# Proof of the Collatz Conjecture for the Natural Numbers

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**Abstract**

This document proves that the Collatz Conjecture is true for the Natural numbers excluding zero. Use is made of the probability distribution of even and odd numbers in supposed diverging Collatz sequences to establish that Collatz sequences do not diverge, having a finite number of terms, and are bounded. Finally proof by contradiction, the pigeon hole principle and proof by induction are used to prove that the Collatz Conjecture is true via two theorems.

**Keywords** Collatz Conjecture, Syracuse Conjecture, $3x + 1$ Conjecture, Ulam conjecture, Hailstone sequence or Hailstone numbers.

## 1 Introduction

The Collatz Conjecture is named after the mathematician Lothar Collatz, who introduced it in 1937. Current and past research is presented in [1], [2], [3]. The solution has proved elusive and the famous mathematician P. Erdos remarked that "Mathematics may not be ready for such problems."[4] By 2020, the conjecture had been verified by computer for all starting values up to $2^{68}$.[5] However a mathematical proof that would prove that the conjecture is true for all Natural numbers greater than zero has yet to be proven.

## 2 Definitions

**Definition 2.1 (Natural Numbers)** *The set of Natural numbers,* $\mathbf{N}$ *referred to in this document does not include zero.*

The Collatz function, $C : \mathbf{N} \to \mathbf{N}, n \in \mathbf{N}$ is shown in Equation 2.1.

$$C(n) = \begin{cases} n/2, & \text{if } 2 \mid n. \\ 3n+1, & \text{otherwise.} \end{cases} \tag{2.1}$$

**Definition 2.2 (generates)** *The phrase n generates denotes the Collatz sequence of numbers iteratively calculated with a starting value of n where the next term in the sequence is obtained by using the Collatz function on the proceeding one.*

**Definition 2.3 (Collatz Conjecture)** *The Collatz Conjecture asserts that Collatz sequences generated from the set of Natural numbers has a term equal to 1 or phrased differently reaches 1. A Collatz sequence as defined herein is deemed terminated upon reaching 1.*

**Definition 2.4 (The kth term of a Collatz sequence)**

*$C^k(n)$ where $k, n \in \mathbf{N}$ is defined as the kth term of a sequence generated from n.*

**Definition 2.5 (cycling)** *A Collatz sequence which repeats one of the numbers in the sequence will cycle through numbers already in the sequence and is defined as cycling.*

As an example Equation 2 shows the number 7 generating a Collatz sequence that reaches 1. Therefore this accords with the Collatz Conjecture.

$$(7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1) \tag{2.2}$$

# 3 Probability that the Parity of $\mathbf{C}^{\infty}(\mathbf{n})$ is Even

**Lemma 3.1** *If $n \in 2\mathbf{N}$ is randomly chosen then $P\{n/2 \in 2\mathbf{N}\}$ is $\frac{1}{2}$.*

**Proof**

Consider the set of even numbers, $\{2, 4, 6, 8..\}$. Upon dividing the elements of this set by 2 the result is $\{1, 2, 3, 4, ...\}$. Note that half of the terms are even and half are odd. Thus for a randomly selected even number n, this implies that the probability that $\frac{n}{2}$is even or odd is $\frac{1}{2}$. $\square$

The following Theorem is developed from [6].

**Theorem 3.2** *$P\{C^{\infty}(n) \in 2\mathbf{N}\}$ is $\frac{2}{3}$.*

**Proof**

The top portion of Figure 1 shows a probability tree for even and odd terms that are generated from a random starting number n in a Collatz sequence. $P\{C(n) \in 2\mathbf{N}\}$ is the probability that C(n) is even. This will be written simply as $P\{C(n)\}$. This implies the probability of an odd C(n) is $1 - P\{C(n)\}$. Also note that $P\{C(n)\} = \frac{1}{2}$ by lemma 3.1.

The next level of the tree represents $C^2(n)$.

