

Understanding Universal Disjunction

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Abstract

We explore the different meanings of the two versions of universal disjunction — a basic but frequently overlooked aspect of quantificational logic. A semantic analysis is provided to illuminate the twelve distinct interpretations that differentiate two disjunctive predicate sentences.

In Quantificational Logic (QL), there are two distinct forms of universal disjunction. (Here, we consider only the simple case involving two predicates.) One form consists of individually quantified sentences joined by disjunction, e.g., $\forall x(Bx) \vee \forall x(Rx)$, which corresponds to the English sentence: Everything is blue or everything is red. The other distributes the universal quantifier over a disjunction, e.g., $\forall x(Bx \vee Rx)$, rendered in English as: Everything is blue or red. These sentence forms are not logically equivalent, and their asymmetric entailment is a basic tenet of QL.

$$\forall x(Bx \vee Rx) \not\vdash \forall x(Bx) \vee \forall x(Rx). \quad (1)$$

Equation (1) illustrates that a universally distributed disjunction does not entail a disjunction of individually quantified sentences, whereas the converse entailment is valid. Lemmon provides a well-known example to demonstrate

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this distinction.[1] Consider a domain consisting of the positive integers. If every number is either even or odd, it does not follow that all numbers are even or that all are odd. While the example is compelling, it remains incomplete. Beyond Lemmon, deeper elucidations of this asymmetry are rare, leaving room for a more comprehensive account.

Our investigation is limited to the following notation with the usual rules for a well formed formula.

$$\forall, \exists, B, R, x, a, \neg, \wedge, \vee, \rightarrow, (,).$$

This limited set of symbols allows an infinite number of formulae, but as the majority are logically equivalent redundancy is also infinite. This contrasts the finite set of logically distinct meanings. A semantics is introduced to account for this finite set.

It will be helpful to introduce the concept of a semantic tile. Unlike an atomic sentence a semantic tile is syntactically complex whilst also a semantic atom. This means it is a proposition only entailed by logically equivalent sentences or those that express contradiction. Logically equivalent sentences that are semantic atoms are the same tile.

Meanings expressible in our limited fragment may be distinguished using a truth function which we can present on a 4×4 array. A white (ivory) tile is a Boolean 1 (true) and a black tile is a Boolean 0 (false). Each tile on the array also represents a semantic tile. The 16 tiles account for 2^{16} possible meanings that are combinations of the four propositions shown in Figure 1.

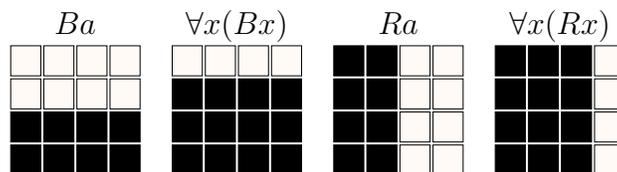


Figure 1

The basic 4×4 array is insufficient to express distributed universal disjunction; for this we need the 32 tile array of Figure 2.

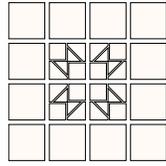


Figure 2

The double triangle (hourglass) is a single tile. Its precise shape is arbitrary, but the configuration of the additional five tiles that make up each of the four central square cells is fixed. As Figure 2 resembles a mosaic, we refer to such grid patterns as ‘mosaics.’ The full set of 32 tiles allows for 2^{32} distinct interpretations (a number approaching 4.3 billion), though we continue to focus on the two forms of universal disjunction. The mosaics corresponding to these, when both predicates are positive, are shown in Figure 3.

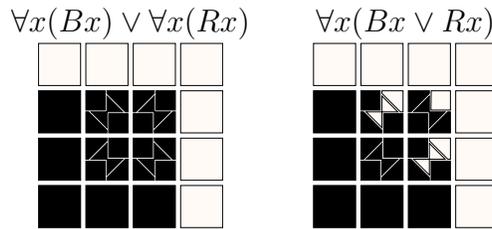


Figure 3

Where predicates are negated the mosaic *points* to the respective corner of the array as shown at Figure 4.

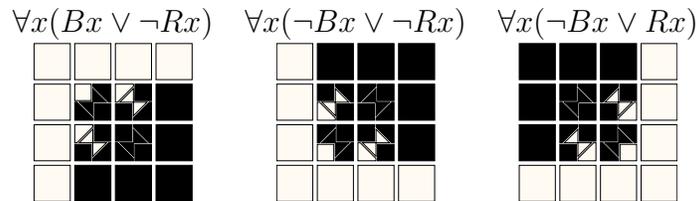


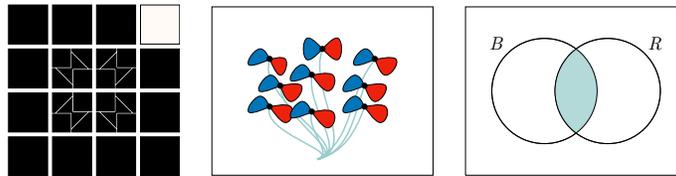
Figure 4

To illustrate universal disjunction we restrict ourselves to Figure 3. Each tile represents an infinite number of well formed formulae. To keep things

simple, we consider one formula with the simplest syntax for each proposition. To interpret the formula the domain is flower stalks. The stalks have blue (B) or red (R) petals and each meaning is illustrated with an English sentence(s), a picture and a Venn diagram. The additional arrow on a Venn diagram makes more sense once realised it points out a counterclaim when stalk a has petals of uniform colour. Twelve propositions (P1 to P12) fully account for Figure 3 and provide a deeper delve into the difference between the two forms of universal disjunction.

