The breather-breather interaction in the Sine-Gordon model

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Abstract. In this paper, we investigate the interaction of two breathers in the Sine-Gordon model. We derive an explicit analytic expression for the two-breather solution of the Sine-Gordon equation and study its dynamics. We show that the breathers behave like classical particles of equal masses upon collision, but with the momentum continuously transferred via their fields. By suitably averaging the oscillations of the solution we derive analytic expressions for the trajectories of the two breathers. It is shown that in the non-relativistic limit, the interaction potential between the two breathers has the same form as the velocity-dependent interaction potentials used for Machian unified theories of gravity and inerita.

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1. Introduction

A breather soliton model of elementary particles [1] is suggested by Machian unified theories of gravity and inertia[2-9]. In such a theory, particles are breather-solitons in the gravitational field, or more generally, a unified field of which the gravitational one is a part. The particles are then part of the field itself, namely the breather soliton. A universe consisting of N particles is described by an N-breather solution to the underlying non-linear field equation, which in the simplest possible form reads

$$\Box\varphi + V(\varphi) = 0. \tag{1.1}$$

This N-breather solution then includes the N particles, the fields generated by them as well as the resulting motion due to the fields. For such a breather soliton model of elementary particles, it is necessary to show that all known properties of the elementary particles are exhibited by the solitons. Especially, the correct behaviour under collisions and the fact that indeed the mutual interaction due to the soliton's fields are included in the multi-soliton solutions, has to be demonstrated. Although the correct three-dimensional soliton equations remain to be found, it is desirable to study the collision and mutual interaction of breather solitons already in toy-models, such as the 1 dimensional Sine-Gordon model. It is based on the Sine-Gordon equation

$$\Box\varphi + \frac{1}{d^2}\sin(\varphi) = 0. \tag{1.2}$$

This equation possesses breather and N-breather solutions [10], with the well-known one-breather solution given by

$$\varphi(x,t) = 4\arctan\left(\cot(q)\frac{\cos\left(\frac{\gamma\sin(q)}{d}(ct - x\beta)\right)}{\cosh\left(\frac{\gamma\cos(q)}{d}(x - vt)\right)}\right). \tag{1.3}$$

Here, v is the breather's velocity and q is a real parameter. d is proportional to the 1/e radius and thus the size of the breather. $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ are defined in the usual way. In the following, we want to study the interaction between two such breathers. This problem has so far only been studied for resting breathers of constant relative phase, using approximate solutions for large separations between the breathers [11,12]. Those results are not applicable to the case of two breathers moving with different velocities, since they necessarily oscillate at different frequencies due to the relation

$$\omega = \frac{\sin(q)\gamma}{d}c \tag{1.4}$$

between the breathers velocity and the oscillation frequency. This implies that their relative phases wont stay constant during their motion. Further, the assumption of a large separation between them is obviously invalid when a collision occurs. Instead, we use exact two-breather solution of the Sine-Gordon equation, which is derived in the appendix. Using the asymptotic expansion of this solution we will show that the breathers behave like classical particles of equal masses under collision. We will determine the trajectory of the breathers by averaging out the oscillations and calculate the maxima of the two breather solution φ , as is described in the next two sections.

Since a breather soliton nature of the elementary particles is suggested by Machian unified theories of gravity and inertia [2-9], it is particularly interesting to analyze the form of the interaction potential between the two breathers in the classical, non-relativistic limit. Those Machian theories are classical and non-relativistic, including up to second order terms in β. They are built on velocity dependent gravitational potentials which only contain relative quantities, like the Weber-potential [2-6]

$$V_{Weber} = -\frac{Gm_1m_2}{r_{12}}(1 - \frac{\dot{r}_{12}^2}{2c^2}), \tag{1.5}$$

or the Riemann-potential [8,9]

$$V_{Riemann} = -\frac{Gm_1m_2}{r_{12}}(1 - \frac{\mathbf{v}_{12}^2}{2c^2}). \tag{1.6}$$

Here, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, G is the gravitational constant and c the speed of light. The velocity dependent part of the potentials then gives rise to the phenomenon of inertia, instead of the usual Newtonian kinetic energy. It is desirable to show that breather-breather interaction potentials in the classical, non-relativistic limit (the lowest non-vanishing order in β) reduce to the form exhibited by (1.5-1.6), which is

$$V = f(r_{12})(1 + \alpha\beta_{12}^2), \tag{1.7}$$

with α some dimensionless parameter of order of unity. This, we want to show in the fourth section of our paper for the interaction of two Sine-Gordon breathers.

2. The averaging procedure

The one breather solution of the Sine-Gordon equation is given by (A.4)

$$\varphi(x,t) = 4\arctan\left(\cot(q)\frac{\cos(b)}{\cosh(a)}\right) \tag{2.1}$$

with

$$a = \frac{\gamma\cos(q)}{d}(x - vt - x_0)$$
$$b = \frac{\gamma\sin(q)}{d}(ct - x\beta) + \delta_0$$

The trajectory of a breather (or particle) can be defined as the line on which the successive maxima and minima of the oscillations lie.

