Study on Goldbach prime pairs and its distribution for integers of form $2(n+1)^2$

Imran Ansari

Abstract

In this article, study related to the distribution of the Goldbach prime pairs for the even integers of form $2(n+1)^2$ has been carried out with the help of other conjectures(/postulate), that is, Legendre's conjecture and Bertrand's postulate. From the obtained results, various possible conjectures have been proposed/suggested. In brief, first the term, Goldbach first prime pair (p_f, p'_f) was defined and for ascertaining the interval in which Goldbach first prime pair would always be present for all the values of $2(n+1)^2$, where, $n = 1, 2, 3, ...,$ was suggested. The proposed conjectures on distribution were tested for all values of $n = 1, 2, 3, ..., 1,000,000$ using algorithm. The expression, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - n(n+1)$ showed that, the Goldbach first prime pair would always existing for all the n values, where, $n = 1, 2, 3, ... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$. The expression, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ showed that, the Goldbach first prime pairs would always existing for all the n values, where, $n = 1, 2, 3, ... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\}$. While, for the expression, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$, were true for all the $n = 1, 2, 3, ...,$ values. Another observation was found that, in the proposed conjecture 1, the gap between Goldbach first prime pair, that is, $(p'_f - p_f)$ was found to be always less than its corresponding n value after $n = 2538$, hence, $\text{Gap}(p'_f - p_f) << n$, where, $n = 2539, 2540, 2541, ...$

Keywords— Goldbach conjecture, Prime pairs, Number theory

1 Introduction

Goldbach's well-known conjecture states that *"Every even integer greater than* 2 *can be written as the sum of two prime numbers"* [1]. Goldbach's conjecture deals with the possibility of writing every even integer greater than 2 as sum of two prime numbers (p, p'), however, the conjecture does not deal with the distribution of such two prime numbers (p, p') or the interval in which it will certainly be falling for all the positive integers or for some selective positive integers. The Legendre's conjecture states that, *"There is a prime number between* n^2 *and* $(n+1)^2$ *for every positive integer* n*"* [2]. The Bertrand's postulate, which has been proven and states that *"for any* $n > 1$*, there is always at least one prime* p *such that* $n < p < 2n$*"* [3]. The Legendre's conjecture and Bertrand's postulate deals with the distribution of the prime numbers. Hence, with the idea of finding the possibility of ascertaining the interval in which there will be certainty that Goldbach prime numbers (p, p' pairs) would be present, in this article, systemic approaches have been shown to obtained the mathematical expressions which shows the possibility of the presence of the Goldbach prime pairs for the even integers of form $2(n+1)^2$, where, $n = 1, 2, 3, ...$

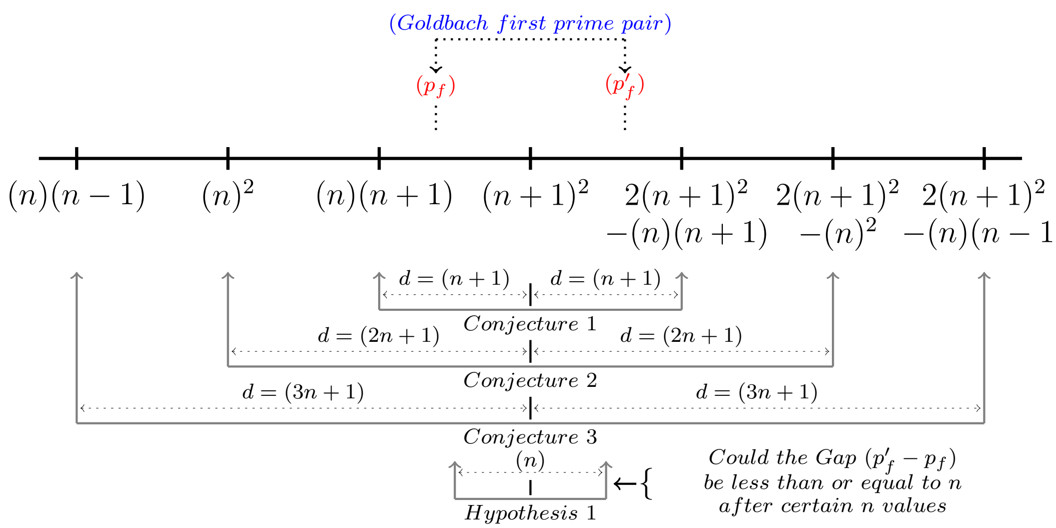

Figure 1: Schematic illustration of the proposed conjectures/hypothesis.

Conjecture 1: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$.

Conjecture 2: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\}$.

Conjecture 3: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ...$

Hypothesis 1: The Gap $(p'_f - p_f)$ between the Goldbach first prime pair for the even integer $2(n+1)^2$ is always less than the corresponding value of n, *where*, $n = 2539, 2540, 2541, ...$

Further, based on the results, the new conjectures were proposed for the even integers of form $2(n+1)^2$, *where*, $n = 1, 2, 3, ...$ and the distribution of its Goldbach first prime pairs. Further, the proposed new conjectures were tested using the computer algorithm till $n = 1, 2, 3, ..., 1,000,000$. Very first the **Conjecture 2** ($(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$) was proposed and tested, based on its results the narrow and broad interval were tried, which became **Conjecture 1** ($(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$) and **Conjecture 3** ($(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$) respectively. Moreover, based on the observations, **Hypothesis 1** has also been suggested. The proposed conjectures and its schematic illustration have been outlined in **Figure 1** for the quick overview. The detailed step-by-step explanation have been given in the further sections.

2 Goldbach conjecture, Legendre's conjecture and Bertrand's postulate

In this section, step-by-step, some of the useful concepts which thought to be necessary for the explanation and understanding of the proposed conjectures have been explained/described.

2.1 Goldbach conjecture, Goldbach prime pairs and Goldbach first prime pair

In this section, taking Goldbach conjecture as benchmark, various definitions and statements have been outlined which thought to be necessary in explaining the conjectures. Further, notes and remarks have been given at appropriate places wherever it was thought to be necessary for clearer explanation of concepts.

Note 1: In this article the term integer means positive integer only.

Statement 1 (Goldbach conjecture): *"Every even integer greater than* 2 *can be written as the sum of two prime numbers."*

Further, from the **Statement 1**, we can easily understand that, for every even integer, $2k$, there will always be two prime numbers exists such that their sum is equal to $2k$. Hence, the Goldbach conjecture can be expressed in the sense of existence of prime numbers as shown in **Statement 2**.

Statement 2: There is always existing prime numbers p *and* p' such that, $2k = p + p'$, *where* $k = 2, 3, 4, ...$

Lets understand **statement 1/2** with the help of number-line. For example, all the possible ways that even integer "16" can be written as the sum of two numbers are: $(1+15), (2+14), (3+13), (4+12), (5+11), (6+10), (7+9)$ and $(8+8)$, which can be illustrated on the number-line as shown in **Figure 2**. Now, the pair of the prime-numbers whose sum gives the sum 16 needs to be from these list of pairs only as these are all the possible ways of writing 16 as sum of two numbers.

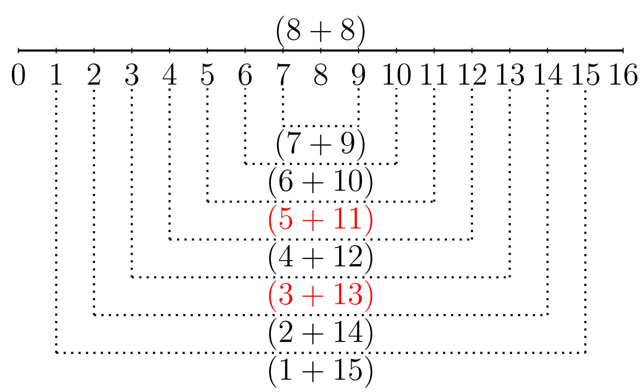

Figure 2: Schematic illustration of all pairs of writing "16" as sum of two integers on the number-line, that is, $(1+15), (2+14), (3+13), (4+12), (5+11), (6+10), (7+9)$ and $(8+8)$.

