A Generalized Contructive Proof for Brouwer Fixed-Point Theorem on D^2 and D^3

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Abstract

This article present a constructive proof by analyzing decompositions of continuous vector field. The original proof of Brouwer's theorem relies on a contradiction argument, which, while effective, does not offer a constructive method for locating the fixed point. Through projecting arbitrary vector field the basis of the vector field, it can be proved there exists zero points on both of the basis. The article will also generalize the proof from 2D to 3D dimensions. The method is also valid under surjective and scaling map.

1 Introduction

This article provides a direct proof of Brouwer's Fixed-Point Theorem by decomposing the vector field into n components in D^n (for dimensions $n = 2$ and $n = 3$). This article explains why a fixed point must exist by using the theory of intersections of surfaces and lines. The proof is valid for any continuous map that maps a compact convex set to itself. Additionally, the article includes a series of code experiments that verify the proof through numerical results.

2 Background

The fixed-point theorem is a fundamental result in mathematics that asserts the existence of points that remain invariant under a given function. One of the most well-known fixed-point theorems is Brouwer's Fixed-Point Theorem[2], which states that any continuous function mapping a compact convex set to itself has at least one fixed point. This theorem is particularly significant in topology and has applications across various fields, including economics, game theory, and differential equations.

Theorem 2.1 (Brouwer's Fixed-Point Theorem)**.** *Let D be a compact convex subset of $\mathbf{R}^n$. If $f : D \to D$ is a continuous function, then there exists at least one point $x \in D$ such that $f(x) = x$.*

Original Proof of Brouwer's Fixed-Point Theorem: Brouwer's original proof employs topological arguments to demonstrate the existence of a fixed point for continuous mappings. The essence of the proof relies on the assumption that if a continuous mapping f does not have a fixed point, then a retraction r can be constructed to project the entire disk onto its boundary. However, it is known that there exists a homotopy f_0 in D^2 that can be continuously deformed to a constant loop. If we compose this homotopy with the retraction r, it implies the existence of a homotopy from S^1 (the boundary of the disk) to a constant loop, which is a contradiction in topology. Therefore, the assumption that f has no fixed point must be false, confirming that at least one fixed point exists within the disk.

3 Continuous Vector fields representation

For any continuous map f mapping $D^n \to D^n$, X is a set of points in D^n, f can be identified by a continuous vector field $\vec{F}(X)$. The vector field has the following relations:

$$\vec{F}(X) = f(X) - X$$

,for a single point positions p in X,

$$\vec{F}(p) = f(p) - p$$

,$\vec{F}(p)$ represent the vector pointing from the original position p to where $f(p)$ has send p.

4 Basis of Projection

In this section, we explore how a continuous vector field on a disk can be represented in terms of its components along the coordinate axes. By examining the projection of the vector field onto each axis, we can better understand the conditions under which the vector field has a fixed point, i.e., where the vector field vanishes.

Let p be a point on the disk D^n (a disk of dimension n). The continuous mapping $\vec{F}(p)$ can be expressed as a linear combination of the standard basis vectors $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$, where $a^i(p)$ represents the real-valued function corresponding to the i-th component of the vector field at point p. Thus, we can write:

$$\vec{F}(p) = \sum_{i=1}^{n} a^i(p)\hat{e}_i$$

In two dimensions, this simplifies to:

$$\vec{F}(p) = a^1(p)\hat{e}_1 + a^2(p)\hat{e}_2$$

The projection function P_i is a map that sends a vector in $\mathbf{R}^n$ to $\mathbf{R}$, extracting the coefficient of the i-th basis vector. That is:

$$P_i\left(\vec{F}(p)\right) = a^i(p)$$

By projecting the vector field $\vec{F}(p)$ onto each coordinate axis using the function P_i, we isolate the i-th component of the vector. This allows us to reduce the problem of finding a fixed point (where the entire vector vanishes) to finding where all the components of the vector field are zero.

If a fixed point of the vector field exists, it means the vector field must vanish at some point p. In other words, for every $i = 1, \ldots, n$, the projection of the vector field onto the i-th axis must be zero:

$$P_i\left(\vec{F}(p)\right) = a^i(p) = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

This implies that the vector field itself is zero at p, i.e., $\vec{F}(p) = 0$.