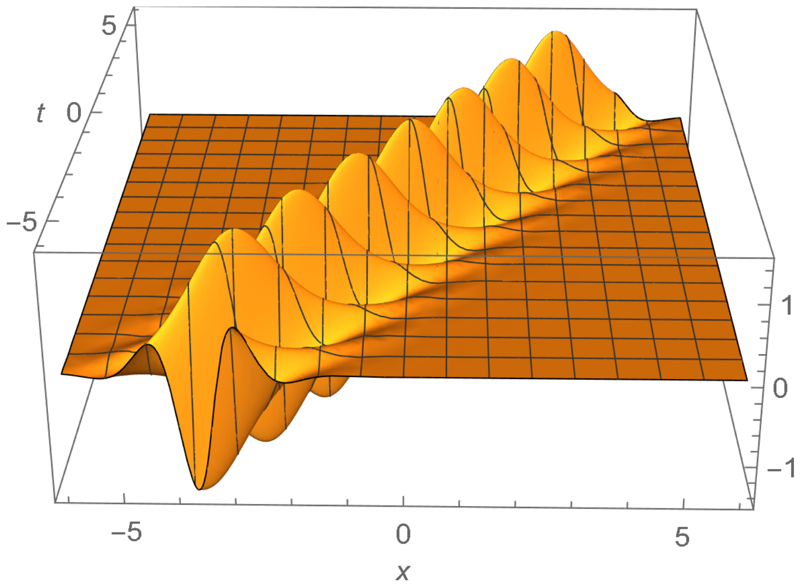


FIGURE 1. The one-breather solution (2.1) for the parameters $\beta = 0.6$, $q = 1.2$, $d = 0.2$. The line on which the successive maxima and minima lie can be defined as the trajectory of the breather

This line can be found by averaging out the oscillations, setting the oscillatory function equal to a constant

$$\cos(b) = \xi = const.$$

and then calculate the maxima via

$$\frac{\partial\varphi}{\partial x} = 0 \tag{2.2}$$

This yields the condition

$$x = vt + x_0 \tag{2.3}$$

which is the free breather's trajectory. Alternatively, if one would've have determined the maxima of (2.1) directly without setting the oscillatory function to a constant, one would've obtained the equation

$$\tan(q)\beta\tan(b) = \tanh(a) \tag{2.4}$$

This transcendent equation gives the solutions of the discrete maxima and minima of the breathers oscillations. The line, on which they lie is again (2.3). Indeed, if we plug this in (2.4), the right side vanishes leaving us with an additional condition which yields the discrete maxima and minima of the oscillations, given by

$$x_n = \frac{d\beta}{\sin(q)\sqrt{1-\beta^2}}(\pi n - \delta_0) + \frac{x_0}{1-\beta^2}$$

If we are interested only in the trajectory (the line, on which these extrema lie), it is sufficient to calculate (2.2) with the oscillatory functions set to a constant.

3. The trajectories of the two-breather solution

This procedure we now want to apply to the two-breather solution, which is given by

$$\varphi = 4\arctan\left(\frac{f_i}{f_r}\right)$$

with f_i and f_r given by (A.7, A.8) (see appendix A for a derivation). Since two breathers of different $q_1 \neq q_2$ still behave like particles of the same masses under collisions, we restrict ourselves to the case $q_1 = q_2 =: q$ here. For simplicity, we also set the initial positions of the two breathers $(x_0)_1 = (x_0)_2 = 0$; keeping a non-zero value for them would only change the specific position of the collision, which doesn't impact our discussion. Because we're only interested in the trajectories and not the discrete positions of the single maxima, we again set the oscillatory functions to a constant. Here, we will distinguish two cases: The case where both breathers move in phase close to their collision point, and the one where they move out of phase close to the collision point. Over the whole motion, one cannot specify a specific relative phase, since, as was already pointed out, both breathers oscillate with different frequencies, according to (1.4). Since the relative phase that impacts the interaction most will be the one where the breathers are closest, we will distinguish both cases by the relative phase at this point. An example of both an in-phase (Fig. 2) and out-of-phase (Fig. 3) collision of two breathers is plotted below. In both cases, one can see how the two breathers move in-phase and out of phase respectively close to the collision point. The further one goes away from that point, the more their relative phases change due to their different oscillation frequencies.

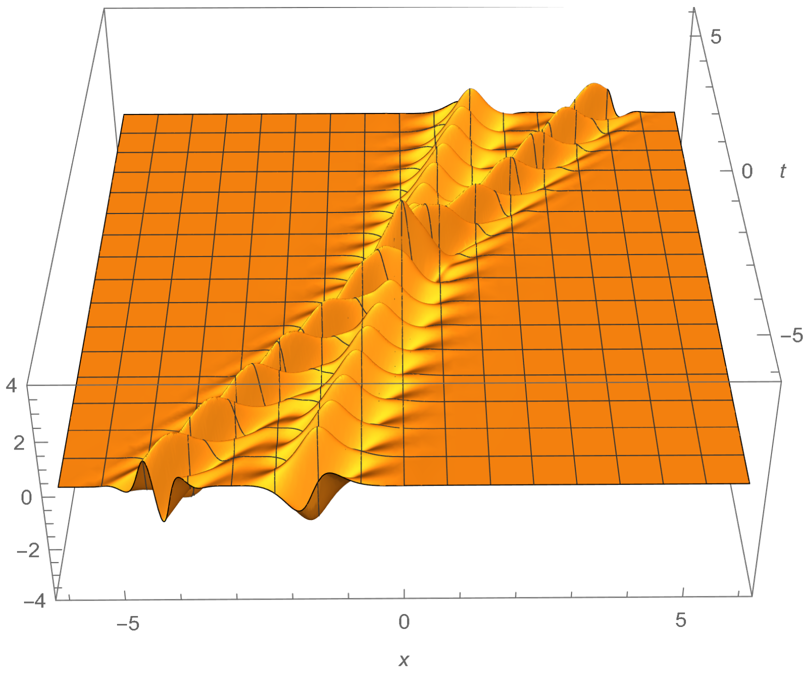


FIGURE 2. The two-breather solution with both breathers being in phase at the collision point. It is plotted for the parameters $\beta_1 = 0.7$, $\beta_2 = 0.3$, $d = 0.1$ and $q = 1.2$.