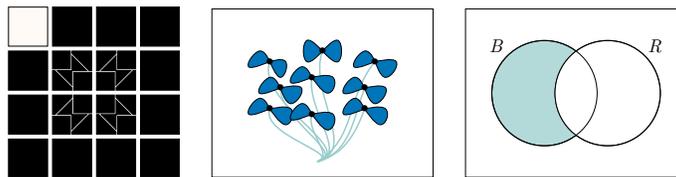
P1. $\forall x(Bx) \wedge \forall x(Rx)$

Every stalk has a blue petal and a red petal.



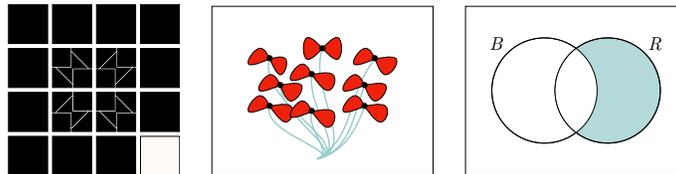
P2. $\forall x(Bx) \wedge \forall x(\neg Rx)$

Every stalk has a blue petal, none has a red petal.



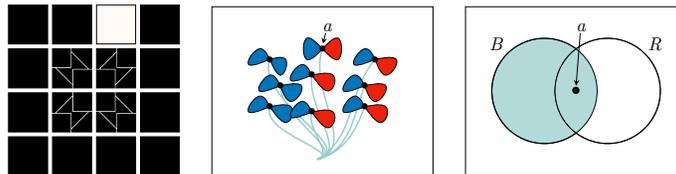
P3. $\forall x(Rx) \wedge \forall x(\neg Bx)$

Every stalk has a red petal, none has a blue petal.



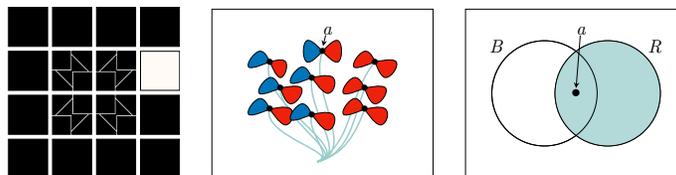
P4. $\forall x(Bx) \wedge \exists x(\neg Rx) \wedge Ra$

Every stalk has a blue petal.
Some stalks do not have a red petal, but stalk *a* does.



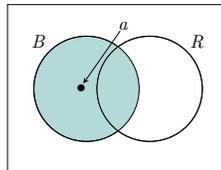
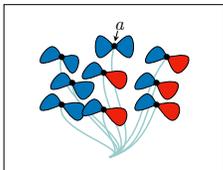
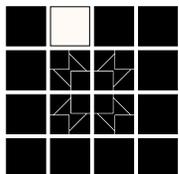
P5. $\forall x(Rx) \wedge \exists x(\neg Bx) \wedge Ba$

Every stalk has a red petal.
Some stalks do not have a blue petal, but stalk *a* does.



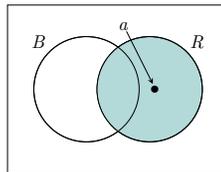
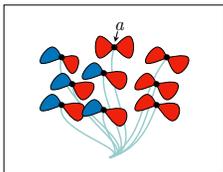
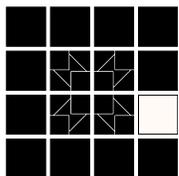
$$P6. \forall x(Bx) \wedge \exists x(Rx) \wedge \neg Ra$$

Every stalk has a blue petal.
Some stalks have a red petal, but stalk a does not.



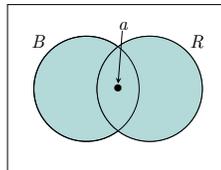
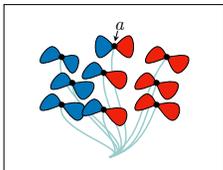
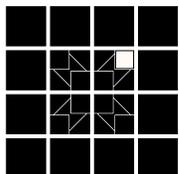
$$P7. \forall x(Rx) \wedge \exists x(Bx) \wedge \neg Ba$$

Every stalk has a red petal.
Some stalks have a blue petal, but stalk a does not.