In the specific case where $n = 2$, the vector field $\vec{F}(p)$ is simply composed of two components, and the projection function extracts either the x or y component. Therefore, the condition for a fixed point simplifies to both the x and y components being zero:

$$a^1(p) = 0 \quad \text{and} \quad a^2(p) = 0.$$

This means that a vector is zero if and only if both its components are zero.

The argument generalizes naturally to higher dimensions. In n-dimensions, the vector field has n components, and the condition for a fixed point is that all n components must simultaneously vanish. This means:

$$a^1(p) = a^2(p) = \cdots = a^n(p) = 0 \quad \text{at the fixed point.}$$

5 Analyze the zero components on basis

This chapter will start discussing the situations of using 2D disks first,then in next chapter it can be generalized to higher dimension.

Lemma 5.1. *Let $\vec{F}$ be a continuous vector field on a disk D^n, mapping $D^n \to D^n$, and let $P_i(\vec{F}(p))$ denote the projection of the vector field $\vec{F}(p)$ onto the i-th axis. Then for each $i \in \{1, \ldots, n\}$, there exist at least one point where $P_i(\vec{F}(p)) = 0$ on a path lie in D^n which the boundary is two intersection points of D^n and the i-th axis.*

Proof. We begin by considering the two-dimensional case, i.e., a disk $D \subset \mathbf{R}^2$, where the vector field $\vec{F}$ is continuous. The basis choose for two dimension disk is simply in cartesian coordinate, $\hat{e_x}$,$\hat{e_y}$. Let $p \in \partial D$ be a point on the boundary of the disk. We represent p in polar coordinates as $p(\theta)$, where $\theta \in [0, 2\pi]$ parameterizes the boundary ∂D.

Consider the projection of $\vec{F}(p)$ onto the x-axis, $P_x(\vec{F}(p))$. By the boundary condition and continuity of the map f, we know that at the rightmost point of the boundary ,denoted as x_0, $(\theta = 0)$ where the vector field $\vec{F}(p)$ must point inward or remain stable, implying

$$P_x(\vec{F}(x_0)) \leq 0$$

. Similarly, at the leftmost point x_0', $(\theta = \pi)$

$$P_x(\vec{F}(x_0')) \geq 0$$

Since $P_x(\vec{F}(p))$ is a continuous function on any path $\mathcal{C}$ start from the point x_0 and end with x_0', and the sign of the function changes between them, by the Intermediate Value Theorem (IVT) [1], there exists a point $x \in \mathcal{C}$ where $P_x(\vec{F}(x)) = 0$.

Next, consider the projection onto the y-axis, $P_y(\vec{F}(p))$. At the top of the disk $\theta = \frac{\pi}{2}$, the vector field $\vec{F}(p)$ must again point inward or remain stable, implying $P_y(\vec{F}(\frac{\pi}{2})) \leq 0$, and at the bottom $\theta = \frac{3\pi}{2}$, we have $P_y(\vec{F}(\frac{3\pi}{2})) \geq 0$.

By the IVT, there exists a point $y \in \mathcal{C}$ where $P_y(\vec{F}(y)) = 0$ for every path $\mathcal{C}$ start from $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{2}$.

For higher dimensions, consider a disk D^n, where a continuous map f and its corresponding vector field $\vec{F}$ can be decomposed into components along the standard basis vectors. If we denote the points where the i-th axis intersects the boundary of the disk as θ_i and θ_i', with θ_i representing the point on the positive side of the i-th axis and θ_i' on the negative side, the following relation holds:

On the loop passing through θ_i and θ_i' of D^n, we have:

$$P_i\left(\vec{F}(\theta_i)\right) \leq 0 \quad \text{and} \quad P_i\left(\vec{F}(\theta_i')\right) \geq 0,$$

where $P_i(\vec{F}(p))$ is the projection of the vector field $\vec{F}(p)$ onto the i-th coordinate axis. By the Intermediate Value Theorem, there must exist at least one point $\theta \in [\theta_i, \theta_i']$ where $P_i\left(\vec{F}(\theta)\right) = 0$. Since this argument holds for each axis, there are at least two distinct points on ∂D^n where the projection function $P_i\left(\vec{F}(p)\right)$ vanishes, i.e., $P_i\left(\vec{F}(p)\right) = 0$.

