Model $\mathbf{P}(\boldsymbol{\varphi})_4$ Quantum Field Theory. A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields. Part II. The field operators and the approximate vacuum

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Abstract. A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant.

1.Introduction

Remind that extending the classical real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate representative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on classical nonstandard real analysis, we refer to [5]-[7].The technique of nonstandard analysis (NSA) in constructive quantum field theory (QFT) originally were considered in P. J. Kelemen and A. Robinson papers [8],[9]. The methods of nonstandard analysis are demonstrated for the construction of the nonstandard $\lambda{:}\varphi_2^4{:}$ model. J. Glimm and A. Jaffe's results [10],[11] were analysed from the nonstandard point of view. For further information on methods of classical nonstandard analysis in QFT, we refer to [12],[13]. However methods of classical nonstandard analysis cannot resolve this problem in physical dimension $d = 4$, in particular for the case of simplest scalar QFT model with interaction $:\varphi_2^4:$, see concise explanation in ref. [15 Introduction, Remark 1.4] and ref.[17 sect.1, Remark 1.4]. Cardinally novel approach has been developed in author papers [14]-[19].This approach based on non-conservative extension of the model theoretical NSA. In this paper we consider a some-what different hyperfinite cut-off theory, namely the $\lambda\varphi_4^4$ theory in a periodic box. This gives a cut-off interaction which is translation invariant, and therefore it is useful for the study of the vacuum state. In a hyperfinite interval we prove that the total Hamiltonian is self #-adjoint and has a complete set of normalizable eigenstates.

Abbreviation 1.11In this paper we adopt the following canonical notations. For a standard set E we often write E_{st}. For a set E_{st} let ${}^{\sigma}E_{\mathrm{st}}$ be a set${}^{\sigma}E_{\mathrm{st}} = \{{}^*x | x \in E_{\mathrm{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_+^{\uparrow} = L_+^{\uparrow}$, etc.
Let ${}^*\mathbb{R}_{\approx}$, ${}^*\mathbb{R}_{\approx+}$, ${}^*\mathbb{R}_{\mathrm{fin}}$, ${}^*\mathbb{R}_{\infty}$, and ${}^*\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^*\mathbb{R}_{\mathrm{fin}} = {}^*\mathbb{R}\setminus{}^*\mathbb{R}_{\infty}$, ${}^*\mathbb{C} = {}^*\mathbb{R} + \mathrm{i}{}^*\mathbb{R}$, ${}^*\mathbb{C}_{\mathrm{fin}} = {}^*\mathbb{R}_{\mathrm{fin}} + \mathrm{i}{}^*\mathbb{R}_{\mathrm{fin}}$. Note that there is a natural imbedding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$, see ref. [5].
Notation 1.1 We denote #- completion of the non- Archimedean field ${}^*\mathbb{R}$ by ${}^*\mathbb{R}_c^{\#}$, see ref. [18],[19].
Abbreviation 1.2 Let ${}^*\mathbb{R}_{c\approx}^{\#}$, ${}^*\mathbb{R}_{c+}^{\#}$ ${}^*\mathbb{R}_{c\mathrm{fin}}^{\#}$, ${}^*\mathbb{R}_{c\infty}^{\#}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers in a non- Archimedean field ${}^*\mathbb{R}_c^{\#}$. Note that there is a natural imbedding ${}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^{\#}$, see ref. [18].
Notation 1.2 We denote by ${}^*\widetilde{\mathbb{R}}_c^{\#}$ special extension of a non-Archimedean field ${}^*\mathbb{R}_c^{\#}$, see ref. [19] and section 22 in this paper.
Abbreviation 1.3 Let ${}^*\mathbb{R}_{c\approx}^{\#}$, ${}^*\mathbb{R}_{c+}^{\#}$ ${}^*\mathbb{R}_{c\mathrm{fin}}^{\#}$, ${}^*\mathbb{R}_{c\infty}^{\#}$ denote the sets of infinitesimal hyperreal numbers, positive

infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers in non- Archimedean field ${}^*\mathbb{R}_c^\#$ respectively. Note that ${}^*\mathbb{R}_{c\text{fin}}^\# = {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}_{c\infty}^\#$, ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$, ${}^*\mathbb{C}_{c\text{fin}}^\# = {}^*\mathbb{R}_{c\text{fin}}^\# + i{}^*\mathbb{R}_{c\text{fin}}^\#$.

Abbreviation 1.3 Let ${}^*\widetilde{\mathbb{R}}_{c\approx}^\#$, ${}^*\widetilde{\mathbb{R}}_{c+}^\#$ ${}^*\widetilde{\mathbb{R}}_{c\text{fin}}^\#$, ${}^*\widetilde{\mathbb{R}}_{c\infty}^\#$ denote the sets of infinitesimal hyperreal numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers in non- Archimedean field respectively. Note that ${}^*\widetilde{\mathbb{R}}_{c\text{fin}}^\# = {}^*\mathbb{R}_c^\# \backslash {}^*\widetilde{\mathbb{R}}_{c\infty}^\#$, ${}^*\widetilde{\mathbb{C}}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$, ${}^*\widetilde{\mathbb{C}}_{c\text{fin}}^\# = {}^*\widetilde{\mathbb{R}}_{c\text{fin}}^\# + i{}^*\widetilde{\mathbb{R}}_{c\text{fin}}^\#$, see ref. [19].
Note that there is a natural imbedding ${}^*\mathbb{R}_c^\# \hookrightarrow {}^*\widetilde{\mathbb{R}}_c^\#$, see ref. [19].