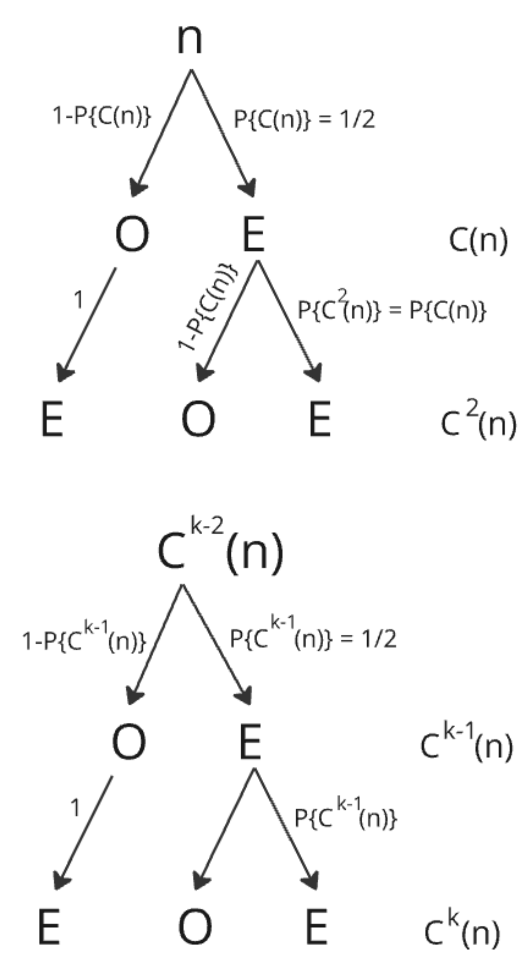

Fig 1. Probability Tree of the Parity of terms in a Collatz sequence.

$P\{C^2(n) \in 2\mathbf{N}\}$ which will be written simply as $P\{C^2(n)\}$ is calculated as

$$\begin{aligned} P\{C^2(n)\} &= (1 - P\{C(n)\}).1 + \frac{1}{2}.P\{C(n)\} \\ &= 1 - \frac{1}{2}P\{C(n)\} \end{aligned} \tag{3.1}$$

Or in general, from observation of the lower portion of Figure 1,

$$P\{C^k(n)\} = 1 - \frac{1}{2}P\{C^{k-1}(n)\} \tag{3.2}$$

where again it is understood that the probability being calculated is for an even number. Applying Equation 3.2 in a recursive manner,

$$\begin{aligned} P\{C^k(n)\} &= 1 - \frac{1}{2}(1 - \frac{1}{2}P\{C^{k-2}(n)\}) \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + \frac{P\{C(n)\}}{2^{k-1}} \end{aligned} \tag{3.3}$$

Taking the limit as k approaches infinity,

$$\lim_{k\to\infty} P\{C^k(n)\} = \lim_{k\to\infty}\left[1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{P\{C(n)\}}{2^{k-1}}\right] \tag{3.4}$$

Noting that the PC(n) term disappears and that the remaining rhs terms form a geometric series with a = 1 and r = $-\frac{1}{2}$.

$$\begin{aligned} P\{C^{\infty}(n)\} &= \frac{a}{1-r} \\ &= \frac{1}{1-\{-\frac{1}{2}\}} \\ &= \frac{1}{1+\frac{1}{2}} \\ &= \frac{1}{\frac{3}{2}} \\ &= \frac{2}{3} \end{aligned} \tag{3.5}$$

Therefore $P\{C^{\infty}(n) \ \in 2\mathbf{N}\}$ is $\frac{2}{3}$ . $\square$

**Corollary 3.3** *In an infinitely long Collatz sequence, assuming it exists,* $\frac{2}{3}$ *of the terms are even and* $\frac{1}{3}$ *are odd. This is obtained by applying the Fundamental Theorem of Probability in reverse. This presumes that there are no powers of 2 encountered, otherwise the sequence does not have an infinite number of terms.*

# 4 Proof that the Collatz Conjecture is True $\forall\ n \in \mathbf{N}$

**Theorem 4.1** *If the numbers from 1 to n generate Collatz sequences that reach 1, this implies that n+1 generates a sequence that reaches 1.*

**Proof**

**Assumption 4.1** *All of the numbers from 1 to n generate Collatz sequences that reach 1.*

**Remark 4.1** *If the numbers from 1 to n can be shown to generate Collatz sequences that reach 1 then if any number larger than n in the course of the generation of the sequence results in a term that is between 1 to n then it will continue along an already established Collatz sequence and will reach 1.*

There are 4 possibilities with regard to a Collatz sequence generated from n+1:

(1) The sequence generated from n+1 diverges.

