A simple method for Solving Optimal Control Problems by Legendre approximations

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ABSTRACT

In control system synthesis, the use of orthogonal functions such as Chebyshev polynomials, Lagrange polynomials, Legendre polynomials and Fourier series has recently attracted special attention.

An important objective of applying these functions and polynomial sequences is to avoid the complexity as possible in considering optimal control problems and to fix the solution of algebraic equations, thus simplifying the problem consideration.

In this paper, the Legendre approximation method for solving optimal control problems is proposed.

Using the Gauss-Legendre quadrature method, the given integration problem is transformed into a polynomial series, and Legendre approximations for the control and state variables are performed to consider the given problem as a nonlinear programming problem.

keywords:Spectral Approach, Legendre Polynomial, Optimal Control Problem

Declarations of interest: A new proposal for easy solving controller design for uncertain and nonstationary plants in algebraic methods.

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1. Introduction

In this paper, using Gauss-Legendre quadrature ,we propose a simple method to solve the constrained optimization problem easier than the preceding method.

To date, a number of papers related to Legendre approximation of optimal control problems have been presented [1,3, 5,8].

In 2012, a paper using the penalized local quadratic interpolation approach as a method for solving constrained optimization problems using Legendre approximation was presented, where the pseudo-spectral integral-differential matrix was mentioned [6~10].

Since then, several papers have discussed the approximation process using pseudo-spectral integral-differential matrices, which are difficult to consider and difficult to understand because of the use of complex formulas [11~16].

Hence, we have considered a simpler approach from a practical point of view.

First, we chose the Gauss- Legendre quadrature method as an easy-to-realistic way to satisfy the accuracy in the integral calculation and apply direct comparison techniques using initial conditions, boundary conditions and constraints to implement Legendre approximations.

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Several methods have been introduced, including robust and Chebyshev approximations, in connection with quadrature selection, among which the advantage of Gauss-Legendre quadrature is that for any function the node selection is symmetrical and the weight for each node is not changed, so it is easy to reduce the computation and the process.

As an example of the application of this paper, we have compared the results with those presented in the previous works, especially since all dynamical systems can be decomposed into integral components, we have added examples applied to second-order integral subcomponents and examples applied to simple tracking systems.

We believe that the approach presented here has a simple and easy-to-implement merit compared to the previous methods, and thus is an advance in Legendre approximation theory.

The organization of this paper is as follows.

In Section 2, we set the optimal control problem with linear terminal constraints.

In Section 3, we consider the Gauss-Legendre quadrature method.

In Section 4, we consider the formulation of Legendre approximation for OCP.

Section 5 presents numerical calculations and applications.

2. Setting Optimal Control Problems with Linear Terminal Constraints

Consider the problem of finding the control u(t) that gives the minimum to the objective function as follows:

$$J = h(x, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, T) + \int_0^T g(x, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, u, \tau) d\tau \tag{1}$$

$$F(x, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, u, \tau) = 0, \qquad 0 \le \tau \le T \tag{2}$$

Where

$$x^{(r)} = \frac{d^r x}{d\tau^r}, r = \overline{1, n}$$ **Error! Reference source not found.**

The initial and termination constraint are as follows.

$$\begin{aligned} &\mathrm{L}(\mathrm{x}(0), \mathrm{x}^{(1)}(0), \mathrm{x}^{(2)}(0), \ldots, \mathrm{x}^{(n-1)}(0)) = 0 \\ &\mathrm{M}(\mathrm{x}(\mathrm{T}), \mathrm{x}^{(1)}(\mathrm{T}), \mathrm{x}^{(2)}(\mathrm{T}), \ldots, \mathrm{x}^{(n-1)}(\mathrm{T})) = 0 \end{aligned} \tag{3}$$

where T is assumed to be given.

To approximate a given problem, we perform the following transformation to map the interval $\tau \in [0,T]$ to the interval $t \in [-1,1]$

$$t = \frac{2\tau}{T} - 1 \tag{4}$$

Hence, the optimal control problem is given as

$$\mathrm{J} = (\mathrm{x}, \mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(n-1)}, 1) + \frac{\mathrm{T}}{2} \int_{-1}^{1} g(\mathrm{x}, \mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(n-1)}, \mathrm{u}, \mathrm{t}) dt \tag{5}$$

$$F(x, \frac{T}{2} x^{(1)}, \ldots, \left(\frac{T}{2}\right)^n x^{(n)}, u, t) = 0, \quad -1 \le t \le 1 \tag{6}$$

$$L(x(-1), \frac{T}{2} x^{(1)}, \ldots, \left(\frac{T}{2}\right)^n x^{(n)}(-1)) = 0 \tag{7}$$

$$M(x(1), \frac{T}{2} x^{(1)}(1), \ldots, \left(\frac{T}{2}\right)^n x^{(n)}(1)) = 0 \tag{8}$$

Using (7) and (8), we can determine the coefficients of Legendre approximation.

