonometric identity (Yeo [2007]) converts the cosine function into tangent functions

$$\cos(2y) = \frac{1-\tan^2 y}{1+\tan^2 y}. \tag{2.3}$$

Replace y by $x/2$ in (2.3) and find

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$$\cos(x) = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}. \tag{2.4}$$

Substitute the right-hand side of (2.4) into the right-hand side of (2.2)

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\sin^2 x+\cos x}\mathrm{d}x &= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\sec^2\left(\frac{x}{2}\right)}{\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}-2}\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\left[1+\tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)-2\left[1+\tan^2\left(\frac{x}{2}\right)\right]}\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\left[1+\tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)-2-2\tan^2\left(\frac{x}{2}\right)}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{x^2\left[1+\tan^2\left(\frac{x}{2}\right)\right]}{1+3\tan^2\left(\frac{x}{2}\right)}\cdot\frac{\sec^2\left(\frac{x}{2}\right)}{2}\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\sec^4\left(\frac{x}{2}\right)}{1+3\tan^2\left(\frac{x}{2}\right)}\mathrm{d}x.
\end{aligned} \tag{2.5}$$

Make $\tan(x/2)=u \Rightarrow \mathrm{d}u=\left[\sec^2(x/2)/2\right]\mathrm{d}x$. Therefore, $x=2\arctan u$. The limits of integration are: $x=\pi/2 \Rightarrow u=1$ e $x=0 \Rightarrow u=0$. Apply this change of variable to (2.5)

$$\begin{aligned}
&\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x^2\sec^4\left(\frac{x}{2}\right)}{1+3\tan^2\left(\frac{x}{2}\right)}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{x^2\left[1+\tan^2\left(\frac{x}{2}\right)\right]}{1+3\tan^2\left(\frac{x}{2}\right)}\cdot\frac{\sec^2\left(\frac{x}{2}\right)}{2}\mathrm{d}x \\
&= \int_0^1 \frac{(2\arctan u)^2\left(1+u^2\right)}{1+3u^2}\mathrm{d}u \\
&= 4\int_0^1 \frac{\left(1+u^2\right)\arctan^2 u}{1+3u^2}\mathrm{d}u \\
&= 4\int_0^1 \frac{\left(1+u^2\right)u^2}{1+3u^2}\left(\int_0^1 \frac{1}{1+u^2t^2}\mathrm{d}t\right)^2\mathrm{d}u \\
&= 4\int_0^1 \frac{\left(1+u^2\right)u^2}{1+3u^2}\int_0^1 \frac{1}{1+u^2t^2}\mathrm{d}t\int_0^1 \frac{1}{1+u^2v^2}\mathrm{d}v\mathrm{d}u \\
&= 4\int_0^1 \frac{\left(1+u^2\right)u^2}{1+3u^2}\int_0^1\int_0^1 \frac{1}{(1+u^2t^2)(1+u^2v^2)}\mathrm{d}t\mathrm{d}v\mathrm{d}u \\
&= 4\int_0^1\int_0^1\int_0^1 \frac{\left(1+u^2\right)u^2}{(1+3u^2)(1+u^2t^2)(1+u^2v^2)}\mathrm{d}t\mathrm{d}v\mathrm{d}u \\
&= 4\int_0^1\int_0^1\int_0^1 \frac{\left(1+z^2\right)z^2}{(1+3z^2)(1+x^2z^2)(1+y^2z^2)}\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we conclude that

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{1+\sin^2 x+\cos x}\mathrm{d}x = 4\int_0^1\int_0^1\int_0^1 \frac{\left(1+z^2\right)z^2}{(1+3z^2)(1+x^2z^2)(1+y^2z^2)}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

This is the desired result. □

Exercise 2. Prove the following integral representations

$$\int_0^1\int_0^1\int_0^1 \frac{1+z^2}{(1+x^2)(1+y^2)}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\pi^2}{12}$$

and

$$\int_0^1 \int_0^1 \int_0^1 \frac{z^2}{(1+x^2)(1+y^2)} \mathrm{d}x \mathrm{d}y \mathrm{d}z = \frac{\pi^2}{48}.$$

3. Conclusion

We have shown that the integral involving trigonometric functions can be represented equivalently as a triple integral involving only rational polynomials. This transformation highlights the usefulness of substitution techniques and trigonometric identities for rewriting very intricate integrals. The result obtained exemplifies how analytical approaches can be employed to solve integration problems that, at first glance, would seem intractable. Therefore, this work contributes to the understanding of integral representations and opens doors for future investigations into integrals in a similar way.

References

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