(2) The sequence generated from n+1 does not diverge but cycles at a number greater than n such that no term in the sequence is below n+1.

(3) The sequence generated from n+1 does not diverge and does not cycle above n but does not reach 1.

(4) The sequence reaches 1.

**Remark 4.2** *If (1) , (2) and (3) are false then the only option left is (d), and thus Theorem 4.1 is true.*

**Proposition 4.2** *The sequence generated from n+1 diverges.*

**Proof**

If a Collatz sequence diverges then it is infinitely long. This implies in such a sequence, if it exists, there are no numbers that are a power of 2, otherwise on encountering this term the sequence would proceed directly to 1 and thus not be divergent. Encountering a power of 2 would upset the probabilities that were calcuated in Theorem 3.2 therefore it is important to point out that this situation does not exist in the case of an infinite sequence.

Consider a Collatz sequence where $\alpha$ is the fraction of the terms that are odd and 1-$\alpha$ is the fraction of those that are even. Each odd term increases the first number in the sequence by approximately 3 and each even term decreases the initial term by 2. Assume that the odd terms instead of increasing the first term by approximately 3, instead increases it by 3.9 without the addition of 1. This simplifies the algebra and allows an upper bound to be calculated. Note that for $n > 1$ (trivial Collatz sequence) $3.9n > 3n + 1$. We can then form an upper bound for the kth term. As there are k terms, (n+1) is multiplied by 3.9 $\alpha k$ times and divided by two $(1 - \alpha)k$ times. This can be written as

$$C^k(n+1) < (n+1) \times \frac{3.9^{\alpha \cdot k}}{2^{(1-\alpha)\cdot k}} \tag{4.1}$$

Taking limits as k approaches infinity and noting by Theorem 3.2 that $\alpha$ the proportion of odd terms approaches $1 - \frac{2}{3} = \frac{1}{3}$.

$$\begin{aligned}
\lim_{k\to\infty} C^k\,(n+1) &< \lim_{k\to\infty}\,(n+1) \times \frac{3.9^{\frac{1}{3}\cdot k}}{2^{\frac{2}{3}\cdot k}} \\
&= (n+1) \times \lim_{k\to\infty} \left(\frac{3.9}{4}\right)^{\frac{1}{3}\cdot k} \\
&= (n+1) \times 0 = 0 \text{ where } k, n \in \mathbf{N}
\end{aligned} \tag{4.2}$$

This is a contradiction. The term at infinity is $\notin \mathbf{N}$. Therefore the reverse

of what we assumed is true. A Collatz sequence has a finite number of terms. It also implies that Collatz sequences are bounded. Without loss of generality assume the number of terms are even. Odd terms are always followed by even terms so at most k/2 terms can be odd. Each odd even pair has the net effect of increasing the starting number n+1 by 3/2 approximately. Therefore certainly $(n+1)2^{k/2}$ exceeds any term in the sequence and is therefore an upper bound. If the number of terms are odd then $(n+1)2^{(k+1)/2}$ can be used as an upper bound.
We can conclude that a Collatz sequence does not diverge, has a finite number of terms, and is bounded.

Therefore proposition 4.2 is false. □

**Proposition 4.3** *The sequence generated from n+1 does not diverge but cycles at a number greater than n such that no term in the sequence is below n+1.*

**Proof**

If n+1 is even then upon dividing by 2 the next term in the sequence is less than n+1 which implies the sequence reaches 1 proving for even n+1 the sequence does not cycle above n.

