

# A pathology of the Von Neumann ordinals

David Selke

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## Abstract

We show that the definition of the Von Neumann ordinal  $\omega_2$  fails to have the properties of an ordinal. In the notation that uses braces to denote ordinals, the “depth” or “height” of nesting of these braces becomes infinite leading to objects which have no top level of braces and which therefore have their elements undefined, leading to the conclusion that they are not sets at all, let alone ordinals. The least of these objects occurs before (that is, within)  $\omega_2$ .

## 1 Introduction

Foundational results in set theory, such as diagonalization and Cantor’s Theorem, preceded axiomatization and have survived the formalization of the field. Our approach recalls those early and lasting proofs, which with ordinary (but not extraordinary) mathematical rigor showed the richness and promise of set theory. This result shows that the class of Von Neumann ordinals is much smaller than previously thought based on the failure of its notation in terms of braces.

## 2 Overview

The number of braces used to write an ordinal such as  $3 = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}$  increases as the ordinal increases. Some pairs of braces are “beside” others as in Figure 1 (we will not concern ourselves with these), while other pairs of braces are “inside” another pair. As the ordinal number grows without bound, the depth of these latter (“red” in the figures) braces also grows without bound, and the brace depth becomes infinite before reaching  $\omega_2$ . This leads to objects of the form  $\dots\{\{\{\}\}\}\dots$  which have no top level. Because ordinals are sets and the top level of these objects is undefined (there is no top, just an ever increasing sequence), the elements of such an object (which for sets appear in the top level) are undefined, so these objects with infinite brace depth/height cannot be sets. Therefore they cannot be ordinals, even though we will show that they occur inside the Von Neumann ordinal  $\omega_2$ .

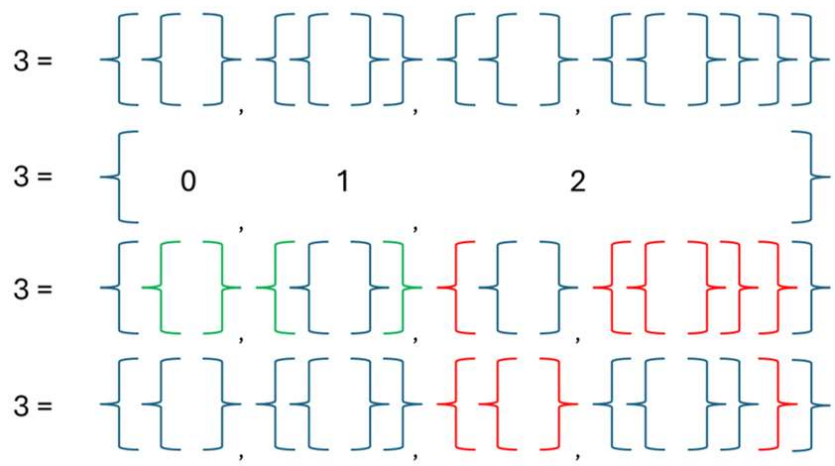


Figure 1: The ordinal 3 displayed in brace-notation. The green colored braces in the third row are “beside” rather than inside each other. The red braces are nested “inside” each other and so contribute to the total brace depth of 4 (red braces in row 3 plus the outermost braces).

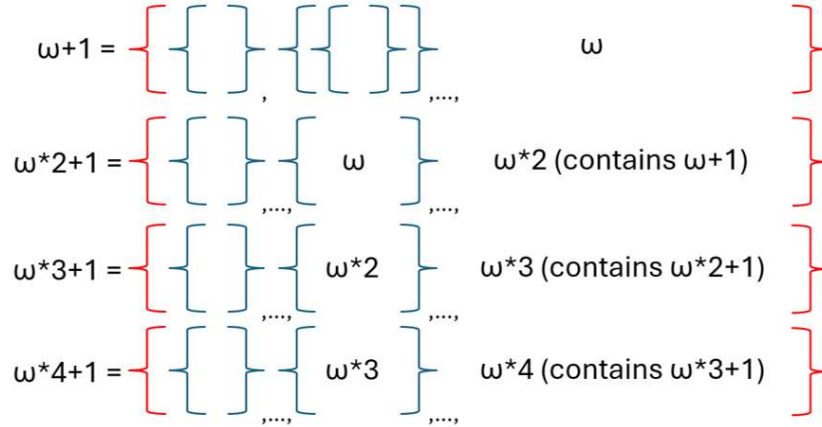


Figure 2: The first few pairs of limit and successor ordinals. This initial segment shows a finite number of steps (4), but if continued throughout  $\omega_2$  the brace depth would become infinite as argued in the Proof section.

### 3 Proof

Consider just the pairs of limit ordinals and their immediate successors. Count the braces that the successors such as  $\omega + 1$  add, counting only the “step” by which the successors’ outer braces enclose their predecessor limit. There are  $\aleph_2$  of these pairs in  $\omega_2$ . Although a descending chain in these braces is finite as a descending chain in a well ordering, ascending chains need not be.

The reverse of the well ordering containing all the braces is not a well ordering, which means that the set of these limits and successors is not finite. This can be seen because the reverse ordering has no least element in some of its subsets, because the (non-reversed) well ordering has no greatest element in some of its subsets. The cardinality of the pairs of limits and successors is  $\aleph_2$ , and there is a separate, nested brace pair for each of the successor ordinals (that which surrounds its limit ordinal), as in Figure 2. Also, because of the ordering of the ordinals, the pairs of limit and successor ordinals do all “stack” to an infinite brace depth since the outermost braces of each of these successor ordinals does enclose all the braces of each lesser successor ordinal.

## 4 Discussion

It might be objected that a transfinite number of stages does not imply an infinite ascending chain of braces, with the example that  $\omega$  contains all the finite brace depths but no more than that. In reply it could be said that, if one increases a finite number by adding one transfinitely, it is hardly possible that the result could still be a finite number after this operation. What we increment in this argument is just the number of braces “deep,” or “high,” that occurs in the brace-notation of the ordinal while reaching the “top level” where the elements of  $\omega_2$  reside. The analogy of  $\omega$  containing all finite heights/depths of braces but no infinite ones does not apply because most initial segments of  $\omega_2$  have length  $\aleph_1$  and incrementing any number  $\aleph_1$  times makes the result at least  $\aleph_1$  in cardinality.

Attempts to prove that the height of nested braces never reaches infinity, using transfinite induction, seem to suffer from a vague predicate. (A similar argument in the natural numbers can prove that every number is small! [1]) In this case, it is hard to see which countable ordinal is the least that has a brace-depth of  $\aleph_0$ , nevertheless, when the induction reaches  $\omega_1$ , there are uncountably many nested braces. This uncountable number could not be finite.

Finally, it has been argued that the growth of the ordinals produces transfinitely many ordinals “beside” one another, at the top level of each ordinal, but not “inside” each other. However, by the transitivity of the ordinals, for a given ordinal each lesser ordinal also appears “inside” the given ordinal, that is, within the outer braces of the given ordinal, which increases the depth of nesting by at least one.

## 5 Conclusion

It was shown that  $\omega_2$ , if defined as a Von Neumann ordinal, does not exist as a set or as an ordinal because its elements are undefined. Some objections were answered. Further work should investigate other definitions of ordinals to find a fall-back definition. Certainly Cantor and Russell knew what they meant when they spoke of ordinals. Unfortunately the very concreteness that must have made the Von Neumann ordinals attractive has also limited their application to somewhat small ordinals with this discovery.

## References

- [1] Boolos, George, *Logic, Logic, and Logic*, (Harvard University Press, Cambridge, Massachusetts, 1999). pp. 354-355.