The integer, 16, is an even number, hence can also be written as $2k = 16$. Hence, the value of k will be 8, as $k = 16/2 = 8$. Further, since every positive integer is of form $2k$, it can also be written in the form of $2k = k + k$. Hence, integer, 16, can be expressed as $8 + 8$.

$$2k = k + k \tag{1}$$
$$Hence,\ for\ k = 8 : 2(8) = 8 + 8 \tag{2}$$
$$16 = 8 + 8 \tag{3}$$

Now, in the equation 1, adding and subtracting i from the right-hand side will not change the value of left-hand side ($2k$), hence, the equation 1 can be written as,

$$2k = k + k - i + i \tag{4}$$
$$2k = (k - i) + (k + i) \tag{5}$$

Hence, for $n = 8$, the equation 4 and 5 will be,

$$2(8) = 8 + 8 - i + i \tag{6}$$
$$16 = (8 - i) + (8 + i) \tag{7}$$

Now, in the equation 7, the value of i, we cannot take greater than 8, because, doing so, makes the term $(8 - i)$ negative. And also, we do not need to take the value of i equal to 8, which makes the term $(8 - i) = 0$. Hence, we need to take the values of i from 0... to ...$n - 1$, that is, I = { 0,1,2,3,4,5,6,7 }, so for each value of i, equation 7 can be written as,

$$for\ i = 0 :\ 16 = (8 - 0) + (8 + 0) = 8 + 8 \tag{8}$$
$$for\ i = 1 :\ 16 = (8 - 1) + (8 + 1) = 7 + 9 \tag{9}$$
$$for\ i = 2 :\ 16 = (8 - 2) + (8 + 2) = 6 + 10 \tag{10}$$
$$for\ i = 3 :\ 16 = (8 - 3) + (8 + 3) = 5 + 11 \tag{11}$$
$$for\ i = 4 :\ 16 = (8 - 4) + (8 + 4) = 4 + 12 \tag{12}$$
$$for\ i = 5 :\ 16 = (8 - 5) + (8 + 5) = 3 + 13 \tag{13}$$
$$for\ i = 6 :\ 16 = (8 - 6) + (8 + 6) = 2 + 14 \tag{14}$$
$$for\ i = 7 :\ 16 = (8 - 7) + (8 + 7) = 1 + 15 \tag{15}$$

Hence, from the above example, if we need to generate all the pairs in which the given even integer can be written as sum of two numbers can be given in the following expression,

$$2(k) = f(k) = k \pm (i),\ where,\ i = 0, 1, 2, 3, ...k - 1 \tag{16}$$

Further, it is noted that, the value of i starts from $i = 0$ and ends at $i = k - 1$, hence there are total $k - 1 + 1$ (plus 1 for the item 0) items in the set I, and $k - 1 + 1 = k$. Hence, it can be stated that, there will be k pairs of numbers in which even integer can be written as sum of two integers. Moreover, the distance (d) between the each pair will be $d = 2(i)$, as $d = (k + i) - (k - i) = i + 1 = 2(i)$. Hence, these facts can be put as a statement in the following way.

Statement 3: If $k = 1, 2, 3, ...$, there are total k pairs of integers to write $2(k)$ as sum of two integers, and all the pairs are obtained by the function $f(k)$,
$2(k) \ = \ f(k) \ = k \pm (i), \ where, \ i = 0, 1, 2, 3, ..., k - 1.$
And, the distance (d $value$) between each pairs is $d = (k + i) - (k - i) = 2(i)$.

Further, from the **statement 3**, the **statement 3a** can be derived as below.

Statement 3a: Based on **Statement 3**, all the pairs to write the $2(k)$ as sum of two integers are always located at equal distance from the k.

Further, it can also be understand that, **Figure 1**, is nothing but the graphical representation of the **Statement 3** for the even integer $2k = 2(8) = 16$. Further, from the **equation 16**, it is also true that, $(k - i) < k < (k + i)$.

Note 2: If $k = 2, 3, 4, ...$ then for any value of k, the expression $(k-i) < k < (k+i)$ is always true for all the values of i, where $i = 0, 1, 2, 3, ..., k - 1$.

Hence, from the **Statement 2/3** and **Note 2**, the Goldbach conjecture can also be written as,

Statement 4: If $k = 2, 3, 4, ...$ then for every integer $2k$, there is always existing atleast one i from the values $0, 1, 2, 3, ..., (k - 1)$ such that, $(k - i) = p < (k) < (k + i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers.

Now, the definition of Goldbach prime pairs can also be derived from the **Statement 4** as below,

Definition 1: Goldbach prime pairs (G_{2k}): If $k = 2, 3, 4, ...$ then for integer $2k$, there is always existing atleast one or more i from $0, 1, 2, 3, ..., (k-1)$ such that, $(k - i) = p < (k) < (k + i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pairs p, p' are defined as the Goldbach prime pairs for $2k$.

Now, from the **Definition 1**, it also implies that, for each Goldbach prime pair for integer $2k$, there will be only one and one i value associated with that particular Goldbach prime pair. Hence, it also implies that, for value $2k$, number of Goldbach prime pairs will equal to number of i values, $where$, $(k - i) = p$ and $(k + i) = p'$.

Further, we can put this discussion in the form of Definition as below,

Definition 2: Total number of Goldbach prime pairs (G^T_{2k}): If $k = 2, 3, 4, ...$ then for integer $2k$, there is always existing atleast one or more i from $0, 1, 2, 3, ..., (k-1)$ such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pairs p, p' are defined as the Goldbach prime pairs. And, Total Number of Goldbach prime pairs (G^T_{2k}) = Total number of i value for which Goldbach prime pairs are existing.

Now, to define the Goldbach first prime pair (G^f_{2k}) for even integer $2k$, lets see the **Figure 1** again. We knew that for integer 16, there are two Goldbach prime pairs, that is, $(5 + 11)$ and $(3 + 13)$. The $(5 + 11)$ is associated with the i value 3 and $(3 + 13)$ is associated with the i value 5 as per the **Definition 1**. There is no Goldbach prime pair for i values less then 3. Hence, the Goldbach prime pair which is associated with the least i value among all the Goldbach prime pair for that particular even integer is defined as the Goldbach first prime pair (G^f_{2k}). In other words, the Goldbach prime pairs which is most near to the k value on the number-line is defined as the Goldbach first prime pair. In another example, as shown in the **Figure 3**, integer 50 has total 4 Goldbach prime pairs.

Definition 3: Goldbach first prime pair (G^f_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, there is always existing atleast one or more i from $1, 2, 3, ..., (k-1)$, such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pair p, p' which is associated with the lowest i value (note: $i \neq 0$) is defined as Goldbach first prime pair (G^f_{2k}). In other words, the Goldbach first prime pair is the prime pair which has the lowest d *value*, that is, $d = (k+i) - (k-i) = 2(i)$. To distinguished it, this pair is denoted as (p_f, p'_f).

For example, as shown in **Figure 3**, for the integer 50, the total Goldbach prime pairs are 4, that is, $(19+31), (13+37), (7+43), (3+47)$. Among them, the d *value* for $(19 + 31)$ is $d = 31 - 19 = 12$, which is lowest among all the Goldbach prime pairs. Hence for integer 50, the Goldbach first prime pair is $(19 + 31)$. While, $(3 + 47)$ is the last Goldbach prime pair.

Note 3: All the integers, $2k$, where the k is the prime numbers, will automatically becomes sum of two prime numbers, for example, integer, $26 = 2(13) = 13 + 13$. In this article, we are not counting these pairs as way of writing the even integer as sum of two prime numbers, and has not been counted as Goldbach prime pair or Goldbach first prime pair as these are the obvious prime pairs. Hence, in **Definition 3**, $i \neq 0$ is taken and k value was started from the 4 as 4 and 6 has the only one Goldbach prime pair, that is, $6 = (3 + 3)$, $4 = (2 + 2)$.

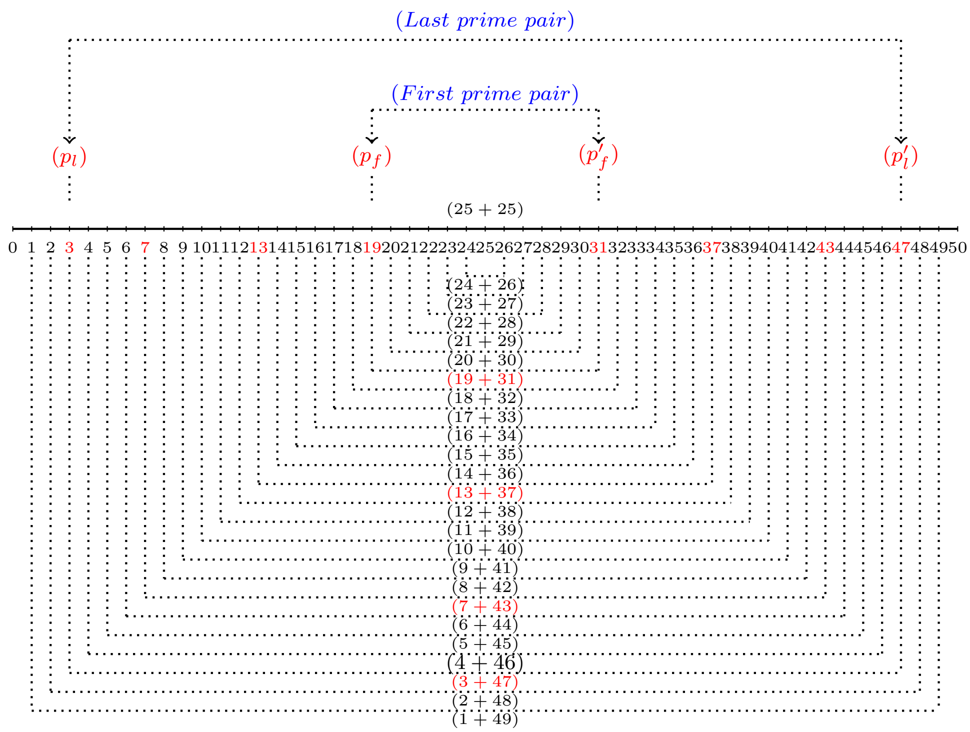

Figure 3: Goldbach prime pairs and Goldbach first prime pair for integer "50".

