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UNIFORM CONTINUITY

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ABSTRACT. This article discusses the notion of uniform continuity, its relation with the derivative of differentiable functions and Lipschitz continuity or even more weakly Hölder continuity related in some way to how wildly the function oscillates. It also discusses its connection with compactness for the very large general class of functions - continuous functions. Further, a few properties of uniform continuous function especially with regards to unique continuous extension of functions are discussed.

1. INTRODUCTION

If a function is uniformly continuous then the rate at which $f(y)$ approaches $f(x)$ as $y \to x$ does not depend upon x. One might expect that if the function is differentiable and the derivative is bounded then the function is uniformly continuous because the rate at which $f(y)$ approaches $f(x)$ as $y \to x$ is bounded. This is true and is proved below. However, due to a remarkable theorem of Heine[1], continuity is all that one needs to impose on a function defined on compact sets for it to be uniformly continuous.
All the functions in the following sections are defined on $E \subset \mathbb{R}$ and are real valued unless stated otherwise.

Definition 1. *Let* $f : E \to \mathbb{R}$ *be a function.* f *is uniformly continuous if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - f(y)| < \epsilon$ *for all* $x, y \in E$ *for which* $|x - y| < \delta$[1][2].

2. A SUFFICIENT CONDITION FOR UNIFORM CONTINUITY OF DIFFERENTIABLE FUNCTIONS

Theorem 2. *If* f *is differentiable and* f' *is bounded on* E *then* f *is uniformly continuous.*

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Proof. Let $x, y \in E$. Using mean value theorem, there exists a c between x and y such that $|f(x)-f(y)| = |f'(c)||x-y| \leqslant M|x-y|$ where $|f'| \leqslant M$ since f' is bounded. Uniform continuity follows by choosing a $0 < \delta < \epsilon/M$. □

Remark 3. *A uniformly continuous function need not be differentiable and thus need not have a bounded derivative. So, the above condition is a sufficient condition and not a necessary condition. For example* $f(x) = |x|^{1/2}$ *is uniformly continuous on* $\mathbb{R}$ *and its slope goes to infinity near the origin. The function is not differentiable at the origin. In fact, every Hölder continuous function and thus every Lipschitz continuous function is uniformly continuous(next theorem).*
The function $f(x) = x^2$ *is uniformly continuous on any bounded interval but not uniformly continuous on* $\mathbb{R}$ *since the derivative is unbounded as* $x \to \infty$.

3. Another Sufficient condition or uniform continuity

Theorem 4. *If* $f : E \to \mathbb{R}$ *is a function such that* $|f(x)-f(y)| \leqslant C|x-y|^\lambda$ *for all* $x, y \in E$ *for some positive constants* C *and* λ *then* f *is uniformly continuous. Such functions are called Hölder continuous.*

Proof. Similar to the above theorem. Let $0 < \delta < (\epsilon/C)^{1/\lambda}$. □

4. A rather surprising connection between compactness and uniform continuity

Theorem 5 (Heine). *If* E *is compact and* f *is continuous then* f *is uniformly continuous on* E.

1[2]. Let $\epsilon > 0$ be given. Since f is continuous at each point $x \in E$, there exist $\delta(x) > 0$ such that $|f(y) - f(x)| < \epsilon/2$ whenever $|y - x| < \delta(x)$. Let $E(x)$ be the set of all points y such that $|y - x| < \frac{1}{2}\delta(x)$. $\{E(x)|x \in E\}$ is an open cover of E since $x \in E(x)$ and each $E(x)$ is open. Since E is compact this open cover has a finite subcover, say $\{E(x_1), E(x_2), ..., E(x_n)\}$. Let $\delta = \frac{1}{2}\min_{1 \leqslant i \leqslant n} \delta(x_i)$. Of course δ is positive since it is a minimum of finitely many positive quantities.
Let x and y be points in E such that $|y - x| < \delta$. $x \in E(x_m)$ for some

integer $m \in 1, 2, ..., n$ since $x \in E$ and $E(x_i)$ is an open cover of E. So, $|x - x_m| < \frac{1}{2}\delta(x_m)$. Also by the triangle inequality,

$$|y - x_m| \leqslant |y - x| + |x - x_m| < \delta + \frac{1}{2}\delta(x_m) < \delta(x_m).$$