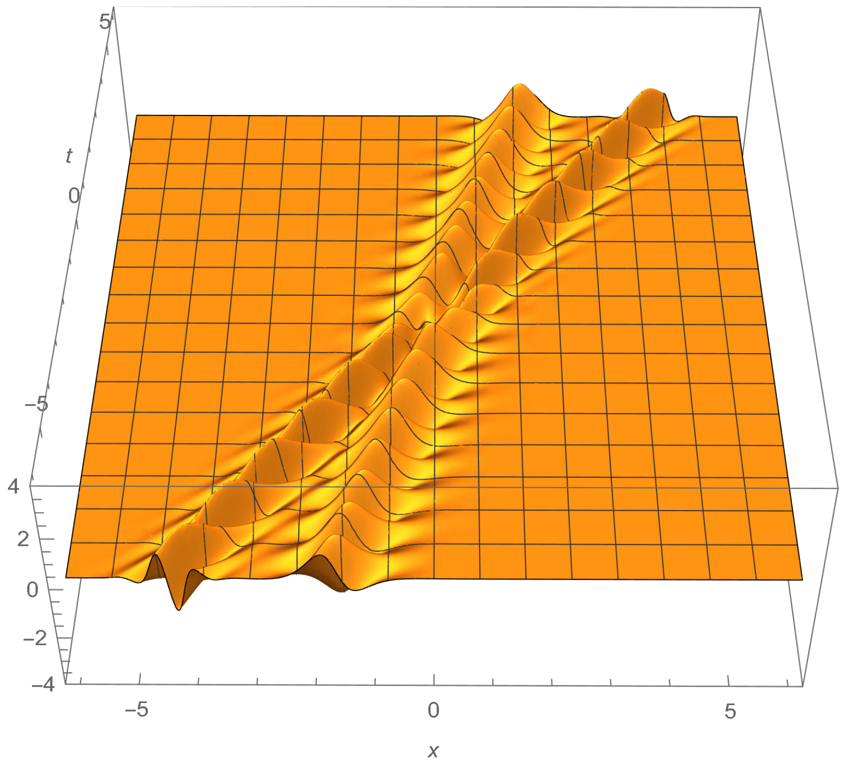


FIGURE 3. The two-breather solution with both breathers being out of phase at the collision point. It is plotted for the parameters $\beta_1 = 0.7$, $\beta_2 = 0.3$, $d = 0.1$ and $q = 1.2$.

Now, both breathers will collide close to the origin, there we have

$$b_1 \approx \delta_1, \qquad b_2 \approx \delta_2$$

For in-phase oscillation at this point, we have $\delta_2 = \delta_1$ and thus we set

$$\cos(b_1) = \cos(b_2) = \xi \tag{3.1}$$

For out-of-phase oscillation we have $\delta_2 = \delta_1 + \pi$ and thus we set

$$\cos(b_1) = -\cos(b_2) = \xi \tag{3.2}$$

3.1. **Out-of-phase oscillation.** We first deal with the case of out-of-phase oscillation. If we plug (3.2) into (A.7,A.8) and use the half-angle formulas for the hyperbolic functions, we obtain the expressions

$$f_i = \sinh\left(\frac{a_1+a_2}{2}\right)(2u_1 w_2 C^2 \xi \sinh\left(\frac{a_1-a_2}{2}\right) -$$

$$8(u_1 w_2)^2 C\sqrt{1-\xi^2}\cosh\left(\frac{a_1-a_2}{2}\right)) \tag{3.3}$$

$$f_r = -E - F\cosh(a_1+a_2) - G\cosh(a_1-a_2) \tag{3.4}$$

with

$$E = 2\xi^2 u_1 u_2 (D^2 - B) + 8(u_1 w_2)^2 D(1-\xi^2) \tag{3.5}$$

$$F = w_1 w_2 (D^2 + B) - 4(u_1 w_2)^2 D \tag{3.6}$$

$$G = w_1 w_2 (D^2 + B) + 4(u_1 w_2)^2 D \tag{3.7}$$

$$a_k = \frac{\gamma_k \cos(q)}{d}(x - v_k t), \quad k = 1, 2$$

Now, we can transform into the center of mass frame moving with

$$v = \frac{\gamma_1 v_1 + \gamma_2 v_2}{\gamma_1 + \gamma_2}$$

In this frame, we have

$$a_1 + a_2 = ax \tag{3.8}$$

$$a_1 - a_2 = bt \tag{3.9}$$

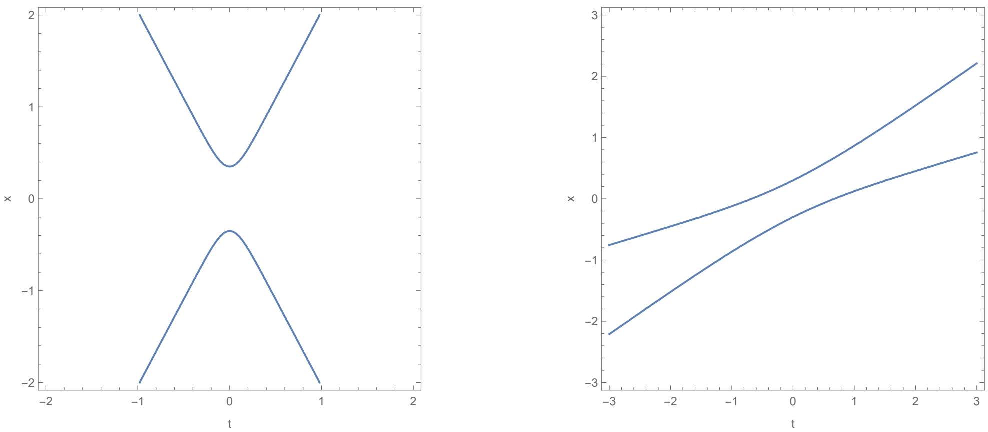
with

$$a = \frac{\gamma_1 + \gamma_2}{\gamma}\frac{\cos(q)}{d}$$

$$b = \frac{2\gamma_1\gamma_2\gamma}{\gamma_1 - \gamma_2}\frac{\cos(q)}{d}v_{12}$$