Now, as per the **Statement 3** and **Definition 1**, all the pair of writing $2(k)$ as sum of two integers will be located in the range 1 to $2k$ only, as putting $i = k - 1$ in the **expression 17** will give this range, where the value of $k - 1$ gives the very last pairs, that will always be $(1, 2k - 1)$. As previously stated, this last pair is not considered as the prime pairs as 1 is not a prime.

$$(k - i) < \quad (k) \quad < (k + i) \tag{17}$$

$$(k - (k - 1)) < \quad (k) \quad < (k + (k - 1)) \tag{18}$$

$$1 < \quad (k) \quad < (2k - 1) \tag{19}$$

Hence, we can write that, all the Goldbach prime pairs is located in the range of 1 to 2k. Hence, the Goldbach first prime pair will also be between these range/interval $(1 \ to \ 2k)$ only.

Definition 4: Distribution of Goldbach prime pair (G_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, all the Goldbach prime pairs are located in the range/interval of $1 < (k) < 2k$.

Definition 5: Distribution of Goldbach first prime pair (G^f_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, the Goldbach first prime pair is located the range of $1 < (k) < 2k$.

Now, **Definitions 4/5**, have been derived for all the even integers greater than 7 as we have taken $2(k)$, *where*, $k = 4, 5, 6, ...$ Hence all the above is true for all the even numbers or any set of even numbers or any selected even numbers that is greater than 7 on the number-line. For example, instead of $2k$, if we take $2(n)^2$, *where*, $n = 2, 3, 4, ...$ that is nothing but even integer $8, 18, 32, ...$ Broadly, $2k = 2(Integer)$, and $2(Integer)$ will always gives the even integer. Hence, we can put any mathematical expression at the place of k, whose value is from $k = 4, 5, 6, ...$ For example, all the previous statements and definitions are true to following expression of k as well, for example, $k = n \ or \ (n + 1) \ or \ (n + 1)^2$ etc.

$$for\ n = 4, 5, 6, ... : \ 2(n) \ \Rightarrow \ is\ even\ integer \tag{20}$$

$$hence,\ for\ n = 3, 4, 5, ... : \ 2(n + 1) \ \Rightarrow \ is\ even\ integer \tag{21}$$

$$and\ for\ n = 1, 2, 3, 4, ... : \ 2(n + 1)^2 \ \Rightarrow \ is\ even\ integer \tag{22}$$

Now, we can write $2(n+1)^2 = 2[n^2 +2(n)+1] = 2k$, where k is from $k = 2, 3, 4, ...$ only, hence, $2(n + 1)^2$ is nothing but even integer.
Hence, from the **Definition 5**, the distribution of the Goldbach first prime pair for even integers of form $2(n + 1)^2$, *where*, $n = 1, 2, 3, ...$ can be shown as,

$$1 < p < \quad (k) \quad < p'_f < 2k \tag{23}$$

$$1 < p_f < \ (n + 1)^2 \ < p'_f < 2(n + 1)^2 \tag{24}$$

Hence, as per the equation 24, the distribution of the Goldbach first prime pair for $2(n + 1)^2$ can be given as per the **Definition 6**.

Definition 6: Distribution of Goldbach first prime pair for $2(n + 1)^2$ ($G^f_{2(n+1)^2}$): If $n = 1, 2, 3, 4, ...$ then for integer $2(n + 1)^2$, the Goldbach first prime pair will be in the range of $1 < p_f < (n + 1)^2 < p'_f < 2(n + 1)^2$.

2.2 Legendre's conjecture

As mentioned in the introduction section, the Legendre's conjecture states that, *there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n.* Hence, we can write as,

$$(n)^2 < p < (n + 1)^2, where\ n = 1, 2, 3, ... \tag{25}$$

2.3 Bertrand's postulate

The Bertrand's postulate, which has been proven and the less restrictive form states that *for any* $n > 1$, *there is always at least one prime* p *such that* $n < p < 2n$. Hence,

$$(n) < p < 2(n), \ where \ n = 1, 2, 3, ... \tag{26}$$

Now, if we put the value of n as $2(n + 1)^2$, hence,

$$(n + 1)^2 < p < 2(n + 1)^2, \ where \ n = 1, 2, 3, ... \tag{27}$$

3 Establishing inter-connection between three conjecture(/postulate)

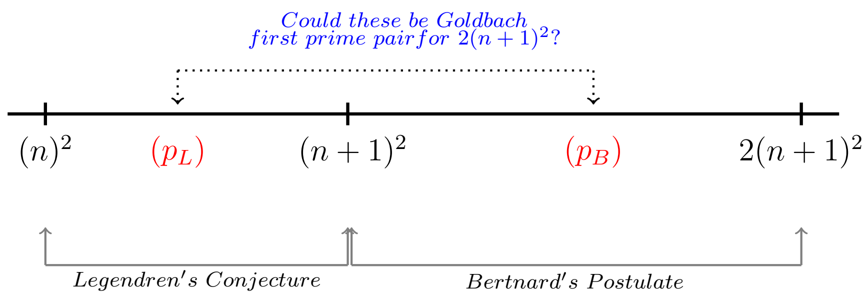

Figure 4: Schematic illustration of the inter-connection of the Legendren's conjectures, Bertnard's postulate, and Goldbach conjecture.

Now, in above mentioned three equations (equation, 24, 25, 27) we can clearly see, in these expressions, the term $(n+1)^2$ is common, hence we can make the common expression as shown in Figure 4.
Re-writing all equations.

$$\begin{aligned} Goldbach\ conjecture \ &: \ 1 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 \\ Legendre's\ conjecture \ &: \ (n)^2 < p_L < (n+1)^2 \\ Bertrand's\ postulate \ &: \ (n+1)^2 < p_B < 2(n+1)^2 \\ where\ n = 1, 2, 3, ... \end{aligned}$$

Hence, lets first write, Legendre's and Bertrand's conjecture as shown below and lets compare it with the Goldbach's expression.

$$\begin{aligned} Legendre's\ +\ Bertrand's &: \quad (n)^2 < p_L < (n+1)^2 < p_B < 2(n+1)^2 \\ Goldbach\ conjecture &: \quad 1 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 \end{aligned}$$

Hence, above expressions will be written in one combined expression as below,

$$(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 \tag{28}$$

Now, as shown in the **Definition 3**, $(k-i) = p_f\ and\ (k+i) = p'_f$, Hence, if the p_f will be in the range, $(n)^2 < p_f < (n+1)^2$, then p'_f will be in the range $(n+1)^2 + (n+1)^2 - (n)^2 = 2(n+1)^2 - (n)^2$, Hence replacing $2(n+1)^2$ with the $2(n+1)^2 - (n)^2$ as we do not need to extend the interval till $2(n+1)^2$.
Hence, equation 28 could be written as follows,

$$\begin{aligned} (n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2 \\ where,\ n = 1, 2, 3, 4, ... \end{aligned} \tag{29}$$

4 Analysis Part 1: Checking the validity of the conjectures

4.1 For expression: $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$

The expression of **Equation 29**, has been checked till $n = 1,000,000$, using the code and results have been outlined in the **Annexure II**. Out of 1,000,000 values following 9 values were found to be falling out of the interval.

$$n = 1, 2, 3, 4... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\} \tag{30}$$

Hence, the above results have been checked till $n = 1,000,000$, which has been found true after 141 and till 1,000,000, and may be true beyond $n = 1,000,000$. Hence, we can formulate the conjecture based on these results as **Conjecture 2**.

Conjecture 2: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\}$.

4.2 For expression: $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$

Further, since total 9 values were found to be deviating in conjecture 2, the interval was slightly widened in hope to incorporate these values by changing the left-hand side interval from $(n)^2$ to $(n)(n+1)$. Hence, the expression of the conjecture is to be written as,

$$(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1) \quad \text{where, } n = 1, 2, 3, 4, ... \tag{31}$$

The expression of the equation 31 was checked for the n=1,000,000, and none of the values were falling out of the interval (**Annexure III**). Hence, the equation 31 is to be written in the form of conjecture as below,

Conjecture 3: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ...$

4.3 For expression: $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$

Further, from the observation of the Conjecture 2/3, it was observed that, beyond one particular n value, the conjecture holds true. Hence, the question arises, can we shorten the interval more around the $(n+1)^2$, although with the expense of some initial n values which falls outside these interval, but becomes true for all the number n thereafter. Hence, the interval $(n)^2$, changed to $(n)(n+1)$, and the expression can be written as below,

$$(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)^2 \quad \text{where, } n = 1, 2, 3, 4, ... \tag{32}$$

The expression of the equation 32 was checked for the n=1,000,000, and the 32 values were found to be falling out of the range (**Annexure I**), $n = 1, 2, 3, 4... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418.\}$ While, from the 419, the equation was found to be true. Hence, in the form of the conjecture, it can be written as below,

Conjecture 1: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ...$ $- \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$.