Definition 1.1 The Schwartz space of essentially rapidly decreasing ${}^*\widetilde{\mathbb{C}}_c^\#$- valued test functions on ${}^{*}{}^*\widetilde{\mathbb{R}}_c^{\#n}, n \in {}^*\mathbb{N}$ is the function space defined by

$$\breve{\mathrm{S}}_{\text{fin}}^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#n}\right) = \breve{\mathrm{S}}_{fin}^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#n}, {}^*\widetilde{\mathbb{C}}_c^\#\right) =$$

$$\left\{f \in C^{*\infty}\left({}^*\widetilde{\mathbb{R}}_c^{\#n}, {}^*\widetilde{\mathbb{C}}_c^\#\right) | \forall(\alpha,\beta)\left(\alpha \in {}^*\mathbb{N}^n, \beta \in {}^*\mathbb{N}^n\right) \exists c_{\alpha\beta}\left(c_{\alpha\beta} \in {}^*\widetilde{\mathbb{R}}_{c,\text{fin}}^\#\right) \forall x\left(x \in {}^*\widetilde{\mathbb{R}}_c^{\#n}\right)\left[\left|x^\alpha\left(D^{\#\beta} f(x)\right)\right| < c_{\alpha\beta}\right]\right\}.$$

Definition 1.2 Let B be a non-Archimedean Banach space endowed with ${}^*\widetilde{\mathbb{R}}_c^\#$ - valued #-norm $\|\circ\|_\#$. Let A be a linear operator $A: B \to B$. We say that operator A is bounded in ${}^*\widetilde{\mathbb{R}}_c^\#$ if there is positive constant $\delta \in {}^*\widetilde{\mathbb{R}}_{c+}^{\#n}$ such that for any $x \in B$the inequality $\|Ax\|_\# \leq \delta\|x\|_\#$ holds.

Definition 1.3 The Fock space $\mathcal{F}^\#$ is the non-Archimedean Hilbert space #- completion of the symmetric tensor algebra over $L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right)$

$$\mathcal{F}^\# = \mathfrak{C}\left(L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right)\right) = Ext\text{-}\bigoplus_{n=0}^{*\infty}\mathcal{F}_n^\#, \tag{1.1}$$

where $\mathcal{F}_n^\#$ is the space of n non-interacting particles,

$$\mathcal{F}_n^\# = L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right) \otimes_s L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right) \otimes_s \cdots \otimes_s L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right). \tag{1.2}$$

The variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^*\widetilde{\mathbb{R}}_c^{\#3}$ denotes momentum vector. For $\psi = \{\psi_0, \psi_1, \dots\} \in \mathcal{F}^\# = \mathcal{F}_0^\# \oplus \mathcal{F}_1^\# \oplus \cdots$

We define on Fock space $\mathcal{F}^\#$ the ${}^*\widetilde{\mathbb{R}}_c^{\#3}$- valued #-norm $\|\cdot\|_\#$ by $\|\psi\|_\#^2 = Ext\text{-}\sum_{n=0}^{*\infty}\|\psi_2\|_{\#2}^2$, where $\|\cdot\|_{\#2}$is a #-norm in $L_2^\#\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right)$ The no particle space $\mathcal{F}_0^\# = {}^*\mathbb{C}^\#$ is the complex numbers, and

$$\Omega_0 = \{1,0,0,\dots\} \in \mathcal{F}^\# \tag{1.3}$$

is the (bare) vacuum or (bare) no-particle state vector. We define operators N and $H_{0,\varkappa}$ by

$$(N\psi)_n = n\left(Ext\text{-}\prod_{i=1}^n \theta(\|\boldsymbol{k}_i\|, \varkappa)\,\psi_n\right), \tag{1.4}$$

$$\left(H_{0,\varkappa}\psi\right)_n(\boldsymbol{k}_1, \dots, \boldsymbol{k}_n) = Ext\text{-}\sum_{j=1}^n \theta\left(\left\|\boldsymbol{k}_j\right\|, \varkappa\right)\mu\left(\boldsymbol{k}_j\right)\psi_n(\boldsymbol{k}_1, \dots, \boldsymbol{k}_n), \tag{1.5}$$

where $\varkappa \in {}^*\widetilde{\mathbb{R}}_{c+}^\# \backslash {}^*\widetilde{\mathbb{R}}_{\text{fin}+}^\#$ and

$$\theta\left(\left\|\boldsymbol{k}_j\right\|, \varkappa\right) = 1 \text{ if } \left\|\boldsymbol{k}_j\right\| \leq \varkappa \text{ and } \left(\left\|\boldsymbol{k}_j\right\|, \varkappa\right) = 0 \text{ if } \left\|\boldsymbol{k}_j\right\| > \varkappa, \mu\left(\boldsymbol{k}_j\right) = \sqrt{\langle\boldsymbol{k}_j, \boldsymbol{k}_j\rangle + m_0^2} \tag{1.6}$$