□

Lemma 5.2. *In two dimensions, the set of points where* $P_i\left(\vec{F}(p)\right) = 0$ *forms a line that extends from one point on the boundary of the disk to another point on the boundary.*

Proof. From **Lemma 5.1**, we know that for the vector field $\vec{F}$, there exist at least two points on the boundary ∂D^2, which is a circle in two dimensions, where the projection function $P_i(\vec{F}(p)) = 0$. Let these points be denoted as p_i and p_i'. We also denote the rightmost point on the disk as x_0 and x_0' , which in polar coordinate, indicated that $\theta = 0$,and $\theta = \pi$. The goal is to show that the set of points where $P_i(\vec{F}(p)) = 0$ forms a continuous curve (a line) that extends from p_i to p_i'. Firstly, when i indicates the index of x-axis, the proof can be done like follows: Start from x_0 and end with x_0' , a path $\mathcal{C}$ can be constructed on the disk. The set of paths can cover the whole disks.There are totally two paths can be constructed if the path lie on the boundary. Denotes the paths as $\mathcal{A}, \mathcal{B}$.

Now, consider a homotopy h_t, $t \in [0, 1]$, which continuously deforms this path $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$. At each time t, $\mathcal{C}$ expands and deforms and the paths shouldn't intersects with each other.

Since the function $P_i(\vec{F}(p))$ is continuous and $P_i(\vec{F}(x_0)), P_i(\vec{F}(x_0'))$ always have different sign, by the Intermediate Value Theorem[1], there must exist at least one point where $P_i(\vec{F}(p)) = 0$ that extends from p_i to p_i' on each $\mathcal{C}$.

Under the continuous homotopy h_t, the points which has $P_i(\vec{F}(p)) = 0$ must also be continuous, forming a continuous line connecting the two boundary points p_i and p_i' .The proof was the same as on the y-axis Therefore, the line of zero projection values extends from one point on the boundary to another, completing the proof. picture:

□

Claim 5.3. *In two dimensions, the two points on the boundary* ∂D^2 *where* $P_y\left(\vec{F}(p)\right) = 0$ *must lie within the different section of* ∂D^2 *that has points where* $P_x\left(\vec{F}(p)\right) = 0$ *as its boundary.*

Proof. Let p_1 and p_1' be the points on the boundary ∂D^2 where $P_x\left(\vec{F}(p)\right) = 0$, which are guaranteed to exist by Claim 4.1. These points must lie on opposite sides of the x-axis, meaning p_1 is on the upper semicircle of ∂D^2 and p_1' is on the lower semicircle, since P_x changes sign between these two points. At p_1, because p_1 lies on the upper half of the boundary, the vector field $\vec{F}(p_1)$ must point inward or along the boundary. Thus, $P_y\left(\vec{F}(p_1)\right) \leq 0$, as the y-component cannot point outward from the disk. Similarly, at p_2, which lies on the lower half of ∂D^2, the vector field $\vec{F}(p_1')$ must also point inward or along the boundary. Therefore, $P_y\left(\vec{F}(p_1')\right) \geq 0$. Now, consider the continuous function $P_y\left(\vec{F}(p)\right)$ on the boundary ∂D^2. Since P_y is continuous, by the Intermediate Value Theorem (IVT), there must be at least one point between p_1 and p_2 where $P_y\left(\vec{F}(p)\right) = 0$.

Therefore, the points where $P_y\left(\vec{F}(p)\right) = 0$ either coincide with p_1 and p_1' or lie strictly between them on the boundary section between p_1 and p_1'. This completes the proof. □

Theorem 5.4. *In a two dimension disk, the lines* $P_y\left(\vec{F}(p)\right) = 0$ *and* $P_x\left(\vec{F}(p)\right) = 0$ *must have at least one intersection corresponds to the point where* $\vec{F}(p) = 0$ *.*