5 Analysis Part 2: Analysis of Gap $(p'_f - p_f)$

In the previous part systemic study has been carried out to study the distribution of the goldbach first prime pair. The interval of the proposed Conjecture 1 is $(n+1)^2 \pm (n+1)$, while for the conjecture 2, it is $(n+1)^2 \pm (2n+1)$ and for conjecture 3, it is $(n+1)^2 \pm (3n+1)$. Further, from these results, we found that, if we tighten the interval around $(n+1)^2$, few n values falls out of the interval, however, after one particular n value, the conjecture becomes true, as we have seen previously. Now, with the objective of even further narrow-down the interval, the interval $(n+1)^2 \pm (n/2)$ were tried. In other words, it can be stated that, the $Gap(p'_f - p_f)$ is lower than the corresponding value of n or not. Hence, the following expression were tested using computer code,

$$Gap\ (p'_f - p_f) < n, where,\ n = 1, 2, 3, 4, ... \tag{33}$$

As, shown in the **Annexure IV**, after the $n = 2539$, the $Gap(p'_f - p_f)$ is found to be less than the corresponding value of n. Hence, the Hypothesis 1 was proposed, which has been tested till $n = 1,000,000$ and found to be true.

Hypothesis 1: *The Gap $(p'_f - p_f)$ between the Goldbach first prime pair for the even integer $2(n+1)^2$ is always less than the corresponding value of n, where,* $n = 2539, 2540, 2541, ...$

6 Limitation

The present study has been done on very limited number of n values, that is, $1,000,000$ due to the limited computation power. Further, as mentioned in the introduction section, this study is focused on the even integers of form $2(n+1)^2$, not all the form of integers.

7 Future Perspective

This study can be extended for the other power value as well. For example, in the previous sections, we have seen for the $Power = 2$. In the following table, the Hypothesis 1 has been checked for $Power = 2, 3, ..., 10$. Here, we can see, as the power is increasing the value of the n from where the expression, $\boldsymbol{Gap}(p'_f - p_f)$, holds true is gradually increasing.

Table 1: Data for proposed Hypothesis 1 with various power in equation $2(n+1)^{Power}$

Power	$\boldsymbol{Gap}\ (p'_f - p_f) < n$
$2(n+1)^2$	$n = 2539, 2540, 2541, ...1,000,000$
$2(n+1)^3$	$n = 8614, 8615, 8616, ...1,000,000$
$2(n+1)^4$	$n = 20869, 20870, 20871, ...1,000,000$
$2(n+1)^5$	$n = 32072, 32073, 32074, ...1,000,000$
$2(n+1)^6$	$n = 72593, 72594, 72595, ...1,000,000$
$2(n+1)^7$	$n = 81440, 81441, 81442, ...1,000,000$
$2(n+1)^8$	$n = 131482, 131483, 131484, ...1,000,000$
$2(n+1)^9$	$n = 160744, 160745, 160746, ...1,000,000$
$2(n+1)^{10}$	$n = 257987, 257988, 257989, ...1,000,000$
.	.
.	.
$2(n+1)^n$	$n = ?, ?, ?, ..., n$

For example, for $Power = 2$, the value of n from where the expression starts true is 2539, while for the $power = 3$, the value of n is 8614. How these values are increasing as the power increasing and what could be for the $power = n$, that is, $2(n+1)^n$. Hence, this could be different set of study, the relationship of power and the first prime pair gap.

References

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Annexure I
(For Conjecture 1)

Conjecture 1: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$ and $p + p'_f = 2(n+1)^2$ and $(n+1)^2 - p = p'_f - (n+1)^2$, where, $n = 1, 2, 3, 4, ...$ $- \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106,$ $117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$.

Note: $G^{Interval}_{2(n+1)^2}$ means the number of Goldbach prime pairs present in that particular "*Interval*". For example, $G^{Interval}_{2(n+1)^2}$ for the interval $(n+1)^2 \pm (n+1)$ means that, number of Goldbach prime pairs present in the interval $(n+1)^2 \pm (n+1)$.

Table 2: Data for proposed conjecture 1

n	*n(n+1)*	*p*	*(n+1)*^{2}	*p'*	*2(n+1)*^{2} − *n(n+1)*	*2(n+1)*^{2}	*p + p'*	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
1	2	3	4	5	6	8	8	1	1
2	6	7	9	11	12	18	18	2	1
3	12	13	16	19	20	32	32	2	1
4	20	19	25	31	30	50	50	4	0
5	30	31	36	41	42	72	72	6	1
6	42	37	49	61	56	98	98	3	0
7	56	61	64	67	72	128	128	3	1
8	72	79	81	83	90	162	162	10	2
9	90	97	100	103	110	200	200	8	1
10	110	103	121	139	132	242	242	8	0
11	132	139	144	149	156	288	288	17	2
12	156	157	169	181	182	338	338	9	1
13	182	193	196	199	210	392	392	11	1
14	210	223	225	227	240	450	450	27	2
15	240	241	256	271	272	512	512	11	1
16	272	271	289	307	306	578	578	12	0
17	306	317	324	331	342	648	648	27	2
18	342	349	361	373	380	722	722	14	1
19	380	379	400	421	420	800	800	21	0
20	420	439	441	443	462	882	882	39	3
21	462	421	484	547	506	968	968	17	0
22	506	487	529	571	552	1058	1058	19	0
23	552	521	576	631	600	1152	1152	36	0
24	600	619	625	631	650	1250	1250	28	2
25	650	661	676	691	702	1352	1352	22	1
26	702	719	729	739	756	1458	1458	48	1
27	756	757	784	811	812	1568	1568	25	1
28	812	829	841	853	870	1682	1682	24	2
29	870	881	900	919	930	1800	1800	75	1
30	930	883	961	1039	992	1922	1922	30	0
31	992	1009	1024	1039	1056	2048	2048	25	2
32	1056	1087	1089	1091	1122	2178	2178	68	3
33	1122	1063	1156	1249	1190	2312	2312	35	0

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Table 2: Data for proposed conjecture 1 (contd...)

n	$n(n+1)$	p	$(n+1)^2$	p'	$2(n+1)^2-n(n+1)$	$2(n+1)^2$	$p+p'$	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
34	1190	1213	1225	1237	1260	2450	2450	56	2
35	1260	1291	1296	1301	1332	2592	2592	68	2
36	1332	1291	1369	1447	1406	2738	2738	37	0
37	1406	1429	1444	1459	1482	2888	2888	40	1
38	1482	1511	1521	1531	1560	3042	3042	93	5
39	1560	1579	1600	1621	1640	3200	3200	54	1
40	1640	1669	1681	1693	1722	3362	3362	43	2
41	1722	1741	1764	1787	1806	3528	3528	103	1
42	1806	1831	1849	1867	1892	3698	3698	42	1
43	1892	1879	1936	1993	1980	3872	3872	52	0
44	1980	2011	2025	2039	2070	4050	4050	125	3
45	2070	2089	2116	2143	2162	4232	4232	51	1
46	2162	2179	2209	2239	2256	4418	4418	49	1
47	2256	2297	2304	2311	2352	4608	4608	117	3
48	2352	2281	2401	2521	2450	4802	4802	64	0
49	2450	2383	2500	2617	2550	5000	5000	76	0
50	2550	2593	2601	2609	2652	5202	5202	130	1
51	2652	2689	2704	2719	2756	5408	5408	63	3
52	2756	2767	2809	2851	2862	5618	5618	56	1
53	2862	2879	2916	2953	2970	5832	5832	135	1
54	2970	3001	3025	3049	3080	6050	6050	99	2
55	3080	3109	3136	3163	3192	6272	6272	78	1
56	3192	3191	3249	3307	3306	6498	6498	151	0
57	3306	3271	3364	3457	3422	6728	6728	76	0
58	3422	3463	3481	3499	3540	6962	6962	73	2
59	3540	3593	3600	3607	3660	7200	7200	198	4
60	3660	3709	3721	3733	3782	7442	7442	74	2
61	3782	3769	3844	3919	3906	7688	7688	78	0
62	3906	3931	3969	4007	4032	7938	7938	197	4
63	4032	4093	4096	4099	4160	8192	8192	76	1
64	4160	4219	4225	4231	4290	8450	8450	130	2
65	4290	4349	4356	4363	4422	8712	8712	195	2
66	4422	4357	4489	4621	4556	8978	8978	89	0
67	4556	4597	4624	4651	4692	9248	9248	98	2
68	4692	4733	4761	4789	4830	9522	9522	199	4
69	4830	4831	4900	4969	4970	9800	9800	147	1
70	4970	5023	5041	5059	5112	10082	10082	99	1
71	5112	5179	5184	5189	5256	10368	10368	204	2
72	5256	5227	5329	5431	5402	10658	10658	105	0
73	5402	5449	5476	5503	5550	10952	10952	106	2
74	5550	5591	5625	5659	5700	11250	11250	286	3
75	5700	5701	5776	5851	5852	11552	11552	111	1
76	5852	5851	5929	6007	6006	11858	11858	142	0
77	6006	6079	6084	6089	6162	12168	12168	244	4
78	6162	6211	6241	6271	6320	12482	12482	121	1
79	6320	6379	6400	6421	6480	12800	12800	159	2
80	6480	6553	6561	6569	6642	13122	13122	245	2
81	6642	6619	6724	6829	6806	13448	13448	130	0
82	6806	6871	6889	6907	6972	13778	13778	120	2
83	6972	7043	7056	7069	7140	14112	14112	306	3
84	7140	7213	7225	7237	7310	14450	14450	183	2
85	7310	7333	7396	7459	7482	14792	14792	136	1
86	7482	7561	7569	7577	7656	15138	15138	280	6
87	7656	7699	7744	7789	7832	15488	15488	153	1