and $v_{12} = v_1 - v_2$, $\gamma = 1/\sqrt{1-\beta^2}$ We can thus write

$$\frac{f_i}{f_r} = \frac{\sinh\left(\frac{ax}{2}\right)g(t)}{E + F\cosh(ax) + G\cosh(bt)} \tag{3.10}$$



(A) In the center of mass frame

(B) In the lab frame

FIGURE 4. The trajectory of the two breathers for the out-of-phase case. It is plotted for the parameters $\beta_1 = 0.7$, $\beta_2 = 0.3$, $d = 0.1$ and $q = 1.2$.

$$g(t) = -2u_1 w_2 C^2 \xi \sinh\left(\frac{bt}{2}\right) + 8(u_1 w_2)^2 C\sqrt{1-\xi^2}\cosh\left(\frac{bt}{2}\right)$$

We can now calculate again (2.2). Taking into account that

$$\frac{\partial \varphi}{\partial x} = 0 \leftrightarrow \frac{\partial f_i/f_r}{\partial x} = 0$$

since the arctan is a strictly monotonic function, and f_i/f_r given by (3.10), we obtain[1])

$$x_\pm(t) = \pm\frac{1}{a}\cosh^{-1}\left(\frac{G}{F}\cosh(bt) + k\right) \tag{3.11}$$

$$k = 2 + \frac{E}{F} \tag{3.12}$$

Eq. (3.11) is the trajectory of our two breathers in the center of mass frame. The solution is plotted for the same values as in Fig. (3).

The asymptotics can be calculated using the formula $\cosh^{-1}(x) \approx \ln(2x)$ as $|x| \to \infty$ giving

$$x_+(t) \approx \frac{b}{a}|t| + \frac{1}{a}\ln\left(\frac{G}{F}\right) \tag{3.13}$$

$$x_-(t) \approx -\frac{b}{a}|t| - \frac{1}{a}\ln\left(\frac{G}{F}\right) \tag{3.14}$$

One can see that, like for classical particles the breathers move towards each other with opposite velocities of equal magnitude

$$v_* = \frac{b}{a} = \frac{2\gamma_1\gamma_2\gamma^2}{\gamma_1^2 - \gamma_2^2} v_{12}$$

[1]A third solution is x=0, which corresponds to the minimum between both breathers along the trajectory of the center of mass. This solution we don't need here.

before the collision, swap their velocities upon collision, and move away from each other afterwards, again with opposite velocities of the same, equal magnitude. Unlike classical particles, however, the collision is not a discrete process in time, but a continuous transfer of momentum via the fields of the particles (the breathers), as can be seen by (3.11). The fields generated by the particles transfer the momentum from the faster particle to the slower one until they have swapped their velocities.

Multiplying (3.11) by a and transforming back into the lab frame with the help of the equations (3.8-3.9) leads to an implicit equation for x given by

$$a_1 + a_2 = \pm \cosh^{-1}(\frac{G}{F}\cosh(a_1 - a_2) + k) \tag{3.15}$$

This equation is not solvable analytically for x in general. It is plotted numerically, again for the same values as in Fig. (3). The asymptotics can again be calculated analytically using $\cosh^{-1}(x) \approx \ln(2x)$, giving

$$x_+(t) \approx \begin{Bmatrix} v_2 t + \frac{d}{2\cos(q)\gamma_2}\ln\left(\frac{G}{F}\right) & t \to -\infty \\ v_1 t + \frac{d}{2\cos(q)\gamma_1}\ln\left(\frac{G}{F}\right) & t \to +\infty \end{Bmatrix} \tag{3.16}$$

$$x_-(t) \approx \begin{Bmatrix} v_1 t - \frac{d}{2\cos(q)\gamma_1}\ln\left(\frac{G}{F}\right) & t \to -\infty \\ v_2 t - \frac{d}{2\cos(q)\gamma_2}\ln\left(\frac{G}{F}\right) & t \to +\infty \end{Bmatrix}. \tag{3.17}$$

Those expressions could've also been obtained by directly transforming (3.13-3.14) back to the lab frame. They agree with those obtained directly from the asymptotics of the exact two-breather solution derived in appendix B (A.14-A.15). Indeed, using the definitions (3.5-3.7) and the identity

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1}(x)$$

we can write the phase shifts as

$$\tanh^{-1}(\frac{4u_1u_2D}{D^2+B}) = \frac{1}{2}\ln\left(\frac{G}{F}\right).$$

which agrees with (3.16-3.17).