3. Gauss-Legendre quadrature method

For the integral, we use the following Gauss-Legendre quadrature formula.

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} A_i f(\xi_i) \tag{9}$$

where ξ_i is a node point selected for approximating the integrals, and A_i is a weight to the each node point.
The advantage of Gauss-Legendre quadrature is that the nodes ξ_i and weight A_i do not change even if the function is changed.

Table 1. Selecting node points and weights in gauss-legendre quadrature

N	node point ξ_i	Weight A_i	N	node point ξ_i	Weight A_i
1	0.577350	1.000000	5	-0.906180	0.236927
3	0.000000	0.88		-0.538469	0.478629
	-0.774597	0.555556		0	0.568889
	0.774597	0.555556		0.538469	0.478629
4	-0.861136	0.347855		0.906180	0.236927
	-0.339981	0.652145	6	**Error! Reference source not found.**0.932470	0.171324
	0.339981	0.652145		**Error! Reference source not found.**0.661209	0.360762
	0.861136	0.347855		**Error! Reference source not found.**0.238619	0.467914

As can be seen from the table 1, the nodes are symmetrically placed with respect to $\xi_i = 0$, and the weights assigned to the nodes in the symmetrical position are the same.
This feature of the Gauss-Legendre quadrature allows the modular integration computation process to provide convenience and quickness.

4. Legendre Approximations for OCP

Legendre approximations are adopted here to approximate the solution to the problem.

In the previous work, Legendre approximations are performed for higher order derivatives $x^{(n)}$ and the lower order derivatives $x^{(0)}, x^{(1)}, \cdots, x^{(n-1)}$ are also approximated through continuous integration of higher order derivatives.

However, the way to calculate this is complicated by the computational process and complicated computational complexity.

Thus, we here approximate a and approximate x higher order differential $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ through continuous differentiation.

Applying continuous differentiation, we have

$$x^{(1)} = \sum_{i=0}^{N} a_i P^{(1)}{}_i.$$
$$x^{(2)} = \sum_{i=0}^{N} a_i P^{(2)}{}_i \tag{10}$$
$$\cdots\cdots\cdots$$

Similarly, if the approximation for the control variable u(t), the optimal control problem (4)~(7) is transformed into the following constrained optimization problem:

$$\mathrm{J} = \mathrm{h}(\mathrm{x}, \mathrm{x}^{(1)}, \mathrm{x}^{(2)}, \ldots, 1) + \frac{\mathrm{T}}{2}\sum_{i=0}^{N} b_{Ni}\, g\Big(\sum_{j=0}^{N} a_j P_j(t_j), \sum_{j=0}^{N} a_j P_j^{(1)}(t_j), \ldots, \sum_{j=0}^{N} a_j P_j^{(n)}(t_j), u(t_j)\Big) \tag{11}$$

Where B_{Ni} are weight coefficients, and t_j s are node points.

Our goal is to find unknown coefficients a_j that give a minimum to J.

5. Numerical Examples and Application

In this section, we consider three numerical examples to illustrate the effectiveness of the proposed method.

Example 1:

As the most basic equations of motion of the control system, we can take the constant coefficient differential equation as an example.

$$\frac{dy}{dx} - y = x, \quad y(0) = 1 \tag{12}$$

The analytical solution of this problem can be found in Matlab, $\mathrm{y} = 2e^{x} - x - 1$.

We solve this problem by Legendre approximation as follows.

$$\overline{y_N}(x, a) = \sum_{r=0}^{N} a_r P_r(x) \tag{13}$$

when N is selected as 4,

$$\overline{y_N}(x,a)=\sum_{r=0}^{4}a_rP_r(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)+a_4P_4(x)$$

Our objective is to determine the unknown coefficients (i=0, 1, 2, 3, 4).

Five algebraic equations are needed to obtain five unknown coefficients.

We want to obtain these five algebraic equations by comparing the coefficients of x in the initial and residual equations.