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Table 2: Data for proposed conjecture 1 (contd. . .)

n	***n(n+1)***	***p***	***(n+1)***^{2}	***p'***	***2(n+1)***^{2}***– n(n+1)***	***2(n+1)***^{2}	***p + p'***	$G^{T}_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
88	7832	7879	7921	7963	8010	15842	15842	142	1
89	8010	8089	8100	8111	8190	16200	16200	384	3
90	8190	8269	8281	8293	8372	16562	16562	180	3
91	8372	8461	8464	8467	8556	16928	16928	143	2
92	8556	8629	8649	8669	8742	17298	17298	302	3
93	8742	8779	8836	8893	8930	17672	17672	161	1
94	8930	9007	9025	9043	9120	18050	18050	211	3
95	9120	9151	9216	9281	9312	18432	18432	313	1
96	9312	9397	9409	9421	9506	18818	18818	161	1
97	9506	9547	9604	9661	9702	19208	19208	185	2
98	9702	9791	9801	9811	9900	19602	19602	363	4
99	9900	9931	10000	10069	10100	20000	20000	231	3
100	10100	10159	10201	10243	10302	20402	20402	172	1
101	10302	10331	10404	10477	10506	20808	20808	371	2
102	10506	10567	10609	10651	10712	21218	21218	176	2
103	10712	10771	10816	10861	10920	21632	21632	196	3
104	10920	11003	11025	11047	11130	22050	22050	568	5
105	11130	11173	11236	11299	11342	22472	22472	189	2
106	11342	11311	11449	11587	11556	22898	22898	184	0
107	11556	11597	11664	11731	11772	23328	23328	370	1
108	11772	11839	11881	11923	11990	23762	23762	194	2
109	11990	12043	12100	12157	12210	24200	24200	290	2
110	12210	12269	12321	12373	12432	24642	24642	412	4
111	12432	12541	12544	12547	12656	25088	25088	236	4
112	12656	12757	12769	12781	12882	25538	25538	201	3
113	12882	12983	12996	13009	13110	25992	25992	437	5
114	13110	13183	13225	13267	13340	26450	26450	285	2
115	13340	13399	13456	13513	13572	26912	26912	208	1
116	13572	13687	13689	13691	13806	27378	27378	469	7
117	13806	13627	13924	14221	14042	27848	27848	217	0
118	14042	14149	14161	14173	14280	28322	28322	270	2
119	14280	14389	14400	14411	14520	28800	28800	593	4
120	14520	14629	14641	14653	14762	29282	29282	251	2
121	14762	14821	14884	14947	15006	29768	29768	239	1
122	15006	15121	15129	15137	15252	30258	30258	460	3
123	15252	15361	15376	15391	15500	30752	30752	251	3
124	15500	15607	15625	15643	15750	31250	31250	326	4
125	15750	15761	15876	15991	16002	31752	31752	564	1
126	16002	16069	16129	16189	16256	32258	32258	243	1
127	16256	16249	16384	16519	16512	32768	32768	244	0
128	16512	16633	16641	16649	16770	33282	33282	513	4
129	16770	16879	16900	16921	17030	33800	33800	359	1
130	17030	17029	17161	17293	17292	34322	34322	267	0
131	17292	17417	17424	17431	17556	34848	34848	561	4
132	17556	17551	17689	17827	17822	35378	35378	316	0
133	17822	17923	17956	17989	18090	35912	35912	263	3
134	18090	18217	18225	18233	18360	36450	36450	710	7
135	18360	18451	18496	18541	18632	36992	36992	293	2
136	18632	18679	18769	18859	18906	37538	37538	276	1
137	18906	19037	19044	19051	19182	38088	38088	586	4
138	19182	19309	19321	19333	19460	38642	38642	274	3
139	19460	19597	19600	19603	19740	39200	39200	450	3
140	19740	19843	19881	19919	20022	39762	39762	595	4
141	20022	19819	20164	20509	20306	40328	40328	283	0

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Table 2: Data for proposed conjecture 1 (contd...)

n	***n(n+1)***	***p***	***(n+1)***^{2}	***p'***	***2(n+1)***^{2}***– n(n+1)***	***2(n+1)***^{2}	***p + p'***	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
142	20306	20389	20449	20509	20592	40898	40898	350	2
143	20592	20719	20736	20753	20880	41472	41472	589	4
144	20880	21019	21025	21031	21170	42050	42050	410	4
145	21170	21313	21316	21319	21462	42632	42632	309	1
146	21462	21601	21609	21617	21756	43218	43218	739	7
147	21756	21871	21904	21937	22052	43808	43808	316	3
148	22052	22129	22201	22273	22350	44402	44402	313	3
149	22350	22469	22500	22531	22650	45000	45000	839	3
150	22650	22741	22801	22861	22952	45602	45602	315	1
151	22952	23041	23104	23167	23256	46208	46208	328	2
152	23256	23371	23409	23447	23562	46818	46818	677	4
153	23562	23689	23716	23743	23870	47432	47432	442	4
154	23870	24007	24025	24043	24180	48050	48050	442	5
155	24180	24281	24336	24391	24492	48672	48672	735	4
156	24492	24517	24649	24781	24806	49298	49298	346	2
157	24806	24841	24964	25087	25122	49928	49928	345	1
158	25122	25261	25281	25301	25440	50562	50562	718	6
159	25440	25579	25600	25621	25760	51200	51200	468	3
160	25760	25903	25921	25939	26082	51842	51842	442	3
161	26082	26237	26244	26251	26406	52488	52488	712	4
162	26406	26497	26569	26641	26732	53138	53138	337	3
163	26732	26839	26896	26953	27060	53792	53792	373	2
164	27060	27211	27225	27239	27390	54450	54450	1113	4
165	27390	27529	27556	27583	27722	55112	55112	381	2
166	27722	27817	27889	27961	28056	55778	55778	376	2
167	28056	28219	28224	28229	28392	56448	56448	897	4
168	28392	28549	28561	28573	28730	57122	57122	406	3
169	28730	28879	28900	28921	29070	57800	57800	539	2
170	29070	29231	29241	29251	29412	58482	58482	807	2
171	29412	29581	29584	29587	29756	59168	59168	411	3
172	29756	29761	29929	30097	30102	59858	59858	407	1
173	30102	30259	30276	30293	30450	60552	60552	813	4
174	30450	30553	30625	30697	30800	61250	61250	644	3
175	30800	30871	30976	31081	31152	61952	61952	439	2
176	31152	31321	31329	31337	31506	62658	62658	831	6
177	31506	31627	31684	31741	31862	63368	63368	414	1
178	31862	31963	32041	32119	32220	64082	64082	413	2
179	32220	32377	32400	32423	32580	64800	64800	1135	8
180	32580	32719	32761	32803	32942	65522	65522	409	3
181	32942	33049	33124	33199	33306	66248	66248	556	2
182	33306	33457	33489	33521	33672	66978	66978	875	7
183	33672	33751	33856	33961	34040	67712	67712	437	2
184	34040	34183	34225	34267	34410	68450	68450	588	3
185	34410	34589	34596	34603	34782	69192	69192	913	7
186	34782	34849	34969	35089	35156	69938	69938	518	1
187	35156	35281	35344	35407	35532	70688	70688	465	3
188	35532	35671	35721	35771	35910	71442	71442	1104	5
189	35910	36013	36100	36187	36290	72200	72200	644	3
190	36290	36469	36481	36493	36672	72962	72962	480	3
191	36672	36857	36864	36871	37056	73728	73728	929	5
192	37056	37189	37249	37309	37442	74498	74498	456	3
193	37442	37579	37636	37693	37830	75272	75272	471	3
194	37830	38011	38025	38039	38220	76050	76050	1394	7
195	38220	38371	38416	38461	38612	76832	76832	578	2

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Table 2: Data for proposed conjecture 1 (contd...)