3.2. **In-phase oscillation.** The derivation for this case is analogous to the out-of-phase case. Plugging (3.1) into (A.7-A.8) yields

$$\frac{f_i}{f_r} = \frac{\cosh\left(\frac{ax}{2}\right)\tilde{g}(t)}{-E + F\cosh(ax) + G\cosh(bt)} \tag{3.18}$$

$$\tilde{g}(t) = 2u_1w_2C^2\xi\cosh\left(\frac{bt}{2}\right) - 8(u_1w_2)^2C\sqrt{1-\xi^2}\sinh\left(\frac{bt}{2}\right)$$

in the center of mass frame and from this one obtains for the trajectory

$$x_\pm(t) = \pm\frac{1}{a}\cosh^{-1}(\frac{G}{F}\cosh(bt) - k) \tag{3.19}$$

These solutions agree with the ones for the out-of-phase case, just with $k \to -k$. Therefore, also the asymptotics agree with the ones obtained for the out-of-phase case. However, in this case, we also need to consider the solution with x=0. This is because at some time t, the right side of (3.19) will vanish and both solutions degenerate. At this point, both breathers merge and propagate as one single breather along the trajectory of their common center of mass until at some time t, they reemerge and propagate as two single breathers again. This behavior can be seen in fig. 2 near the collision point: There no longer exist two separate maxima/minima of the two breathers, but only one combined maximum/minimum due to constructive interference between them. The points, at which the two breathers merge and reemerge respectively can be found from the condition

$$\cosh^{-1}(\frac{G}{F}\cosh(bt) - k) = 0$$

which, upon solving for t is equivalent to

$$t = \pm\frac{1}{b}\cosh^{-1}(\frac{E+3F}{G})$$

The "lifetime" of the composite breather is thus

$$\tau = \frac{2}{b}\cosh^{-1}(\frac{E+3F}{G})$$

In the lab frame, the corresponding expressions read

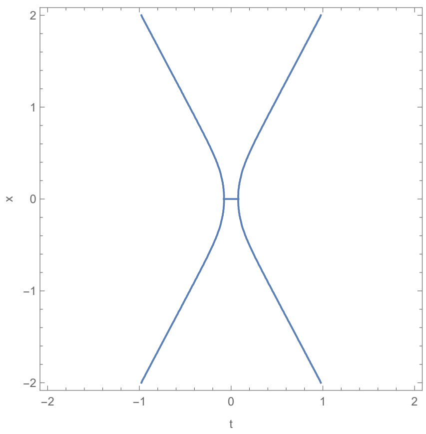
$$a_1 + a_2 = \pm\cosh^{-1}(\frac{G}{F}\cosh(a_1 - a_2) + k) \tag{3.20}$$

for the (implicit) equation for the trajectory and

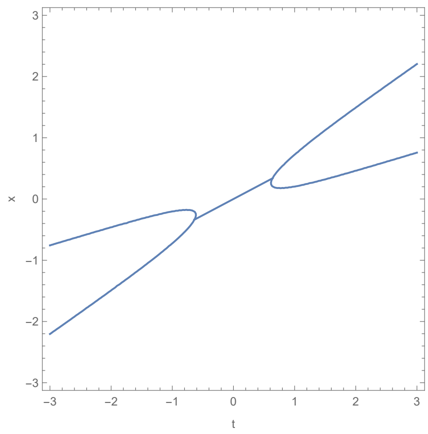
$$t = \pm\frac{d}{\cos(q)}\frac{\gamma_1+\gamma_2}{2\gamma_1\gamma_2 v_{12}}\cosh^{-1}(\frac{E+3F}{G})$$

for the times of the merging and reemerging of the two breathers.

The discontinuity of the velocity of the breather trajectories at the two points t is an artifact of our definition of the trajectories. When the two breathers approach the point t, the minimum of the function φ between them will suddenly have a higher value than the two former maxima. This point corresponds to the discontinuity in the derivatives of the breather trajectories. Nevertheless, the breathers move with a well-defined, finite velocity through these points. It is just the position of the maximum that shifts discontinuously.



(A) In the center of mass frame



(B) In the lab frame

FIGURE 5. The trajectory of the two breathers for the in-phase case. It is plotted for the parameters $\beta_1 = 0.7$, $\beta_2 = 0.3$, $d = 0.1$ and $q = 1.2$. The merge/reemerge points are at $t = \pm 0.637$ in the lab frame and $t = \pm 0.078$ in the cms frame.

4. THE CLASSICAL, NON-RELATIVISTIC LIMIT AND CONNECTION TO THE MACHIAN THEORIES

We now want to analyze the motion of the two breathers in the classical, non-relativistic limit. This means we have $\beta_1, \beta_2 << 1$ and $r >> d$ [2]) with r the relative separation between the two breathers

$$r := x_1 - x_2 \tag{4.1}$$

We will show that the trajectory of the two breathers in this case corresponds to the motion in a Machian interaction potential of the form

$$V = f(r)(1 + \alpha\beta_{12}^2) \tag{4.2}$$

with α some dimensionless parameter of the order of unity.

Since the in phase case only differs from the out-of-phase case by a substitution $k \to -k$, we can just deal with the out-of-phase case and later make this substitution in the result to get the formulas for the in-phase case. In the non-relativistic limit we have $\gamma_1 \approx \gamma_2 \approx 1$ and thus

$$a_1 + a_2 = 2x - (v_1 + v_2)t$$

$$a_1 - a_2 = -v_{12}t$$

[2]The second condition comes from the fact that we're considering the classical limit, thus our two particles have to be far enough away from each other so that their separation r is beyond a typical quantum mechanical scale like the Bohr radius $r >> a_0$. d is now the 1/e radius of the breather and thus our particle. We therefore have $d \approx r_p$, with r_p the radius of the proton. Both together imply the claimed condition.

