The integral equation for the relic black-body

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Abstract

We consider the possibility that black-body photons are created by the energetical sources in space. it leads to the integral equation for the energy distribution of sources. The Planck formula for photons is, in this situation, valid, for the special distribution of sources. It means that the black-body is at the crystalline form.

1 Introduction

The relic radiation, or, the cosmic microwave background (CMB) is the thermal radiation in the Big Bang cosmology. The CMB is the oldest light in the universe and it is an emission of uniform, black-body thermal energy coming from all parts of the sky.

The cosmical rays including relic photons were predicted by Gamow as a consequence of the Big Bang. The Mach cone is created when the high energy cosmical particles move with the speed greater than the velocity of sound in cosmical relic photon sea (Pardy, 2013a; 2013b).

The accidental discovery of the CMB in 1964 by American radio astronomers Arno Penzias and Robert Wilson was the culmination of work initiated in the 1940, and earned the discoverers the 1978 Nobel Prize.

Relic photons form so called black-body, which has the distribution law of photons derived in 1900 by Planck (1900, 1901), (Schöpf, 1978). The derivation was based on the investigation of the statistics of the system of oscillators inside of the black-body. Later Einstein (1917) derived the Planck formula from the Bohr model of atom where electrons have the discrete energies and the energy of the emitted photons are given by the Bohr formula $\hbar\omega = E_i - E_f$, E_i, E_f are the initial and final energies of electrons.

2 The Einstein derivation of the Planck formula

The Einstein derivation of the Planck black-body spectral formula was his personal reaction at the Planck derivation which was so called heuristic intuitive derivation . The distribution of the black-body photons was derived by Planck (1900) from modification of the thermodynamical entropy and the system of quantum oscillators. Einstein (1917) derived the Planck formula from the Bohr model of atom which was based on two postulates: 1. every atom can exist in the discrete series of states in which electrons do not radiate even if they are moving at acceleration (the postulate of the stationary states), 2. transiting electron from the stationary state to other, emits the energy according to the law $\hbar\omega = E_m - E_n$, called the Bohr formula, where E_m is the energy of an electron in the initial state, and E_n is the energy of the final state of an electron to which the transition is made and $E_m > E_n$.

Let us remark still that the Bohr theory does not involve the physical mechanism of creation of photons and the adequate model of photon. However, it follows from quantum theory of fields, that photon is excited state of vacuum and at the same time also an electron is the excited state of vacuum, which follows from the elementary experimental equation $\gamma + \gamma \rightleftharpoons e^+ + e^-$ (Berestetzkii et al., 1999). At present time we know from the most general quantum field theory that all matter and antimatter in universe are excited states of vacuum.

Einstein introduced coefficients of spontaneous and stimulated emission A_{mn}, B_{mn}, B_{nm}. In case of spontaneous emission, the excited atomic state decays without external stimulus as an analogue of the natural radioactivity decay. The energy of the emitted photon is given by the Bohr formula. In the process of the stimulated emission the atom is induced by the external stimulus to make the same transition. The external stimulus is a black-body photon that has an energy given by the Bohr formula.

If the number of the excited atoms is equal to N_m, the emission energy per unit time conditioned by the spontaneous transition from energy level E_m to energy level E_m is

$$P_{spont.\,emiss.} = N_m A_{mn} \hbar\omega, \tag{5}$$

where A_{mn} is the coefficient of the spontaneous emission.

In case of the stimulated emission, the coefficient B_{mn} corresponds to the transition of an electron from energy level E_m to energy level E_n and coefficient B_{nm} corresponds to the transition of an electron from energy level E_n to energy level E_m. So, for the energy of the stimulated emission per unit time we have two formulas :

$$P_{stimul.\,emiss.} = \varrho_\omega N_m B_{mn} \hbar\omega \tag{6}$$

$$P_{stimul.\,absorption} = \varrho_\omega N_n B_{nm} \hbar\omega. \tag{7}$$

If the black-body is in thermal equilibrium, then the number of transitions from E_m to E_n is the same as from E_n to E_m and we write:

$$N_m A_{mn} \hbar\omega + N_m \varrho_\omega B_{mn} \hbar\omega = N_n \varrho_\omega B_{nm} \hbar\omega, \tag{8}$$

where ϱ_ω is the density of the photon energy of the black-body.