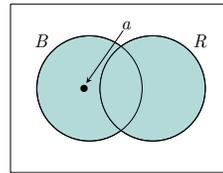
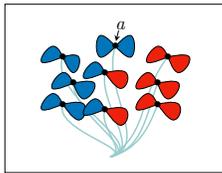
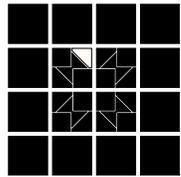
$$P8. \forall x(Bx \vee Rx) \wedge \exists x(\neg Bx) \wedge \exists x(\neg Rx) \wedge Ba \wedge Ra$$

Every stalk has a blue or red petal.
Some stalks do not have a blue petal
and some do not have red,
but stalk a has both.



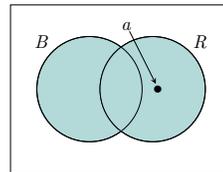
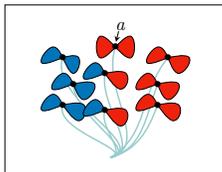
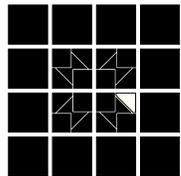
$$P9. \forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Bx) \wedge \neg Ra$$

Every stalk has a blue petal or red petal.
 Some stalks have both a blue petal and a red petal.
 Some stalks do not have a blue petal.
 Stalk a does not have a red petal.



$$P10. \forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Rx) \wedge \neg Ba$$

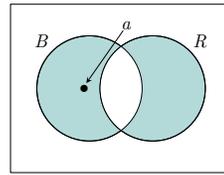
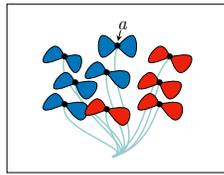
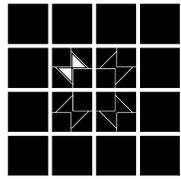
Every stalk has a blue or red petal.
 Some stalks have both a blue petal and a red petal.
 Some stalks do not have a red petal.
 Stalk a does not have a blue petal.



It is easier to make sense of the next two propositions if the the distributed quantifier is rendered as the logically equivalent universal implication.

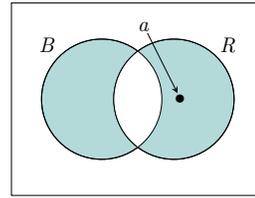
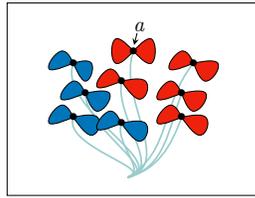
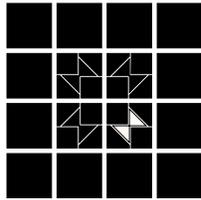
$$P11. \forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Bx) \wedge \exists x(Rx) \wedge \neg Ra$$

If stalks have a blue petal then they do not have a red petal
and if stalks do not have a blue petal they have a red petal.
Some stalks do not have a blue petal.
Some stalks have a red petal, but stalk a does not.



$$P12. \forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Rx) \wedge \exists x(Bx) \wedge \neg Ba$$

If stalks have a blue petal then they do not have a red petal
and if they do not have a blue petal they have a red petal.
Some stalks do not have a red petal.
Some stalks have a blue petal, but stalk a does not.



A comparison between propositions P1–P7 and the full set P1–P12 makes clear the nuances in meaning that Lemmon’s elucidation overlooks. Admittedly, the complete set of twelve propositions has been introduced without derivation. However, the accompanying Venn diagrams provide assurance that all logical possibilities have been accounted for. Furthermore, the approach is grounded in three valid arguments.

$$\vdash \neg(Pn \wedge Pm), \quad (2)$$

where Pn and Pm are any two propositions taken from P1 to P12.

Eq. (2) is the contrary clause that insists no two propositions P1 to P12 may be true together.

$$\forall x(Bx) \vee \forall x(Rx) \dashv\vdash P1 \vee \dots \vee P7. \quad (3)$$

Eq. (3) confirms the disjunction of two universally quantified sentences is equal to the disjunction P1 to P7.

$$\forall x(Bx \vee Rx) \dashv\vdash P1 \vee \dots \vee P12. \quad (4)$$

Eq. (4) confirms distributed universal quantification is equal to the disjunction P1 to P12.

There is also a theorem for the 32 tile mosaic.

$$\vdash P1 \vee \dots \vee P32 \quad (5)$$

Eq. (5) confirms Figure 2 represents a truth table tautology. To prove Eq. (5) it is only necessary to prove the mosaic's top right quartile is a disjunction of eight propositions equivalent to $Ba \wedge Ra$. A principle of symmetry applies to the remaining quartiles.

In conclusion, the advantages of the semantic approach presented here become evident when Figure 3 serves as a quick reminder of why Equation (1) holds.

References

- [1] Edward John Lemmon. *Beginning logic*. CRC Press, 1971.