Thus

$$|f(y) - f(x) \leqslant |f(y) - f(x)| + |f(y) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

So, f is uniformly continuous on E. □

2. Let f be not uniformly continuous so for every $\epsilon > 0$ there exist a $\delta > 0$ such that for all $x, y \in E$ it is true that $|x - y| < \delta$ and $|f(x) - f(y)| \geqslant \epsilon$. Taking $\delta = 1, 1/2, 1/3, ...$, (x_k, y_k) are obtained such that $|x_k - y_k| < 1/k$ for each k. Since $\{x_k\}$ is a sequence in a compact set, it has a limit point and thus a convergent subsequence, say $\{x_{n_k}\}$ converging to, say a. So the subsequence $\{y_{n_k}\}$ of $\{y_k\}$ also converges to a. Since f is continuous, $f(x_{n_k}) - f(y_{n_k}) \to f(a) - f(a) = 0$ which contradicts $|f(x_k) - f(y_k)| \geqslant 1/k$ for all positive integers k. So, f is uniformly continuous. □

5. Properties of uniformly continuous functions

Theorem 6. *If E is bounded and f is uniformly continuous then f is bounded.*[2]

Proof. If $f(E)$ is unbounded there exist a sequence $f(x_k)$ which diverges to $\pm\infty$. So, $|f(x_k)| \to \infty$. Since E is bounded, every sequence in E has a convergent subsequence. Let $\{x_{n_k}\}$ be such a convergent subsequence. $\{x_{n_k}\}$ is Cauchy since $E \subset \mathbb{R}$. Given any $\delta > 0$ there exists $N \in \mathbb{N}$ such that for all $n_i, n_j \geqslant N$ $|x_{n_i} - x_{n_j}| \leqslant \delta$. n_i can be chosen such that $|f(x_{n_i})| > 1 + |f(x_{n_j})|$ and thus $|f(x_{n_i}) - f(x_{n_j})| \geqslant 1$. This contradicts the uniform continuity of f since given $\epsilon = 1$ there exists a $\delta > 0$ such that $x_{n_i} - x_{n_j}| \leqslant \delta$ and $|f(x_{n_i}) - f(x_{n_j})| \geqslant 1$. So f is bounded. □

Theorem 7. *Composition of uniformly continuous functions is uniformly continuous.*[2]

Omitting the proof.

Remark 8. *However uniformly continuous function composed with continuous function, continuous function composed with uniformly continuous function or continuous function composed with continuous function might not be uniformly continuous.*

Theorem 9. *Let $f : X \to Y$ where X and Y are metric spaces. If f is uniformly continuous then f maps Cauchy sequences to Cauchy sequences.*[2]

Proof. Let $\{x_n\}$ be a Cauchy sequence in X and $\epsilon > 0$ be given. Since f is uniformly continuous there exists a $\delta > 0$ such that $d_Y(f(x_n), f(x_m)) < \epsilon$ whenever $d_X(x_n, x_m) < \delta$. There exists an $N \in \mathbb{N}$ such that $d_X(x_n, x_m) < \delta$ for all $n, m \geqslant N$. So, $d_Y(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geqslant N$ and thus the sequence $\{f(x_n)\}$ is Cauchy. □

Remark 10. *Uniform continuity implies continuity, however the metric space might not be Cauchy complete.*

Theorem 11. *Let E be a dense subset of a metric space X. If f is uniformly continuous then there exists a unique continuous extension g from E to X.*[2]