n	***n(n+1)***	***p***	***(n+1)***^{2}	***p'***	***2(n+1)***^{2}***− n(n+1)***	***2(n+1)***^{2}	***p + p'***	$G^{T}_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
196	38612	38767	38809	38851	39006	77618	77618	494	1
197	39006	39199	39204	39209	39402	78408	78408	1110	6
198	39402	39499	39601	39703	39800	79202	79202	494	1
199	39800	39937	40000	40063	40200	80000	80000	652	4
200	40200	40343	40401	40459	40602	80802	80802	1019	2
201	40602	40759	40804	40849	41006	81608	81608	509	1
202	41006	41161	41209	41257	41412	82418	82418	640	3
203	41412	41611	41616	41621	41820	83232	83232	1099	2
204	41820	41911	42025	42139	42230	84050	84050	694	3
205	42230	42409	42436	42463	42642	84872	84872	520	3
206	42642	42839	42849	42859	43056	85698	85698	1107	6
207	43056	43237	43264	43291	43472	86528	86528	577	3
208	43472	43651	43681	43711	43890	87362	87362	625	4
209	43890	44089	44100	44111	44310	88200	88200	1758	9
210	44310	44269	44521	44773	44732	89042	89042	544	0
211	44732	44917	44944	44971	45156	89888	89888	554	1
212	45156	45361	45369	45377	45582	90738	90738	1125	4
213	45582	45751	45796	45841	46010	91592	91592	572	2
214	46010	46171	46225	46279	46440	92450	92450	787	4
215	46440	46649	46656	46663	46872	93312	93312	1145	4
216	46872	47059	47089	47119	47306	94178	94178	705	4
217	47306	47521	47524	47527	47742	95048	95048	568	3
218	47742	47843	47961	48079	48180	95922	95922	1189	3
219	48180	48337	48400	48463	48620	96800	96800	874	3
220	48620	48823	48841	48859	49062	97682	97682	684	5
221	49062	49261	49284	49307	49506	98568	98568	1247	9
222	49506	49711	49729	49747	49952	99458	99458	600	5
223	49952	50131	50176	50221	50400	100352	100352	718	3
224	50400	50599	50625	50651	50850	101250	101250	1612	7
225	50850	51043	51076	51109	51302	102152	102152	611	3
226	51302	51481	51529	51577	51756	103058	103058	612	3
227	51756	51977	51984	51991	52212	103968	103968	1302	5
228	52212	52321	52441	52561	52670	104882	104882	621	1
229	52670	52837	52900	52963	53130	105800	105800	891	3
230	53130	53281	53361	53441	53592	106722	106722	1682	4
231	53592	53791	53824	53857	54056	107648	107648	653	3
232	54056	54217	54289	54361	54522	108578	108578	647	1
233	54522	54713	54756	54799	54990	109512	109512	1393	5
234	54990	55207	55225	55243	55460	110450	110450	896	5
235	55460	55681	55696	55711	55932	111392	111392	672	2
236	55932	56167	56169	56171	56406	112338	112338	1355	5
237	56406	56629	56644	56659	56882	113288	113288	865	5
238	56882	56983	57121	57259	57360	114242	114242	674	4
239	57360	57559	57600	57641	57840	115200	115200	1799	6
240	57840	57991	58081	58171	58322	116162	116162	691	3
241	58322	58549	58564	58579	58806	117128	117128	754	3
242	58806	59029	59049	59069	59292	118098	118098	1369	6
243	59292	59443	59536	59629	59780	119072	119072	700	1
244	59780	60013	60025	60037	60270	120050	120050	1126	3
245	60270	60493	60516	60539	60762	121032	121032	1461	7
246	60762	60961	61009	61057	61256	122018	122018	805	3
247	61256	61381	61504	61627	61752	123008	123008	723	3
248	61752	61991	62001	62011	62250	124002	124002	1424	5
249	62250	62467	62500	62533	62750	125000	125000	956	4

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Table 2: Data for proposed conjecture 1 (contd...)

n	***n(n+1)***	***p***	***(n+1)²***	***p'***	***2(n+1)² − n(n+1)***	***2(n+1)²***	***p + p'***	$G^{T}_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
250	62750	62971	63001	63031	63252	126002	126002	714	4
251	63252	63487	63504	63521	63756	127008	127008	1735	13
252	63756	63781	64009	64237	64262	128018	128018	846	1
253	64262	64453	64516	64579	64770	129032	129032	738	2
254	64770	64997	65025	65053	65280	130050	130050	2148	11
255	65280	65521	65536	65551	65792	131072	131072	749	3
256	65792	65929	66049	66169	66306	132098	132098	756	2
257	66306	66541	66564	66587	66822	133128	133128	1552	5
258	66822	67033	67081	67129	67340	134162	134162	940	7
259	67340	67477	67600	67723	67860	135200	135200	1102	3
260	67860	68071	68121	68171	68382	136242	136242	1591	5
261	68382	68539	68644	68749	68906	137288	137288	779	3
262	68906	69001	69169	69337	69432	138338	138338	794	1
263	69432	69653	69696	69739	69960	139392	139392	1775	4
264	69960	70201	70225	70249	70490	140450	140450	1070	5
265	70490	70729	70756	70783	71022	141512	141512	1001	4
266	71022	71261	71289	71317	71556	142578	142578	1621	5
267	71556	71761	71824	71887	72092	143648	143648	821	2
268	72092	72253	72361	72469	72630	144722	144722	806	3
269	72630	72893	72900	72907	73170	145800	145800	2224	8
270	73170	73189	73441	73693	73712	146882	146882	832	1
271	73712	73951	73984	74017	74256	147968	147968	881	5
272	74256	74527	74529	74531	74802	149058	149058	2190	8
273	74802	74959	75076	75193	75350	150152	150152	824	3
274	75350	75571	75625	75679	75900	151250	151250	1247	3
275	75900	76103	76176	76249	76452	152352	152352	1769	5
276	76452	76441	76729	77017	77006	153458	153458	864	0
277	77006	77191	77284	77377	77562	154568	154568	865	5
278	77562	77783	77841	77899	78120	155682	155682	1821	5
279	78120	78283	78400	78517	78680	156800	156800	1388	5
280	78680	78721	78961	79201	79242	157922	157922	852	2
281	79242	79427	79524	79621	79806	159048	159048	1757	5
282	79806	80071	80089	80107	80372	160178	160178	886	4
283	80372	80629	80656	80683	80940	161312	161312	913	4
284	80940	81157	81225	81293	81510	162450	162450	2556	7
285	81510	81649	81796	81943	82082	163592	163592	1074	3
286	82082	82351	82369	82387	82656	164738	164738	1083	7
287	82656	82891	82944	82997	83232	165888	165888	1815	5
288	83232	83401	83521	83641	83810	167042	167042	973	4
289	83810	83737	84100	84463	84390	168200	168200	1259	0
290	84390	84649	84681	84713	84972	169362	169362	1844	7
291	84972	85159	85264	85369	85556	170528	170528	950	4
292	85556	85669	85849	86029	86142	171698	171698	924	2
293	86142	86381	86436	86491	86730	172872	172872	2261	10
294	86730	87013	87025	87037	87320	174050	174050	1246	3
295	87320	87589	87616	87643	87912	175232	175232	967	7
296	87912	88177	88209	88241	88506	176418	176418	2079	6
297	88506	88801	88804	88807	89102	177608	177608	954	6
298	89102	89371	89401	89431	89700	178802	178802	1092	4
299	89700	89989	90000	90011	90300	180000	180000	2585	7
300	90300	90583	90601	90619	90902	181202	181202	1201	4
301	90902	91159	91204	91249	91506	182408	182408	961	3
302	91506	91807	91809	91811	92112	183618	183618	2008	5
303	92112	92413	92416	92419	92720	184832	184832	1052	4

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Table 2: Data for proposed conjecture 1 (contd...)

n	***n(n+1)***	***p***	***(n+1)^2***	***p'***	***2(n+1)^2− n(n+1)***	***2(n+1)^2***	***p + p'***	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
304	92720	92899	93025	93151	93330	186050	186050	1336	5
305	93330	93553	93636	93719	93942	187272	187272	2147	4
306	93942	94207	94249	94291	94556	188498	188498	1000	4
307	94556	94777	94864	94951	95172	189728	189728	1370	3
308	95172	95479	95481	95483	95790	190962	190962	2056	9
309	95790	96043	96100	96157	96410	192200	192200	1431	5
310	96410	96703	96721	96739	97032	193442	193442	1036	3
311	97032	97301	97344	97387	97656	194688	194688	2277	8
312	97656	97927	97969	98011	98282	195938	195938	1032	3
313	98282	98479	98596	98713	98910	197192	197192	1044	4
314	98910	99191	99225	99259	99540	198450	198450	3369	12
315	99540	99721	99856	99991	100172	199712	199712	1036	4
316	100172	100459	100489	100519	100806	200978	200978	1087	4
317	100806	101107	101124	101141	101442	202248	202248	2174	7
318	101442	101653	101761	101869	102080	203522	203522	1274	2
319	102080	102367	102400	102433	102720	204800	204800	1437	5
320	102720	102983	103041	103099	103362	206082	206082	2168	4
321	103362	103681	103684	103687	104006	207368	207368	1389	5
322	104006	104311	104329	104347	104652	208658	208658	1245	3
323	104652	104953	104976	104999	105300	209952	209952	2214	6
324	105300	105601	105625	105649	105950	211250	211250	1614	6
325	105950	106273	106276	106279	106602	212552	212552	1114	3
326	106602	106921	106929	106937	107256	213858	213858	2272	10
327	107256	107449	107584	107719	107912	215168	215168	1157	2
328	107912	108211	108241	108271	108570	216482	216482	1341	5
329	108570	108893	108900	108907	109230	217800	217800	3361	8
330	109230	109303	109561	109819	109892	219122	219122	1150	2
331	109892	110017	110224	110431	110556	220448	220448	1144	1
332	110556	110879	110889	110899	111222	221778	221778	2393	7
333	111222	111043	111556	112069	111890	223112	223112	1129	0
334	111890	112213	112225	112237	112560	224450	224450	1557	5
335	112560	112771	112896	113021	113232	225792	225792	2792	6
336	113232	113359	113569	113779	113906	227138	227138	1176	3
337	113906	114229	114244	114259	114582	228488	228488	1310	2
338	114582	114901	114921	114941	115260	229842	229842	2413	9
339	115260	115597	115600	115603	115940	231200	231200	1677	7
340	115940	116269	116281	116293	116622	232562	232562	1382	3
341	116622	116959	116964	116969	117306	233928	233928	2537	7
342	117306	117619	117649	117679	117992	235298	235298	1442	8
343	117992	118273	118336	118399	118680	236672	236672	1241	4
344	118680	118967	119025	119083	119370	238050	238050	3415	7
345	119370	119659	119716	119773	120062	239432	239432	1224	4
346	120062	120391	120409	120427	120756	240818	240818	1228	5
347	120756	121039	121104	121169	121452	242208	242208	2573	9
348	121452	121609	121801	121993	122150	243602	243602	1236	3
349	122150	122497	122500	122503	122850	245000	245000	1999	8
350	122850	123143	123201	123259	123552	246402	246402	2724	6
351	123552	123829	123904	123979	124256	247808	247808	1413	4
352	124256	124459	124609	124759	124962	249218	249218	1246	3
353	124962	125303	125316	125329	125670	250632	250632	2592	5
354	125670	126019	126025	126031	126380	252050	252050	1751	5
355	126380	126733	126736	126739	127092	253472	253472	1314	4
356	127092	127301	127449	127597	127806	254898	254898	3288	11
357	127806	127951	128164	128377	128522	256328	256328	1310	4