Consider odd n+1. Note Figure 2 where it is seen that C(n+1) = 3n+4 as n+1 is odd. The next term is (3n+4)/2 as the preceeding term is even. Then there are two possibilities: (3n+4)/4 and the term (9n+14)/2. This implies that odd n+1 generates terms of the form

$$\frac{\alpha n + \beta}{2^{\delta}} \text{ where } \alpha, \beta, \delta, n \in \mathbf{N} \tag{4.3}$$

For proposition 4.2 to be true

$$C^{k}(n+1) = C^{j}(n+1) \text{ where } j, k, n \in \mathbf{N} \text{ and } j > k > 2 \tag{4.4}$$

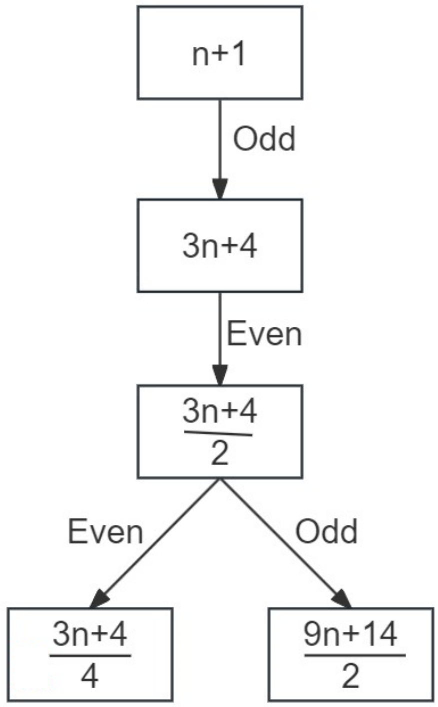


Fig 2. Tree structure showing terms that n+1 may generate

Say $C^k(n+1) = p$ and s cycles later produces p again.

$$C^k(n+1) = p = C^{k+s} \text{ where } k, n, p, s \in \mathbf{N} \tag{4.5}$$

From observation of equations 5,6 and 7

$$p = \frac{\alpha p + \beta}{2^\delta} \text{ where } p, \alpha, \beta, \delta \in \mathbf{N} \tag{4.6}$$

Comparing terms from Equation 8

$$1 = \frac{\alpha}{2^{\delta}} \tag{4.7}$$

$$0 = \frac{\beta}{2^{\delta}} \tag{4.8}$$

The only solution which satisfies equations 9 and 10 is when $\beta = 0$ and $\alpha$ is a power of 2, which implies p is a power of 2 which generates subsequent terms that monotonically decrease to 1 contradicting proposition 4.3. Therefore proposition 4.3 is false. □

**Proposition 4.4** *The sequence does not diverge and does not cycle above n but does not reach 1.*

**Proof** We have proven that Collatz sequences are bounded. Assume t equals the number of Natural numbers greater than n but less than an upper bound M and let any number in this range be equal to q. Then generate q to term t+1. Because no number repeats above n, as there is no cycling, in a worst case scenario where all the t numbers in the range were exhausted then by the pigeonhole principle term t+1 has to be in the range from 1 to n which, from remark 4.1 means that the sequence reaches 1. This is in contradiction to Proposition 4.4 which is then false. □

As propositions 4.2 ,4.3 and 4.4 are false then the only conclusion that can be reached is that Theorem 4.1 is true. □

**Theorem 4.5** *Collatz sequences which are generated from the Natural numbers reach 1.*

**Proof**

The Collatz sequence for a starting value of 1 is as follows

$$(1) \tag{4.9}$$

As n = 1 generates a Collatz sequences that reaches 1, by Theorem 4.2 this implies that n+1 = 2 also generates a Collatz sequence that reaches 1. Continuing in this manner then by induction n generates a Collatz sequence that reaches 1 $\forall\, n \in \mathbf{N}$. Therefore Theorem 4.5 is true. □

**Corollary 4.6** *There are no repeated numbers in a Collatz sequence otherwise there would be a cycle and the sequence would never reach 1.*

**Corollary 4.7** *At some point in a Collatz sequence a term is encountered for the first time such that*

$$C^k(n) = 2^j \text{ where } j, n, k \in \mathbf{N} \tag{4.10}$$

*In other words eventually a term in a generated sequence is a power of 2. This is the only way that the sequence could eventually reach 1.*

# 5 Conclusion

For all $n \in \mathbf{N}$:

(a) Collatz sequences reach 1 starting from any Natural number.

(b) Collatz sequences do not diverge, have a finite number of terms and are bounded.

(c) Collatz sequences have distinct terms.

(d) Collatz sequences eventually reach a term that is a power of 2.

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