Plugging this into (3.11) and solving for x yields for the trajectories in the lab frame

$$x_{\pm}(t) = \frac{v_1 + v_2}{2} t \pm \frac{d}{2} \cosh^{-1}(\frac{G}{E} \cosh\left(\frac{v_{12}t}{d}\right) + k) \tag{4.3}$$

For the relative separation (3.1) this gives

$$r = d \cosh^{-1}(\frac{G}{E} \cosh\left(\frac{v_{12}t}{d}\right) + k) \tag{4.4}$$

For the time derivative we obtain

$$\dot{r} = \frac{1}{\sinh(r/d)} v_{12} \sinh\left(\frac{v_{12}t}{d}\right) \tag{4.5}$$

Solving (4.4) for $\cosh(v_{12}t/d)$ and plugging the result into the square of (4.5) yields

$$\dot{r}^2 = v_{12}^2(1 + \frac{1 + k^2 - G^2/E^2 - 2k\cosh\left(\frac{r}{d}\right)}{\sinh\left(\frac{r}{d}\right)^2})$$

In this, we can identify the first term in the brackets as the total energy and the second as the negative interaction potential

$$V(r) = -v_{12}^2 \frac{1 + k^2 - G^2/E^2 - 2k\cosh\left(\frac{r}{d}\right)}{\sinh\left(\frac{r}{d}\right)^2} \tag{4.6}$$

Now, in the classical limit we have $r >> d$. Therefore, in the interaction potential (4.6) we only have to keep the last term proportional to $\cosh(r/d)$. Further, we can approximate

$$\frac{\cosh\left(\frac{r}{d}\right)}{\sinh\left(\frac{r}{d}\right)^2} \approx 2\exp\left(-\frac{r}{d}\right)$$

leaving us with the far field

$$V(r) \approx 4kv_{12}^2 \exp\left(-\frac{r}{d}\right) \tag{4.7}$$

Finally, we have for k in the lowest non-vanishing order β

$$k = 2(1 + \frac{\cot(q)^2}{\beta_{12}^2})$$

Plugging this into (4.6) yields

$$V(r) \approx 8c^2 \cot(q)^2 (1 + \tan(q)^2 \beta_{12}^2) \exp\left(-\frac{r}{d}\right)$$

which has the claimed form (4.2).
For the in-phase case we get accordingly

$$V(r) \approx -4kv_{12}^2 \exp\left(-\frac{r}{d}\right) \tag{4.8}$$

As we can see, this still has the same form, just with a different sign. We can see from (4.7) and (4.8), and the fact that $k > 0$, that the potential is attractive for the in-phase case and repulsive for the out-of-phase case, in agreement with what has already been found in [11,12]

5. Conclusion

We have analyzed the interaction between two Sine-Gordon breathers and found analytic expressions for the trajectories they follow due to their mutual interaction. We have seen that they behave like classical particles with equal masses: They swap their velocities upon collision. However, the momentum transfer is not discrete, but continuous, mediated via their fields. This shows how in a soliton model indeed particles and fields have a unified description. The fields "generated" by the particles are part of the solitons themselves and are included in the soliton solutions. As is the interaction due to these fields: The trajectories found for each of the two breathers correspond to an accelerated motion, caused by the field of the other.

In the classical, non-relativistic limit we've shown that the interaction potential between the two breathers takes the same form as the interaction potentials underlying the Machian unified theories of gravity and inertia. As we stated in the beginning, the soliton model of elementary particles is suggested as a consequence of those Machian theories. Here, we have shown that in turn the interaction potential between two breathers in the Sine-Gordon model indeed reduces to a velocity dependent potential of the same form as used in the Machian theories. This is a further indication that a breather soliton model of elementary particles is indeed the correct quantum-relativistic generalisation of the classic, non-relativistic Machian theories. To show, that the above said is also true for the full, 3 dimensional theory remains a task to be done, once the correct soliton equations are found.

Appendix A (The two-breather solution of the Sine-Gordon equation)

The N-soliton solution to the Sine-Gordon equation

$$\Box\varphi + \frac{1}{d^2}\sin(\varphi) = 0$$

is given by [10]

$$\varphi = \arctan\left(\frac{f_i}{f_r}\right) \tag{A.1}$$

$$f = f_r + if_i = W(\psi_1, ..., \psi_N)$$

Here, f_r, f_i denote the real and imaginary part of f and W is the Wronskian with the entry vector $\psi = (\psi_1, ..., \psi_N)^T$

$$W(\psi_1, ..., \psi_N) = |\psi^{(0)}, \psi^{(1)}, ..., \psi^{(N-1)}|$$
$$= \begin{vmatrix} \psi_1^{(0)} & \psi_1^{(1)} & ... & \psi_1^{(N-1)} \\ \psi_2^{(0)} & \psi_2^{(1)} & ... & \psi_2^{(N-1)} \\ ... & ... & ... & ... \\ \psi_N^{(0)} & \psi_N^{(1)} & ... & \psi_N^{(N-1)} \end{vmatrix} \qquad (A.2)$$

with $\psi_k^{(j)} = \partial^j \psi_k / \partial X^j$. The number N is the number of solitons in the solution. The functions ψ_k are given by

$$\psi_k = a_k \exp\left(\frac{\zeta_k}{2}\right) + b_k \exp\left(-\frac{\zeta_k}{2}\right)$$
$$\zeta_k = \alpha_k X + \frac{1}{\alpha_k} T + \zeta_k^{(0)}$$

$X = (x + ct)/2d$ and $T = (x - ct)/2d$ are the light cone coordinates. α, a and b are complex parameters, $\zeta^{(0)}$ a complex phase.