Proof. it is know that there are four points on boundary of the disk given by claim 4.1. The pair of points ,denoted by p_1, p_1', p_2, p_2' ,with the following properties:

$$P_x\left(\vec{F}(p_1)\right) = 0, P_y\left(\vec{F}(p_1)\right) \leq 0$$

$$P_x\left(\vec{F}(p_1')\right) = 0, P_y\left(\vec{F}(p_1')\right) \geq 0$$

$$P_y\left(\vec{F}(p_2)\right) = 0, P_x\left(\vec{F}(p_2)\right) \geq 0$$

$$P_y\left(\vec{F}(p_2')\right) = 0, P_x\left(\vec{F}(p_2')\right) \leq 0$$

Then, consider the function $P_y\left(\vec{F}(p_i)\right)$, $P_y\left(\vec{F}(p_i')\right)$ has different signs , for p on the lines bounded by p_1, p_1',and p_2, p_2', by the *Intermediate Value Theorem (IVT)*, must always has zero points, which means that the line $P_x\left(\vec{F}(p_i)\right) = 0 = P_y\left(\vec{F}(p_i)\right)$ has intersections by IVT. Thus, the lines where $P_x\left(\vec{F}(p)\right) = 0$ and $P_y\left(\vec{F}(p)\right) = 0$ must intersect, and this intersection corresponds to the fixed point of the vector field. □

6 Fixed point Theorem on D^3

This chapter proves the fixed point theorem in three dimensions, specifically in the 3-ball D^3. The proof is similar to the two-dimensional case (D^2) but involves more complexity due to the additional dimension.

Lemma 6.1. *In three dimensions, the set of points where* $P_i\left(\vec{F}(p)\right) = 0$ *forms a surface,and on* ∂D^3*, form a loop.*

Proof. We will consider the case where the i-th axis corresponds to the y-axis. Once the proof is established for the y-axis, it generalizes to the other axes.

Let y_0 and y_0' denote the rightmost and leftmost points on the boundary ∂D^3. In spherical coordinates, these correspond to $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$, respectively. By **Lemma 5.1**, every path through these boundary points must have at least one zero point for the function $P_y(\vec{F}(p))$.

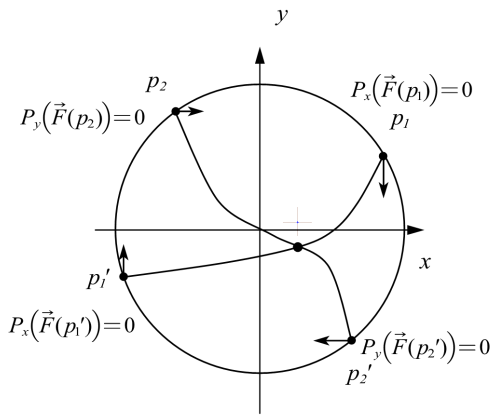

Figure 1: intersection of two lines

Since $P_i(\vec{F}(p))$ is continuous, we can apply a homotopy h_t that deforms paths continuously across the interior of the sphere. As these paths are deformed, the zero points of $P_i(\vec{F}(p))$ also move continuously. Hence, the set of zero points traces out a continuous surface between the boundary points y_0 and y_0'.

Therefore, the set of points where $P_i(\vec{F}(p)) = 0$ forms a surface in D^3, and on the boundary ∂D^3, it forms a loop. □

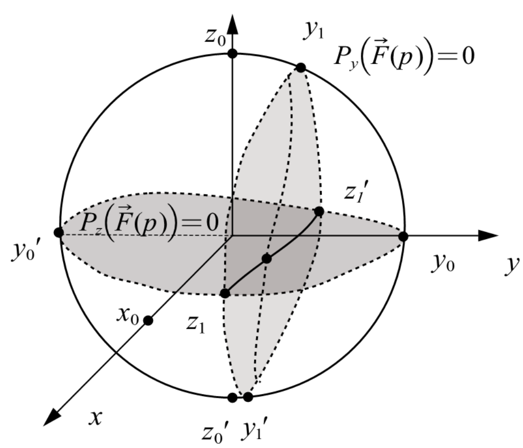

Figure 2: intersection of three surface on D^3

Theorem 6.2. *In three dimensions,there at least one exist intersection point of* $P_i\left(\vec{F}(p)\right) = 0$ *for each* $i = x, y, z$.

Proof. By **Lemma 5.1**, for any path starting from z_0 to z_0' on ∂D^3, there exists at least one point where $P_z\left(\vec{F}(p)\right) = 0$. Similarly, for any path starting from y_0 to y_0', there is a point where $P_y\left(\vec{F}(p)\right) = 0$. Now, take a path lying in the section between y_0 and y_0', which, in spherical coordinates, corresponds to the arc $\phi \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$ and $\theta = 0$. By Lemma 5.1, there must exist a point y_1 on this arc where $P_y\left(\vec{F}(p)\right) = 0$, and this point belongs to ∂D^3.