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Table 2: Data for proposed conjecture 1 (contd. . .)

n	***n(n+1)***	***p***	***(n+1)^2***	***p'***	***2(n+1)^2 − n(n+1)***	***2(n+1)^2***	***p + p'***	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
358	128522	128761	128881	129001	129240	257762	257762	1303	1
359	129240	129593	129600	129607	129960	259200	259200	3470	7
360	129960	130279	130321	130363	130682	260642	260642	1365	3
361	130682	130987	131044	131101	131406	262088	262088	1354	1
362	131406	131759	131769	131779	132132	263538	263538	2979	8
363	132132	132469	132496	132523	132860	264992	264992	1734	8
364	132860	133033	133225	133417	133590	266450	266450	1807	1
365	133590	133949	133956	133963	134322	267912	267912	2758	7
366	134322	134371	134689	135007	135056	269378	269378	1361	2
367	135056	135277	135424	135571	135792	270848	270848	1430	4
368	135792	136133	136161	136189	136530	272322	272322	2835	10
369	136530	136849	136900	136951	137270	273800	273800	1888	5
370	137270	137623	137641	137659	138012	275282	275282	1684	4
371	138012	138319	138384	138449	138756	276768	276768	2845	9
372	138756	138967	139129	139291	139502	278258	278258	1370	4
373	139502	139861	139876	139891	140250	279752	279752	1648	6
374	140250	140611	140625	140639	141000	281250	281250	3717	11
375	141000	141241	141376	141511	141752	282752	282752	1404	3
376	141752	142099	142129	142159	142506	284258	284258	1592	3
377	142506	142871	142884	142897	143262	285768	285768	3410	8
378	143262	143629	143641	143653	144020	287282	287282	1410	5
379	144020	144349	144400	144451	144780	288800	288800	2020	5
380	144780	145109	145161	145213	145542	290322	290322	2903	10
381	145542	145879	145924	145969	146306	291848	291848	1400	5
382	146306	146677	146689	146701	147072	293378	293378	1464	5
383	147072	147409	147456	147503	147840	294912	294912	2897	7
384	147840	148207	148225	148243	148610	296450	296450	2598	9
385	148610	148933	148996	149059	149382	297992	297992	1468	5
386	149382	149767	149769	149771	150156	299538	299538	3008	6
387	150156	150517	150544	150571	150932	301088	301088	1476	3
388	150932	151303	151321	151339	151710	302642	302642	1498	5
389	151710	152077	152100	152123	152490	304200	304200	4339	7
390	152490	152821	152881	152941	153272	305762	305762	1650	3
391	153272	153589	153664	153739	154056	307328	307328	1787	6
392	154056	154439	154449	154459	154842	308898	308898	3023	6
393	154842	155203	155236	155269	155630	310472	310472	1562	3
394	155630	155893	156025	156157	156420	312050	312050	2047	7
395	156420	156799	156816	156833	157212	313632	313632	3404	7
396	157212	157579	157609	157639	158006	315218	315218	1546	4
397	158006	158359	158404	158449	158802	316808	316808	1519	6
398	158802	159193	159201	159209	159600	318402	318402	3951	8
399	159600	159799	160000	160201	160400	320000	320000	2095	3
400	160400	160789	160801	160813	161202	321602	321602	1565	5
401	161202	161569	161604	161639	162006	323208	323208	3212	4
402	162006	162289	162409	162529	162812	324818	324818	1764	4
403	162812	163021	163216	163411	163620	326432	326432	1594	3
404	163620	164011	164025	164039	164430	328050	328050	4223	15
405	164430	164623	164836	165049	165242	329672	329672	2002	3
406	165242	165589	165649	165709	166056	331298	331298	1844	4
407	166056	166457	166464	166471	166872	332928	332928	3466	9
408	166872	167221	167281	167341	167690	334562	334562	1607	5
409	167690	167953	168100	168247	168510	336200	336200	2237	2
410	168510	168899	168921	168943	169332	337842	337842	3269	9
411	169332	169657	169744	169831	170156	339488	339488	1652	3

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Table 2: Data for proposed conjecture 1 (contd...)

n	***n(n+1)***	***p***	***(n+1)^2***	***p'***	***2(n+1)^2− n(n+1)***	***2(n+1)^2***	***p + p'***	$G^{T}_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
412	170156	170497	170569	170641	170982	341138	341138	1990	5
413	170982	171263	171396	171529	171810	342792	342792	3475	7
414	171810	172171	172225	172279	172640	344450	344450	2214	7
415	172640	173053	173056	173059	173472	346112	346112	1778	3
416	173472	173861	173889	173917	174306	347778	347778	3366	7
417	174306	174649	174724	174799	175142	349448	349448	2038	4
418	175142	175129	175561	175993	175980	351122	351122	1670	0
419	175980	176383	176400	176417	176820	352800	352800	5394	18

Annexure II
(For Conjecture 2)

In the Table 2, the n values which falling out of the range are, $\{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$. Hence, as mentioned previously, in second trial, the interval range was widened, to check, whether all the values are falling in the range or not. Hence, we need to check only for those numbers only. As whatever n values are falling in the interval of conjecture 1, will obviously fit in this conjecture as well, the interval of this conjecture is more wider than the previous conjecture and value of the n are same. The data are shown in the Table 3.

Conjecture 2: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ and $p + p'_f = 2(n+1)^2$ and $(n+1)^2 - p = p'_f - (n+1)^2$, where, $n = 1, 2, 3, 4... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\}$.

Table 3: Data for proposed conjecture 2

n	n^2	p	$(n+1)^2$	p'	$2(n+1)^2 - n^2$	$2(n+1)^2$	$p + p'$	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
21	441	421	484	547	527	968	968	17	0
23	529	521	576	631	623	1152	1152	36	0
30	900	883	961	1039	1022	1922	1922	30	0
33	1089	1063	1156	1249	1223	2312	2312	35	0
36	1296	1291	1369	1447	1442	2738	2738	37	0
48	2304	2281	2401	2521	2498	4802	4802	64	0
49	2401	2383	2500	2617	2599	5000	5000	76	0
117	13689	13627	13924	14221	14159	27848	27848	217	0
141	19881	19819	20164	20509	20447	40328	40328	283	0

Annexure III
(For Conjecture 3)

In the Table 3, the n values which were falling out of the range are, $\{21, 23, 30, 33, 36, 48, 49, 117, 141\}$. Hence, further interval widened, to check whether all the values are falling in the proposed interval. The data are shown in Table 4. Based on the data Conjecture 3 was proposed.

Conjecture 3: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ...$

Table 4: Data for proposed conjecture 3

n	***n(n-1)***	***p***	***(n+1)^2***	***p'***	***2(n+1)^2 − n(n-1)***	***2(n+1)^2***	***p + p'***	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
21	420	421	484	547	548	968	968	17	1
23	506	521	576	631	646	1152	1152	36	2
30	870	883	961	1039	1052	1922	1922	30	1
33	1056	1063	1156	1249	1256	2312	2312	35	1
36	1260	1291	1369	1447	1478	2738	2738	37	2
48	2256	2281	2401	2521	2546	4802	4802	64	1
49	2352	2383	2500	2617	2648	5000	5000	76	1
117	13572	13627	13924	14221	14276	27848	27848	217	2
141	19740	19819	20164	20509	20588	40328	40328	283	2

Annexure IV
(For Hypothesis 1)

Hypothesis 1: The Gap $(p'_f - p_f)$ between the Goldbach first prime pair for the even integer $2(n+1)^2$ is always less than the corresponding value of n, *where*, $n = 2539, 2540, 2541, ...$

Table 5: Data for proposed Hypothesis 1

n	$Gap(p'_f - p_f)$
1	2
2	4
3	6
4	12
5	10
6	24
10	36
12	24
15	30
16	36
18	24
19	42
21	126
22	84
23	110
25	30
27	54
29	38
30	156
33	186
36	156
39	42
41	46
43	114
45	54
46	60
48	240
49	234
52	84
53	74
56	116
57	186
61	150
62	76
66	264
69	138
72	204
75	150
76	156
81	210
85	126
87	90

Continued on the next page...