Then, using the Maxwell statistics

$$N_n = De^{-\frac{E_n}{kT}}, \quad N_m = De^{-\frac{E_m}{kT}}, \tag{9}$$

we get:

$$\varrho_\omega = \frac{\frac{A_{mn}}{B_{mn}}}{\frac{B_{nm}}{B_{mn}} e^{\frac{\hbar\omega}{kT}} - 1}. \tag{10}$$

The spectral distribution of the black-body does not depend on the specific atomic composition of the black-body and it means the formula (10) must be so called the Planck formula:

$$\varrho_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}. \tag{11}$$

After comparison of eq. (10) with eq. (11) we get:

$$B_{mn} = B_{nm} = \frac{\pi^2 c^3}{\hbar\omega^3} A_{mn}. \qquad (12)$$

It means that the probabilities of the stimulated transitions from E_m to E_n and from E_n to E_m are proportional to the probability of the spontaneous transition A_{mn}. So, it is sufficient to determine only one of the coefficient in the description of the radiation of atoms.

The internal density energy of the black-body gas is given by integration of the last equation over all frequencies ω, or

$$u = \int_0^\infty \varrho(\omega) d\omega = aT^4; \quad a = \frac{\pi^2 k^4}{15\hbar^3 c^3} \qquad (13)$$

and the pressure of photons inside the black-body follows from the electrodynamic situation inside black-body as follows:

$$p = \frac{u}{3}. \qquad (14)$$

Let us remark that coefficients A_{mn} of the so called spontaneous emission cannot be specified in the framework of the classical thermodynamics, or, statistical physics. They can be determined only by the methods of quantum electrodynamics as the consequences of the so called radiative corrections. So, the radiative corrections are hidden external stimulus, which explains the spontaneous emission.

3 The Planck black-body formed by oscillators

We know that the relation between average energy of oscillator $< E >$ and the spectral density ϱ_ω of radiation of electromagnetic energy is expressed in the form (Sokolov et al., 1962)

$$\varrho_\omega = \frac{\omega^2}{\pi^2 c^3} < E > . \qquad (15)$$

The distribution of particles with energy E in the thermal bath is given according to statistical physics as follows:

$$N(E) = Ae^{-\alpha E}, \qquad (16)$$

where $\alpha = 1/kT, k = 1.38.10^{-16} erg.grad^{-1}$ is the Boltzmann constant.

With regard to these facts, we get that the average energy $< E >$ of the system is as follows:

$$< E >= \frac{A \int_0^\infty e^{-\alpha E} E dE}{A \int_0^\infty e^{-\alpha E} dE} = \tag{17}$$

$$-\frac{\partial}{\partial \alpha} \ln \int_0^\infty e^{-\alpha E} dE = \tag{18}$$

$$= \frac{\partial}{\partial \alpha} \ln \alpha = kT. \tag{19}$$

After inserting $< E >$ from (17-19) into ϱ_ω in (15), we get the formula of Rayleigh-Jeans

$$\varrho_\omega = \frac{\omega^2}{\pi^2 c^3} kT. \tag{20}$$

In case of the quantization of energy of oscillators in harmony with the formula

$$E = n\varepsilon; \ n = 1, 2, ... \tag{21}$$

we are forced to replace the integral in (19) by summation, or

$$< E >= -\frac{\partial}{\partial \alpha} \ln \sum_0^\infty \varepsilon e^{-\alpha n \varepsilon} = \tag{22}$$

$$-\frac{\partial}{\partial \alpha} \ln \frac{\varepsilon}{1 - e^{-\alpha \varepsilon}} = \frac{\varepsilon}{e^{\alpha \varepsilon - 1}}. \tag{23}$$

So, after insertion of eq. (23) in eq. (15), we get

$$\varrho_\omega = \frac{\omega^2}{\pi^2 c^3} \frac{\varepsilon}{e^{\alpha \varepsilon - 1}}. \tag{24}$$

After the historical convention quantum energy was expressed as

$$\varepsilon = \hbar \omega. \tag{25}$$

and it means tha the final Planck formula for the photon distribution is as follows

$$\varrho_\omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar \omega}{kT}} - 1}, \tag{26}$$

where $\hbar = 1,05.10^{-27} erg.s$ is the so called Planck constant.

For $\hbar \omega / kT \ll 1$, we have

$$e^{\frac{\hbar \omega}{kT}} \approx 1 + \frac{\hbar \omega}{kT} \tag{27}$$

4 The integral equation for the relic radiation

Now, let us consider the situation that there is some density distribution $\varrho(E)$ of energy in the black-body and the distribution of particles with energy E in the thermal bath is given according to statistical physics as follows:

$$N(E) = Ae^{-\alpha E}, \tag{28}$$

where $\alpha = 1/kT, k = 1.38.10^{-16} erg.grad^{-1}$ is the Boltzmann constant.