The N-breather solution is obtained from the 2N-soliton solution by setting

$$\psi = (\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}..., \psi_{N1}, \psi_{N2})^T$$

with

$$\psi_{k1} = a_k \exp\left(\frac{\zeta_k}{2}\right) + b_k \exp\left(-\frac{\zeta_k}{2}\right), \qquad \psi_{k2} = a_k^* \exp\left(\frac{\zeta_k^*}{2}\right) - b_k^* \exp\left(-\frac{\zeta_k^*}{2}\right)$$

Without loss of generality, we can set $a_k = b_k = 1$ by absorbing the constants into the phase $\zeta_k^{(0)}$.

proof:

By using $a_k = \exp(\ln(a_k))$ and $b_k = \exp(\ln(b_k))$, we can write

$$\psi_{k1} = \exp\left(\frac{\delta_k}{2}\right)(\exp\left(\frac{\tilde{\zeta}_k}{2}\right) + \exp\left(-\frac{\tilde{\zeta}_k}{2}\right)) = \exp\left(\frac{\delta_k}{2}\right)\tilde{\psi}_{k1}$$

with

$$\delta_k = \ln(a_k b_k)$$
$$\tilde{\zeta}_k^{(0)} = \zeta_k^{(0)} + \ln\left(\frac{a_k}{b_k}\right)$$

The same can be done for ψ_{k2} , giving

$$\psi_{k2} = \exp\left(\frac{\delta_k^*}{2}\right)(\exp\left(\frac{\tilde{\zeta}_k^*}{2}\right) + \exp\left(-\frac{\tilde{\zeta}_k^*}{2}\right)) = \exp\left(\frac{\delta_k^*}{2}\right)\tilde{\psi}_{k2}$$

Now, since δ_k doesn't depend on X, it isn't affected by the differentiations in the Wronskian (A.2). Since the first line in W is proportional to $\exp(\delta_1/2)$, the second to $\exp(\delta_1^*/2)$, and so on, we can pull each of these factors out of the determinant, and write

$$W(\psi) = \exp\left(\sum_{k=1}^{N} Re(\delta_k)\right) W(\tilde{\psi})$$

Now, the factor standing by the new Wronskian on the right side is real and thus it cancels out in the final solution (A.1), which is only dependent on the quotient f_i/f_r.

$\square$

Thus we can write the functions ψ_k as

$$\psi_{k1} = \exp\left(\frac{\zeta_k}{2}\right) + \exp\left(-\frac{\zeta_k}{2}\right), \qquad \psi_{k2} = \exp\left(\frac{\zeta_k^*}{2}\right) - \exp\left(-\frac{\zeta_k^*}{2}\right)$$

where we have suppressed the tilde symbols again for brevity.

The one-breather solution can now obtained by choosing N=1, which yields the solution

$$f_r = 2u\cos(b) \tag{A.3}$$

$$f_i = -2w\cosh(a) \tag{A.4}$$

with

$$a = \zeta_r = \frac{\gamma\cos(q)}{d}(x - vt - x_0), \qquad b = \zeta_i = \frac{\gamma\sin(q)}{d}(ct - x\beta) + \delta_0$$

and we've set

$$x_0 = -\frac{d}{\gamma\cos(q)}\zeta_r^{(0)}, \qquad \delta_0 = \zeta_i^{(0)}$$

Here, v is the velocity of the breather, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Further, we have

$$|\alpha|^2 = \frac{1-\beta}{1+\beta}$$

and

$$u = \alpha_r = |\alpha|\cos(q)$$

$$w = \alpha_i = |\alpha|\sin(q)$$

with q a real parameter. Thus, we can write the one-breather solution with (A.1) and (A.3, A.4) as

$$\varphi = 4\arctan\left(\cot(q)\frac{\cos(b)}{\cosh(a)}\right)$$

where we have used the formula $\arctan(1/x) = \pi/2 - \arctan(x)$.

The two-breather solution is obtained by choosing N=2. Performing the Wronskian with *Mathematica* one obtains

$$f_i = -w_1u_2(C^2 - A)\cosh(a_1)\cos(b_2) - w_2u_1(C^2 + A)\cosh(a_2)\cos(b_1)$$
$$+4u_1u_2w_1w_2C(\sinh(a_1)\sin(b_2) - \sinh(a_2)\sin(b_1)) \qquad (A.5)$$
$$f_r = -w_1w_2(D^2 + B)\cosh(a_1)\cosh(a_2) + u_2u_1(D^2 - B)\cos(b_2)\cos(b_1)$$
$$+4u_1u_2w_1w_2D(\sinh(a_1)\sinh(a_2) + \sin(b_2)\sin(b_1)) \qquad (A.6)$$

with

$$a_k = (\zeta_k)_r = \frac{\gamma_k\cos(q_k)}{d}(x - v_kt - (x_0)_k)$$
$$b_k = (\zeta_k)_i = \frac{\gamma_k\sin(q_k)}{d}(ct - \beta_kx) + \delta_k$$
$$(x_0)_k = -\frac{d}{\gamma_k\cos(q_k)}(\zeta_k^{(0)})_r, \qquad \delta_k = (\zeta_k^{(0)})_i, \qquad k = 1, 2$$

Here, v_k are the velocities of the two breathers 1 and 2, $\beta_k = v_k/c$ and $\gamma_k = 1/\sqrt{1-\beta_k^2}$. Further,

$$|\alpha_k|^2 = \frac{1-\beta_k}{1+\beta_k}$$

and

$$u_k = |\alpha_k|\cos(q_k)$$
$$w_k = |\alpha_k|\sin(q_k)$$

as well as

$$C = -|\alpha_1|^2 + |\alpha_2|^2$$
$$D = |\alpha_1|^2 + |\alpha_2|^2$$
$$A = 2|\alpha_1|^2|\alpha_2|^2(\cos(2q_1) - \cos(2q_2))$$
$$B = 2|\alpha_1|^2|\alpha_2|^2(\cos(2q_1) + \cos(2q_2))$$