Proof. Let $p \in X$. If $p \in E$ then $g(p) = f(p)$. If $p \notin E$ then p is a limit point of E since E is dense in X and thus there exists a sequence x_n in E such that $x_n \to p$. $g(p) = \lim g(x_n) = \lim f(x_n)$ since x_n is a Cauchy sequence and thus $f(x_n)$ is Cauchy since f is uniformly continuous. So, $f(x_n)$ is convergent since $\mathbb{R}$ is Cauchy complete. It will be shown that g is well defined, that is, if there are two sequences s_n and t_n in E converging to x then $\lim g(s_n) = \lim g(t_n)$ and $g : X \to \mathbb{R}$ is continuous. Uniqueness follows from the definition and continuity of g

Let $f(s_n) \to q$. Since f is uniformly continuous, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $d_X(a, b) < \delta$ implies that $|f(a) - f(b)| < \epsilon$. Since $s_n \to p$ there exists $N_1 \in \mathbb{N}$ such that $d_X(s_n, p) < \delta/2$ for all $n \geqslant N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $d_X(t_n, p) < \delta/2$ for all $n \geqslant N_2$ since $t_n \to p$. If $N = max(N_1, N_2)$ then $d_X(s_n, t_n) \leqslant d_X(s_n, p) + d_X(p, t_n) < \delta$ for all $n \geqslant N$ and thus $|f(s_n) - f(t_n)| < \epsilon$. $|f(t_n - q)| \leqslant |f(t_n) - f(s_n)| + |f(s_n) - q| <$

2ϵ for all $n \geqslant N'$ where $N' = max(N, N_3)$ where $|f(s_n) - q| < \epsilon$ for all $n \geqslant N_3$. Such N_3 exists since $f(s_n) \to q$. So, $f(t_n) \to q$. Thus g is well defined.

Let $p \in E$. Since f is uniformly continuous, given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $a, b \in E$, $d_X(a, b) < \delta$ implies $|f(a) - f(b)| < \epsilon/2$. Let $x \in X$ such that $d_X(x, p) < \delta/2$. If $x \in E$ then $g(x) = f(x)$ so $|g(x) - g(p)| = |f(x) - f(p)| < \epsilon/2 < \epsilon$. If $x \notin E$ then x is a limit point of E and thus there is a sequence in E, say s_n converging to x. So, there exists $N_1 \in \mathbb{N}$ such that $d_X(s_n, x) < \delta/2$ for all $n \geqslant N_1$. So, $d_X(s_n, p) \leqslant d_X(s_n, x) + d_X(x, p) < \delta$. So, $|g(s_n) - g(p)| = |f(s_n) - f(p)| < \epsilon/2$. Since $g(x) = \lim f(s_n) = \lim g(s_n)$ that is $g(s_n) \to g(x)$ there exists $N_2 \in \mathbb{N}$ such that $|g(s_n) - g(x)| < \epsilon/2$ for all $n \geqslant N_2$. Let $N = max(N_1, N_2)$. So, $|g(x) - g(p)| \leqslant |g(x) - g(s_n)| + |g(s_n) - g(p)| < \epsilon$ for all $n \geqslant N$. So, g is continuous on E.

Let $p \notin E$. p is a limit point of E. There is a sequence $t_n \to p$. Since f is uniformly continuous, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $d_X(a, b) < \delta$ implies that $|f(a) - f(b)| < \epsilon/3$. Let $x \in X$ such that $d_X(x, p) < \delta/3$. If $x \in E$ then $g(x) = f(x)$ so $|g(x) - g(p)| \leqslant |f(x) - f(t_n)| + |g(t_n) - g(p)| < 2\epsilon/3 < \epsilon$ eventually since for large enough n, x and t_n are in $\delta/3$ nbd of p and thus $d_X(x, t_n) < \delta/3$ which implies $|f(x) - f(t_n)| < \epsilon/3$ and $g(t_n) \to g(p)$. If $x \notin E$ then it is a limit point of E. There is a sequence $s_n \to x$. $s_n \to x$ and $t_n \to p$, so $d_X(s_n, t_n) < \delta/3$ for large enough n. $|g(x) - g(p)| \leqslant |g(x) - g(s_n)| + |f(s_n) - f(t_n)| + |g(t_n) - g(p)| < \epsilon$ eventually since $g(s_n) \to g(x)$ and $g(t_n) \to g(p)$. So, g is continuous on X. □