In this problem, one equation can be obtained by the initial condition, so the remaining four equations are constructed by comparing the constants, the coefficients of x, the square coefficients of x and the cubic coefficients of x.

$$\begin{cases} a_0-\frac{1}{2}a_2+\frac{3}{8}a_4=1 \\ a_1-\frac{3}{2}a_3-a_0+\frac{1}{2}a_2-\frac{3}{8}a_4=0 \\ 3a_2-\frac{15}{2}a_4-a_1+\frac{3}{2}a_3=1 \\ \frac{15}{2}a_3-\frac{3}{2}a_2+\frac{15}{4}a_4=0 \\ \frac{35}{2}a_4-\frac{5}{2}a_3=0 \end{cases} \tag{14}$$

Solving this equation yields the following result.

$$a_0=1.35, a_1=1.2, a_2=\frac{5}{7}, a_3=\frac{2}{15}, a_4=\frac{2}{105}$$

Substituting the obtained coefficients a_i into the test solution yields.

$$\overline{y}_N(x)=1+x+x^2+\frac{1}{3}x^3+\frac{1}{12}x^4$$

The solution thus found is an approximate solution.

It is first assumed that the solution is represented by a linear combination of finite Legendre polynomials.

Next, the coefficients of the square of x in the residual equation were not compared.

The error between analytical solution and test approximate solution is given in the following table.

From the results obtained, it can be seen that this technique is very effective for constant coefficient nonhomogeneous differential equations.

For the example mentioned above, the error is found for the case using Legendre approximation and for the case using Taylor expansion.

Table 2 Comparison of analytical solutions of ordinary coefficient differential equations with solutions by approximations

X	N=4	N=6	N=8	N=10
0.0	0	0	0	0
0.1	1.6948e-07	4.0184e-011	5.5511e-015	0
0.2	5.5163e-06	5.29092e-09	2.8793e-012	8.8818e-016
0.3	4.2615e-05	9.0152e-008	1.1183e-010	9.1260e-014
0.4	0.00018272	6.8417e-007	1.5048e-009	2.1738e-012
0.5	0.00056754	3.3053e-006	1.1328e-008	2.5525e-011
0.6	0.00143760	1.2001e-005	5.9067e-008	1.9130e-010
0.7	0.00316374	3.5779e-005	2.3903e-007	1.0518e-009

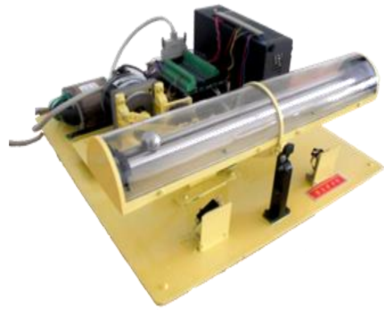
Table 3 Comparison of Legendre approximation and Taylor approximation

x	Legendre approximation error	Taylor approximation error	Error between two approximations
0.1	1.69484628550e-07	1.69484628550e-07	0
0.2	5.5163203396e-06	5.5163203398e-06	-2.2204e-16
0.3	4.2615152006e-05	4.2615152006e-05	0
0.4	0.00018272861587	0.00018272861587	0
0.5	0.00056754140025	0.00056754140025	-2.2204e-16
0.6	0.00143760078101	0.00143760078101	0
0.7	0.00316374827428	0.00316374827428	-4.4408e-16
0.8	0.00628185698493	0.00628185698493	-4.440892e-16
0.9	0.01153122231389	0.01153122231389	-8.881784e-16

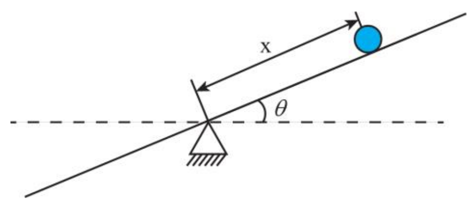
It can be seen from the table that as the order N of the test approximate solution increases, the errors become smaller, as compared with the Taylor expansion, we can see that it is reasonable to apply Legendre approximation to the numerical solution and that the errors by direct comparison technique are very small.

Example 2: All linear control objects can be decomposed into integral or second-order integral components.

Ball control system, international standard experimental device, is represented by second-order integral component.



(a)



(b)

$q_1 = \theta, \quad q_2 = x$

J - rotary inertia moment of the beam

J_b - rotary inertia moment of the ball

Fig 1. ball stabilization device and generalized coordinates
(a)- Ball control device (b) generalized coordinates

The motion equation of this device is as follows.

$$(J + J_b + mx^2)\ddot{\theta} + mgx\cos\theta + 2mx\dot{x}\theta = u$$

Representing above formula to transferfunction

$$\frac{x(s)}{\theta(s)} = -\frac{k}{s^2} \quad , \quad \text{where} \quad k = \frac{mg}{m + \frac{J_b}{r^2}}$$

Therefore, consider the optimal control problem of second-order integral modulation.