Table 5: Data for proposed conjecture 3 (contd...)

n	$Gap(p'_f - p_f)$
93	114
95	130
97	114
99	138
101	146
105	126
106	276
107	134
109	114
117	594
121	126
125	230
127	270
130	264
132	276
136	180
141	690
156	264
157	246
172	336
175	210
183	210
186	240
198	204
204	228
210	504
218	236
228	240
238	276
252	456
262	336
270	504
276	576
280	480
285	294
289	726
292	360
330	516
331	414
333	1026
336	420
348	384
357	426
364	384
366	636
399	402
405	426
418	864
420	504
424	492
426	540
447	534
451	594
460	720
505	654

Continued on the next page...

Table 5: Data for proposed conjecture 3 (contd. . .)

n	$Gap(p'_f - p_f)$
537	546
549	558
556	636
576	840
621	630
642	756
732	744
742	924
1278	1320
1360	1524
1596	1896
2538	2616
2539	822
2540	116
2541	990
2542	240
2543	34
2544	336
2545	186
2546	364
2547	66
2548	276
2549	94
2550	180
.	.
.	.
.	.

Annexure V
(Code for Conjecture 1)

Note: All the code are of Python programming language. Python 3.12 64-bit has been used for computation. All the used Python libraries have been mentioned in the respective codes.

Listing 1: File-1

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 1000000, 1)
print(f"$ n $ & $ n*(n+1) $ & $ p $ & $ (n+1)2 $ & $ p' $ & $
    2(n+1)2 - n*(n+1) $ & $ 2(n+1)2 $ & $ p+p' $ & $ total p-
   pairs $ & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) !=
                                0) & (right_side_check(
                               right_side) != 0):
```

```
                    pair = (left_side,
                        right_side)
                    list_A.append(pair)
                    #n*(n+1)
                    if ((int(math.sqrt(int(
                        number/2))))-1)*(((int
                        (math.sqrt(int(number
                        /2))))-1)+1) <=
                        left_side:
                         list_z.append(pair)
    i=i+1
# n
A = ((int(math.sqrt(int(number/2))))-1)
# p, first component of pair
C = list_A[0][0]
# p', second component of pair
D = list_A[0][1]
# (n+1)
E = (((int(math.sqrt(int(number/2))))-1)+1)
# (n+1)^2
F = (((int(math.sqrt(int(number/2))))-1)+1)
   *(((int(math.sqrt(int(number/2))))-1)+1)
# 2(n+1)^2
G = 2*((((int(math.sqrt(int(number/2))))-1)
   +1)*(((int(math.sqrt(int(number/2))))-1)
   +1))
# 2(n+1)2 - n(n+1)
L = G - A*E
#gap
gap = D - C
t_pairs = len(list_A)
t_l_pairs = len(list_z)
#if A <= 1000:
print(f"$ {A} $ & $ {A*E} $ & $ {C} $ & $ {F}
    $ & $ {D} $ & $ {L} $ & $ {G} $ & $ {C+D}
    $ & $ {t_pairs} $ & $ {t_l_pairs} $ \\\\"
   )
if (C- A*E) < 0:
    print("Pre_fail")
if (L - D) < 0:
    print("Post_fail")
if (C == A*E):
    print("Similar")
if (L == D):
    print("Similar")
```

```
                adding_range(number)
            items_a(number)
accessing_items()
#Code ends
```

Note: After $n = 418$, the above code can be easily modified to not to check Total prime pairs, instead, just check those prime pairs which is falling in the interval to check the hypothesis. These is no need to check all the prime pairs for all the n values.

Listing 2: File-1a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 3: File-1b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure VI
(Code for Conjecture 2)

Listing 4: File-2

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 500000, 1)
print(f"$ n $ & $ n^2 $ & $ p $ & $ (n+1)2 $ & $ p' $ & $ 2(n
    +1)2 - n^2 $ & $ 2(n+1)2 $ & $ p+p' $ & $ total p-pairs $
    & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) !=
                                 0) & (right_side_check(
                               right_side) != 0):
                                    pair = (left_side,
                                        right_side)
                                    list_A.append(pair)
```

```
                    #n^2
                    if ((int(math.sqrt(int(
                        number/2))))-1)*((int(
                        math.sqrt(int(number
                        /2))))-1) <= left_side
                        :
                         list_z.append(pair)

    i=i+1

# n
A = ((int(math.sqrt(int(number/2))))-1)
# p, first component of pair
C = list_A[0][0]
# p', second component of pair
D = list_A[0][1]
# (n+1)
E = (((int(math.sqrt(int(number/2))))-1)+1)
# (n+1)^2
F = (((int(math.sqrt(int(number/2))))-1)+1)
   *(((int(math.sqrt(int(number/2))))-1)+1)
# 2(n+1)^2
G = 2*((((int(math.sqrt(int(number/2))))-1)
   +1)*(((int(math.sqrt(int(number/2))))-1)
   +1))
# 2(n+1)2 - n^2
L = G - A*A

#gap
gap = D - C
t_pairs = len(list_A)
t_l_pairs = len(list_z)

print(f"$ {A} $ & $ {A*A} $ & $ {C} $ & $ {F}
    $ & $ {D} $ & $ {L} $ & $ {G} $ & $ {C+D}
    $ & $ {t_pairs} $ & $ {t_l_pairs} $ \\\\"
   )
if (C- A*A) < 0:
    print("Pre_fail")
if (L - D) < 0:
    print("Post_fail")
if (C == A*A):
    print("Similar")
if (L == D):
    print("Similar")
```

```
            adding_range(number)
        items_a(number)
accessing_items()
#Code ends
```

Listing 5: File-2a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 6: File-2b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure VII
(Code for Conjecture 3)

Listing 7: File-3

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 500000, 1)
print(f"$ n $ & $ n*(n-1) $ & $ p $ & $ (n+1)2 $ & $ p' $ & $
     2(n+1)2 - n*(n-1) $ & $ 2(n+1)2 $ & $ p+p' $ & $ total p-
    pairs $ & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) !=
                                 0) & (right_side_check(
                               right_side) != 0):
                                    pair = (left_side,
                                       right_side)
                                    list_A.append(pair)
```

```
                                   #n*(n-1)
                                   if ((int(math.sqrt(int(
                                      number/2))))-1)*(((int
                                      (math.sqrt(int(number
                                      /2))))-1)-1) <=
                                      left_side:
                                       list_z.append(pair)
          i=i+1

        # n
        A = ((int(math.sqrt(int(number/2))))-1)
        # p, first component of pair
        C = list_A[0][0]
        # p', second component of pair
        D = list_A[0][1]
        # (n-1)
        E = (((int(math.sqrt(int(number/2))))-1)-1)
        # (n+1)^2
        F = (((int(math.sqrt(int(number/2))))-1)+1)
            *(((int(math.sqrt(int(number/2))))-1)+1)
        # 2(n+1)^2
        G = 2*((((int(math.sqrt(int(number/2))))-1)
            +1)*(((int(math.sqrt(int(number/2))))-1)
            +1))
        # 2(n+1)2 - n(n-1)
        L = G - A*E

        #gap
        gap = D - C
        t_pairs = len(list_A)
        t_l_pairs = len(list_z)

        print(f"$ {A} $ & $ {A*E} $ & $ {C} $ & $ {F}
             $ & $ {D} $ & $ {L} $ & $ {G} $ & $ {C+D}
             $ & $ {t_pairs} $ & $ {t_l_pairs} $ \\\\"
            )
        if (C- A*E) < 0:
            print("Pre_fail")
        if (L - D) < 0:
            print("Post_fail")
        if (C == A*E):
            print("Similar")
        if (L == D):
            print("Similar")
    adding_range(number)
```

```
        items_a(number)
accessing_items()
#Code ends
```

Listing 8: File-3a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 9: File-3b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure VII
(Code for Hypothesis I)

Listing 10: File-4

```
#Code starts
from left import *
from right import *
from decimal import *
from math import sqrt

#Creating item list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 1000000, 1)


def accessing_items_from_list_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    if (left_side != 1) and ((left_side % 2)
                        != 0):
                         if (left_side_check(left_side) != 0):
                              if (right_side_check(int(2*(
                                  half_num) - left_side)) != 0):
                                   pair = (left_side,(int(2*(
                                       half_num) - left_side)))
                                   list_A.append(pair)
                                   if len(list_A) >= 1:
                                        break
                    i=i+1
                # n
                A = ((int(sqrt(int(number/2))))-1)
```

```
                # p, 1st component of pair
                C = list_A[0][0]
                # p', second component of pair
                D = list_A[0][1]
                #gap
                gap = D - C
                if (gap >= A):
                    print(f"$ {A} $ & $ {gap} $")
                if A == 1000000:
                    print(f"{A}")
            adding_range(number)
        items_a(number)
accessing_items_from_list_items()
#Code ends
```

Listing 11: File-4a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 12: File-4b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```