Then, we get that the average energy $< E >$ of the system is as follows:

$$< E >= \frac{A\int_0^\infty \varrho(E)e^{-\alpha E}EdE}{A\int_0^\infty \varrho(E)e^{-\alpha E}dE} = \tag{29}$$

$$-\frac{\partial}{\partial\alpha}\ln\int_0^\infty \varrho(E)e^{-\alpha E}dE \tag{30}$$

We can put the question if there is such distribution of $\varrho(E)$, which gives the Planck distribution with eq. (23). So, we have the integral equation for $\varrho(E)$:

$$-\frac{\partial}{\partial\alpha}\ln\int_0^\infty \varrho(E)e^{-\alpha E}dE = \frac{\varepsilon}{e^\alpha - 1}. \tag{31}$$

In case that we are able to give the physical meaning to $\varrho(E)$ in the last equation, we can define general integral equation for $\varrho(E)$, or,

$$-\frac{\partial}{\partial\alpha}\ln\int_0^\infty \varrho(E)e^{-\alpha E}dE = \Gamma(\alpha, \varepsilon), \tag{32}$$

where $\Gamma(\alpha, \varepsilon)$ is average density of cosmical photons determined by cosmical measurement of CMB.

We have seen that in case of the quantization of energy of oscillators with the formula

$$E = n\varepsilon;\ n = 1, 2, ... \tag{33}$$

we have replaced the integral in (19) by summation, or

$$< E >= -\frac{\partial}{\partial\alpha}\ln\sum_0^\infty \varepsilon e^{-\alpha n\varepsilon} = \tag{34}$$

The last formula can be replaced by the integral

$$< E >= -\frac{\partial}{\partial\alpha}\ln\sum_0^\infty \int_{-\infty}^\infty \delta(E - n\varepsilon)\varepsilon e^{-\alpha E}, \tag{35}$$

which after mathematical realization forms

$$< E >= \frac{\varepsilon}{e^{\alpha\varepsilon - 1}}. \tag{36}$$

So in this case the distribution of energy is

$$\varrho(E) = \delta(E - n\varepsilon). \tag{37}$$

In this case we can generalize the equation (35) as follows

$$< E >= -\frac{\partial}{\partial\alpha} \ln \sum_0^\infty \int_{-\infty}^{\infty} \varrho(E - n\varepsilon)\varepsilon e^{-\alpha E} = \Gamma(E, \alpha), \tag{38}$$

which is the integral equation for the determination of the energy distribution $\varrho(E)$ if it is known $\Gamma(E, \alpha)$.

5 The dielectric crystal in he black-body

We suppose here that inside of the Planck black-body there is the dielectric crystal with the index of refraction $n(\omega)$. Such assumption is of the cosmological meaning, because the cosmological background is non homogeneous and it can be expressed also by the index of refraction (Lachieze-Rey et al., 1999). Then, the wave vector of photon inside the dielectric medium is given by known formula

$$q = n(\omega)\frac{\omega}{c}. \tag{39}$$

The number of light modes in the interval $q, q+dq$ inside of the dielectric in the volume V is Vq^2dq/π^2. After differentiation of formula (39) we get with $d\ln\omega = d\omega/\omega$

$$dq = \frac{1}{c}[n(\omega) + \omega\frac{dn(\omega)}{d\omega}]d\omega = \frac{n(\omega)}{c}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}d\omega. \tag{40}$$

Then, it is easy to see that the number of states in the interval $\omega, \omega + d\omega$ of the electromagnetic vibrations in the volume V is

$$Vg(\omega)d\omega = \frac{V}{\pi^2}\left(\frac{n(\omega)}{c}\right)^3 \frac{d\ln[n(\omega)\omega]}{d\ln\omega}d\omega. \tag{41}$$

If we multiply the last formula by the average energy of the harmonic oscillator (41), we get the Planck formula for the black-body with dielectric medium:

$$\varrho(\omega) = \frac{n^3(\omega)\omega^2}{\pi^2 c^3}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1}, \tag{42}$$

where for $n = 1$, we get exactly formula (11).

6 Discussion

The cosmic microwave background (CMB, CMBR), or relic radiation, is radiation that fills all space in the observable universe. With a standard optical telescope, the background space between stars and galaxies is almost completely dark. However, a sufficiently sensitive radio telescope detects a background glow that is almost uniform and is not associated with any star, galaxy, or other object. This glow is strongest in the microwave region of the radio spectrum. The accidental discovery of the CMB in 1965 by American radio astronomers Arno Penzias and Robert Wilson was the culmination of work initiated in the 1940s.

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