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = u(t) \end{cases} \quad t \in [0,3]$$
$$\begin{cases} x_1(0) = 2 \\ x_2(0) = -1 \end{cases}$$
$$\begin{cases} x_1(3) = 0 \\ x_2(3) = 0 \end{cases} \tag{15}$$
$$\mathrm{J} = \frac{1}{2}\int_0^3 u^2(t)dt$$

Let's solve this problem by the method described in this paper.

Let us convert time interval t∈ [0,3] to t∈ [-1,1].

$$t' = \frac{2}{3}t - 1, \quad dt' = \frac{2}{3}dt$$

Then, a given problem is as follows;

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) \end{cases} \qquad t \in [0,3] \tag{16}$$

$$\begin{cases} x_1(-1) = 2 \\ x_2(-1) = -1 \end{cases} \qquad \text{initial condition} \tag{17}$$

$$\begin{cases} x_1(1) = 0 \\ x_2(1) = 0 \end{cases} \qquad \text{end condition} \tag{18}$$

$$J = \int_{-1}^{1} u^2(t)dt \tag{19}$$

The second step is to approximate $x_1(t), x_2(t), u(t)$ by Legendre approximation.

$$x_1(t) = \sum_{i=0}^{4} a_i p_i$$

$$x_2(t) = \sum_{i=0}^{4} b_i p_i \tag{20}$$

$$u(t) = \sum_{i=0}^{2} c_i p_i$$

Where a_i, b_i, c_i are unknown constants , p_i is the i-st order Legendre polynomial.
We transmit the integral to finite series by Gauss-Legendre quadrature.

$$\int_{-1}^{1} u^2(t)dt = \sum_{i=1}^{6} A_i u^2(t_i) \tag{21}$$

Where $A_i (i = \overline{1,6})$ are weight coefficients, $t_i (i = \overline{1,6})$ are collocation points.
As shown in Table 1, A_i and t_i are known constants.
Then, a given problem is given as follows;

$$\int_{-1}^{1} u^2(t)dt = \sum_{i=1}^{6} A_i u^2(t_i) = \sum_{i=1}^{6} A_i (\sum_{j=0}^{2} c_j p_j(t_i))^2 \tag{22}$$

Therefore, we have to find the unknown coefficient c_j that gives the minimum to J.

In order to solve this problem, considering the constraints, initial and final conditions, the given problem becomes a nonlinear quadratic programming problem.

Solving this nonlinear quadratic programming problem, we can obtain the following result.

$$x_1(t) = 2 - t + t^3 / 27$$

$$x_2(t) = -1 + t^2 / 9$$

$$u(t) = \frac{2}{9} t$$

$$J = 0.5$$

For this problem, the error with the exact solution is calculated as table 4.

Table 4 Comparison of exact and approximate solutions.

t_i	$u(t)$		$x_1(t)$		$x_2(t)$	
	Approximate solutions	exact solutions	Approximate solutions	exact solutions	Approximate solutions	exact solutions
0.0	0.000	0.000	2.000	2.000	-1.000	-1.000
0.4	0.098	0.113	1.602	1.592	-0.972	-0.969
0.8	0.177	0.187	1.119	1.117	-0.929	-0.920
1.0	0.222	0.211	1.019	0.999	-0.888	-0.876
1.4	0.301	0.298	0.701	0.697	-0.772	-0.769
1.8	0.407	0.397	0.416	0.402	-0.630	-0.623
2.0	0.424	0.412	0.266	0.257	-0.545	-0.536
2.4	0.503	0.499	0.992	0.987	-0.360	-0.353
2.8	0.622	0.617	0.013	0.011	-0.129	-0.118
3.0	0.666	0.666	0.000	0.000	0.000	0.000

As shown in Table 2, the error is given to be least as 0.01.
Figure 2 shows the comparison curves of x1 by legendre approximation and on the actual plant.

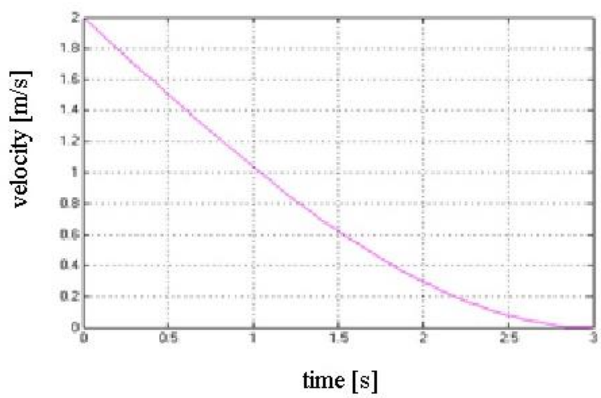


Fig 2. Comparison curve of X1 obtained in both methods

Figure 3 shows the comparison curves of x2 by legendre approximation and on the actual plant.