On this path, because $\phi \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$, y_1 always lies on the upper half of the sphere, we can construct y_1' in the lower half of the sphere use the same way. A path can be constructed like follows:

Start from z_0, then connected with y_1 ,then extend it within the surface $P_y\left(\vec{F}(p)\right) = 0$ to y_1', finally go to z_0'. The path $\mathcal{C}$

$$\mathcal{C} = z_0 \to y_1 \to y_1' \to z_0'$$

can be represented by

$$z_0y_1 \circ y_1y_1' \circ y_1'z_0'$$

A homotopy again can be constructed from y_1 to y_1' within the surface $P_y\left(\vec{F}(p)\right) = 0$. Let $\mathcal{A}$ denote the portion of the path on the hemisphere with positive x-coordinates, and $\mathcal{B}$ denote the portion on the another hemisphere (negative x-coordinates). As the path deforms and sweeps across the surface, by Lemma 5.1 again, there must be points where $P_z\left(\vec{F}(p)\right) = 0$ between y_1 and y_1'. And due to continuity, the zero points of $P_z\left(\vec{F}(p)\right) = 0$ form a line. Also, because path $\mathcal{C}$ also lies on the surface $P_y\left(\vec{F}(p)\right) = 0$, we have this line $\mathcal{L}$ with x, y components of its vector field all zero.

Next is to prove that there always an intersection on this $\mathcal{L}$ where $P_x\left(\vec{F}(p)\right) = 0$. Denote the points on $\mathcal{A}$ and $\mathcal{B}$ where $P_z\left(\vec{F}(p)\right) = 0$ as z_1 and z_1', respectively .Since $z_1 \in \mathcal{A}$ has positive x-coordinates and $z_1' \in \mathcal{B}$ has negative x-coordinates, a path $\mathcal{C}'$ can be constructed:

$$\mathcal{C}' = x_0 \to z_1 \to \mathcal{L} \to z_1' \to x_0'$$

which is equivalently

$$x_0z_1 \circ \mathcal{L} \circ z_1'x_0'$$

By Lemma 5.1, there must exist a point on this path where $P_x\left(\vec{F}(p)\right) = 0$. Therefore, an intersection point can be found on $\mathcal{L}$ where $P_x\left(\vec{F}(p)\right) = 0$, and at this point, the vector field satisfies $P_x\left(\vec{F}(p)\right) = P_y\left(\vec{F}(p)\right) = P_z\left(\vec{F}(p)\right) = 0$, which proves the existence of a fixed point. □

7 Fixed point Theorem on D^4 and higher dimensions D^n

In four dimension,the coordinates are respectively w, z, y, x. As the same, For every path pass from w_0,w_0', like in three dimensions, For every pair of x, y, z collections

Claim

for every pair(w,z,(x,y))

8 Experiment

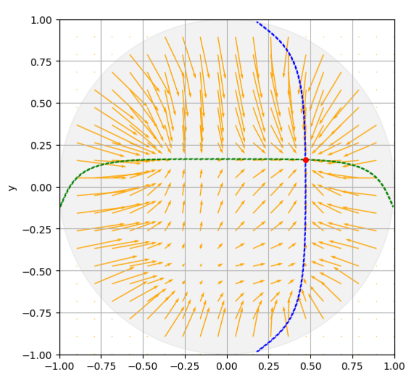

Figure 3: sinusoidal field $U, V =< sin(5x) + 0.4(x - 0.3), cos(5y) - 0.4(y + 0.2) >$

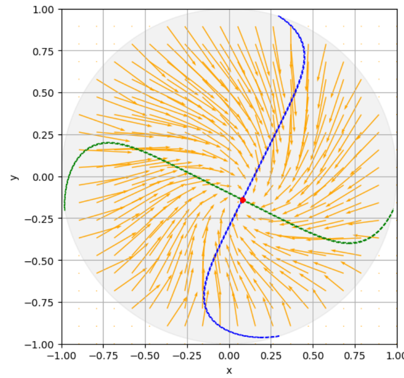

Figure 4: Rotation field add shrink field

References

[1] R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis*. 4th. John Wiley & Sons, 2011.

[2] L. E. J. Brouwer. "Die fixed-points der Funktionaloperationen". In: *Proceedings of the Amsterdam Academy of Sciences* 14 (1911), pp. 39–49.