Corollary 12. *Let $f : E \to Y$ where Y is a Cauchy complete metric space. If f is uniformly continuous then there exists a unique continuous extension $\overline{f}$ from E to $\overline{E}$ which is also uniformly continuous.*

Proof. E is dense in $\overline{E}$. Follows from the above theorem. □

A result on continuous extensions:

Theorem 13. *If $E \subset \mathbb{R}$ is closed and f is continuous then there exists continuous extensions from E to $\mathbb{R}$.*[2]

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$. Let the graph of $g(x)$ be completed over $\mathbb{R}$ by joining the gaps in the graph of $f(x)$ using straight lines on $\mathbb{R} - E$. $g : \mathbb{R} \to \mathbb{R}$ is continuous and is thus a continuous extension of $f : E \to \mathbb{R}$ from the closed set E to $\mathbb{R}$. □

6. An example of a uniformly continuous function associated with any subset of a metric space

A function associated with a metric space which is uniformly continuous:

Definition 14. *Let E be a subset of the metric space (X, d). The distance function is defined as $\rho_E(x) = \inf_{z \in E} d(x, z)$ as x varies over X. Distance between two sets is defined as the infimum of the distances of points of one set to another.*

Theorem 15. *ρ_E is uniformly continuous.*

Proof. $\rho_E(x) \leqslant d(x, z) \leqslant d(x, y) + d(y, z)$ for all $z \in E$ and so $\rho_E(x) - \rho(y) \leqslant \rho_E(x) - d(y, z) \leqslant d(x, y)$. Similarly, $\rho_E(y) - \rho_E(x) \leqslant d(x, y)$. So, $|\rho_E(x) - \rho_E(y)| \leqslant d(x, y)$ and thus ρ_E is uniformly continuous. □

The distance function shows up in analysis. Here is an interesting example:

Example 16. *Distance between two disjoint closed sets of a metric space may be zero. For example: set of positive integers E and the set $F = \{n + 1/n\}$. Sets E and F are closed since they do not have limit points. The distance between them is 0 since $\inf_{n \in \mathbb{Z}^+} 1/n = 0$.*

If the space is Hausdorff(for example $\mathbb{R}^k$), closed sets which are bounded are compact. If one of the closed sets is compact in the above example then the distance between them is always positive.

Theorem 17. *Distance between a closed set F and a compact set K of a metric space X is positive. In fact there exists a $\delta > 0$ such that $d_X(p, q) > \delta$ for all $p \in F$ and $q \in K$.*[2]

First a lemma:

Lemma 18. $\rho_E(x) = 0$ *if and only if* $x \in \overline{E}$.

Proof. Let $\rho_E(x) = 0$. If $x \notin \overline{E}$ then $x \in X - \overline{E}$. $X - \overline{E}$ is open so there exists $r > 0$ such that $d(y, x) < r$ and $y \in X - \overline{E}$. So, $y \notin \overline{E}$. Clearly $z \in closure E$ for all $z \in E$ so $d(z, x) \geqslant r > 0$ and thus $\rho_E(x) = \inf_{z \in E} d(z, x) > 0$ which is a contradiction.
Let $x \in \overline{E}$. If $x \in E$ then $\rho_E(x) = 0$. If $x \notin E$ then x is a limit point of E. Every nbd of x contains a point of E and thus $\rho_E(x) = 0$. □

Proof of the above theorem:

Proof. $\rho_F(p) > 0$ for all $p \in K$ since if $\rho_F(p) = 0$ for some $p \in K$ then $p \in \overline{F} = F$ by the above lemma which is a contradiction since F and K are disjoint. Since ρ_F is uniformly continuous it is continuous and the infimum of ρ_F on K is attained since K is compact. So, there is a point $q \in K$ such that $d(p, q) = \inf_{k \in K} \rho_F(k)$. δ can be chosen to be less than $d(p, q)$. □

References

[1] Eduard Heine. Die elemente der functionenlehre. *Journal für die reine und angewandte Mathematik*, 1872.

[2] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976.