For two equal breathers with $q_1 = q_2 =: q$ (but in general different velocities), we have

$$A = 0$$
$$B = 4|\alpha_1|^2|\alpha_2|^2\cos(2q)$$

With this, we can write the solution as

$$f_i = -u_1w_2C^2(\cosh(a_1)\cos(b_2) + \cosh(a_2)\cos(b_1))$$
$$+4(u_1w_2)^2C(\sinh(a_1)\sin(b_2) - \sinh(a_2)\sin(b_1)) \qquad (A.7)$$
$$f_r = -w_1w_2(D^2 + B)\cosh(a_1)\cosh(a_2) + u_2u_1(D^2 - B)\cos(b_2)\cos(b_1)$$

$$+4(u_1w_2)^2D(\sinh(a_1)\sinh(a_2)+\sin(b_2)\sin(b_1)) \qquad (A.8)$$

6. Appendix B: The asymptotics of the two breather solution

In this appendix we want to derive the asymptotic expressions of the two breather solution (A.1) with (A.5, A.6). By dividing f_i and f_r by $\cosh(a_1)\cosh(a_2)$ we can write them as

$$f_i = -w_1u_2(C^2-A)z_2 - w_2u_1(C^2+A)z_1$$

$$+4u_1u_2w_1w_2C(\tanh(a_1)y_2 - \tanh(a_2)y_1)$$

$$f_r = -w_1w_2(D^2+B) + u_2u_1(D^2-B)z_1z_2$$

$$+4u_1u_2w_1w_2D(\tanh(a_1)\tanh(a_2)+y_1y_2)$$

with

$$z_k = \frac{\cos(b_k)}{\cosh(a_k)}, \quad k=1,2$$

$$y_k = \frac{\sin(b_k)}{\cosh(a_k)}, \quad k=1,2$$

In this form, we can now easily find the asymptotics. Without loss of generality we assume $v_1 > v_2$. This means, that before the collision, breather 1 is left of breather 2, and vice versa after the collision. We can find the asymptotics now by for example first looking at breather 1 before the collision. Here, breather 2 is far enough away so that we have

$$z_2 \approx y_2 \approx 0$$

in the vicinity of 1. In addition, since 1 is left of 2, we have

$$\tanh(a_2) \approx -1$$

This leaves us with

$$f_i \approx -w_2u_1(C^2+A)z_1 + 4u_1u_2w_1w_2Cy_1$$

$$f_r \approx -w_1w_2(D^2+B) - 4u_1u_2w_1w_2D\tanh(a_1)$$

Plugging both into (A.1) yields

$$\varphi_1^{before} \approx 4\arctan\left(\frac{w_2u_1(C^2+A)\cos(b_1) - 4u_1u_2w_1w_2C\sin(b_1)}{w_1w_2(D^2+B)\cosh(a_1) + 4u_1u_2w_1w_2D\sinh(a_1)}\right) \qquad (A.9)$$

If we set

$$\sin(\delta) = 4u_1u_2w_1w_2C \qquad \cos(\delta) = w_2u_1(C^2+A)$$

$$\sinh(\epsilon) = 4u_1u_2w_1w_2D \qquad \cosh(\epsilon) = w_1w_2(D^2+B)$$

we can use the addition formulas for trigonometric and hyperbolic functions to write (A.9) in the form

$$\varphi_1^{before} = 4\arctan\left(\frac{\cos(b_1+\delta_1)}{\cosh(a_1+\epsilon)}\right) \tag{A.10}$$

with the phase and position shifts δ_1 and ϵ given by

$$\tan(\delta_1) = \frac{4u_2w_1C}{C^2+A}$$

$$\tanh(\epsilon) = \frac{4u_1u_2D}{D^2+B}$$

Equation (A.10) is a one-breather solution phase shifted by δ_1 and position shifted by ϵ. In the same way we can obtain

$$\varphi_2^{before} = 4\arctan\left(\frac{\cos(b_2+\delta_2)}{\cosh(a_2-\epsilon)}\right) \tag{A.11}$$

$$\varphi_1^{after} = 4\arctan\left(\frac{\cos(b_1-\delta_1)}{\cosh(a_1-\epsilon)}\right) \tag{A.12}$$

$$\varphi_2^{after} = 4\arctan\left(\frac{\cos(b_2-\delta_2)}{\cosh(a_2+\epsilon)}\right) \tag{A.13}$$

The phase shift δ_2 is given by[3])

$$\tan(\delta_2) = \frac{4u_1w_2C}{C^2+A}$$

We can directly obtain the trajectories of the asymptotics. Taking into account that to the trajectory x_+ belong the asymptotics φ_2^{before} and φ_1^{after} , and to x_- the asmptotics φ_1^{before} and φ_2^{after} we get

$$x_+ \approx \left\{\begin{matrix} v_2t + \frac{d}{\cos(q)\gamma_2}\tanh^{-1}(\frac{4u_1u_2D}{D^2+B}) & t\to-\infty \\ v_1t + \frac{d}{\cos(q)\gamma_1}\tanh^{-1}(\frac{4u_1u_2D}{D^2+B}) & t\to+\infty \end{matrix}\right\} \tag{A.14}$$

$$x_- \approx \left\{\begin{matrix} v_2t - \frac{d}{\cos(q)\gamma_2}\tanh^{-1}(\frac{4u_1u_2D}{D^2+B}) & t\to-\infty \\ v_1t - \frac{d}{\cos(q)\gamma_1}\tanh^{-1}(\frac{4u_1u_2D}{D^2+B}) & t\to+\infty \end{matrix}\right\} \tag{A.15}$$

[3]In the case considered in section 3 we have $q_1 = q_2$ and thus $u_1w_2 = u_2w_1$. Consequently we then also have $\delta_1 = \delta_2$ in this case.

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