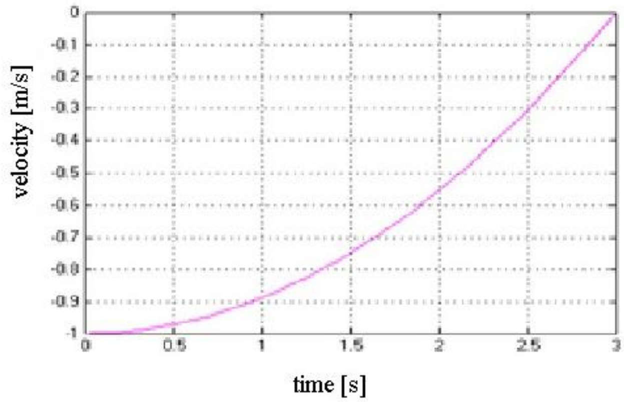


Fig 3. Comparison curve of X2 obtained in both methods

As shown in the figures, when the plant is given initial and final conditions, the operation when inputting the control by Legendre approximation is completely the same.

Both curves in both methods cannot be distinguished in too consistent.

Example 3:

Find a suitable control for minimization of the following optimal control problem.

$$J = 0.5\int_{-1}^{1}[x_1^2(t) + x_2^2(t) + u^2(t)]dt \qquad (24)$$

Subject to:

$$2x_1'(t) = x_2(t) + u(t)$$
$$2x_2'(t) = u(t)$$

with the boundary conditions:

$$x_1(-1) + x_2(-1) = 3, x_2(1) = 1$$

The optimal values of state and control are given in Table 3. As can be seen on Table 3, the proposed method is efficient in solving OCP.

Table 3. Observed state x(t) and control u(t) variables for Example (2)

T	x1(t)	x2(t)	u(t)
-1.000	1.21	1.79	-2.422
-0.809	1.158	1.578	-2.038
-0.309	1.099	1.179	-1.179
0	1.123	1.033	-0.725
0.309	1.196	0.953	-0.307
0.809	1.435	0.957	0.331
1	1.571	1	0.575

Example 4:
In practice, there are more cases of closed-loop control than open-loop systems.
Therefore, let us try the following closed-loop control problem in this way.

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = k \cdot e(t) \end{cases} \qquad t \in [0,0.1] \tag{25}$$

$$\begin{cases} x_1(0) = 0 \\ x_2(0) = 70 \end{cases} \qquad \text{initial condition} \tag{26}$$

$$\begin{cases} x_1(0.1) = 1 \\ x_2(0.1) = 20 \end{cases} \qquad \text{end condition} \tag{27}$$

$$J = \frac{1}{2}\int_0^{0.1} u^2(t)dt \tag{28}$$

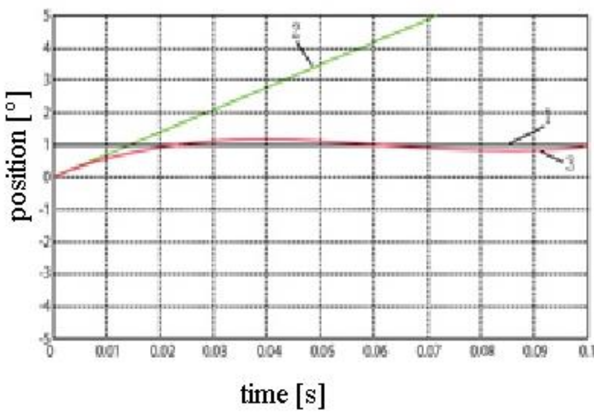


Fig 4. Comparison of step response as adding control and not
1-Step input , 2- output without controller , 3- output with controller

The order of legendre polynomial is the higher, output error of the system is the smaller.

6. Conclusions

In this paper, we consider a simple approach to solving optimal control problems with constrained boundary conditions by Legendre approximation to nonlinear optimal programming.
This method is useful for objects with uncertainties, which are difficult to solve analytical solutions, especially for those with simplified calculations and modularization, and has the advantage of being feasible.
In addition, depending on the characteristics of the plant, controller design also overcomes the conventional approach that has to be complicated and provides the possibility to design controller design in a uniformly matrix algebraic equation.

Simulation results demonstrate that the method by Legendre polynomial approximation is very useful for applications, and direct comparison techniques for solving them can be seen as the simplest method, giving results close enough to the exact solution.
This method can be applied to linear and nonlinear plants, especially those with uncertainties and non-stationary plants.

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