Here N is the number of particles operator, and $H_{0,\varkappa}$ is the free energy operator (the free Hamiltonian). The rest mass of the non-interacting particles is m_0, and $\mu(\boldsymbol{k})$ is the energy of a free particle with momentum vector $\boldsymbol{k}$. We use the standard annihilation and creation operators $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$,

$$(a(\boldsymbol{k})\psi)_{n-1}(\boldsymbol{k}_1, \dots, \boldsymbol{k}_{n-1}) = \sqrt{n}\psi_n(\boldsymbol{k}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{n-1}).$$

As a convenient minimal domain for $a(\boldsymbol{k})$, we use the set $\mathcal{E}^{\#}$ of vectors $\psi \in \mathcal{F}^{\#}$ with $\psi_n = 0$ for large $n \in {}^*\mathbb{N}$ and $\psi_n \in \tilde{S}^{\#}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#\boldsymbol{3}}_c\right)$ for all $n \in {}^*\mathbb{N}$.

$$(a^*(\boldsymbol{k})\psi)_{n+1}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{k}_{n+1}) = {}^{-1/2}\sqrt{n+1}\,Ext\text{-}\sum_{j=1}^{n+1} \delta^{\#}(\boldsymbol{k} - \boldsymbol{k}_j)\, \psi_n(\boldsymbol{k}_1, \ldots, \widehat{\boldsymbol{k}}_j, \ldots \boldsymbol{k}_n). \quad (1.7)$$

Here the variable $\boldsymbol{k}_j$ is omitted. While a*((k) is not an operator, it is a densely defined bilinear form on $\mathcal{E}^{\#} \times \mathcal{E}^{\#}$.
Remark 1.1 Note for a ${}^*\widetilde{\mathbb{C}}^{\#}_c$- valued function or ${}^*\widetilde{\mathbb{C}}^{\#}_c$- valued distribution b we can define ${}^*\widetilde{\mathbb{C}}^{\#}_c$- valued bilinear form $B =$

$$Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}^{\#3}_c \cdot \times \cdot {}^*\widetilde{\mathbb{R}}^{\#3}_c \cdot} b(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m; \boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_n) a^*(\boldsymbol{k}_1) \cdots a^*(\boldsymbol{k}_m) a(-\boldsymbol{k}'_1) \cdots a(-\boldsymbol{k}'_n) d^{\#3}\boldsymbol{k}'_1 \ldots d^{\#3}\boldsymbol{k}'_n. \quad (1.8)$$

The integration helps in (1.8) and B is not only a bilinear form, but often an operator. This is the case if, for example, b is the kernel of a bounded operator B_0 from $\mathcal{F}^{\#}_n$ to $\mathcal{F}^{\#}_m$. In this case

$$\left\|(N_\varkappa + I)^{-\alpha/2} B (N_\varkappa + I)^{-\beta/2}\right\|_{\#} \leq \text{const} \cdot \|B_0\|_{\#}, \quad (1.9)$$

provided that $m + n \leq \alpha + \beta$. The constant depends only on α, β, m and n. Intuitively we think of B as being dominated by $N_\varkappa^{(m+n)/2}$; in particular B is an operator on $D\left(N_\varkappa^{(m+n)/2}\right)$ the domain of $N_\varkappa^{(m+n)/2}$. The inequality (1.9) is one of our basic estimates and in using it we will often dominate $\|B_0\|_{\#}$ by the Hilbert Schmidt #-norm $\|B_0\|_{\#HS} \leq \|b\|_{\#2}, \|B_0\|_{\#HS} = \sqrt{Ext\text{-}\sum_{i \in {}^*\infty} \|Ae_i\|_{\#}}$, and where $\{e_i | i \in {}^*\infty\}$ is an orthonormal basis in $\mathcal{F}^{\#}$. By definition the field with hyperfinite momentum cut-off $\varphi^{\#}_\varkappa(\boldsymbol{x}), \boldsymbol{x} = (x_1, x_2, x_3) \in {}^*\widetilde{\mathbb{R}}^{\#\boldsymbol{3}}_c, \varkappa \in {}^*\widetilde{\mathbb{R}}^{\#}_{c+} \backslash {}^*\widetilde{\mathbb{R}}^{\#}_{\text{fin}+}$ is

$$\varphi^{\#}_\varkappa(\boldsymbol{x}) = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} \left(Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k}) + a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{-1/2} d^{\#3}k =$$

$$= Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}^{\#3}_c} \theta(\|\boldsymbol{k}\|, \varkappa)\left(Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k}) + a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{-1/2} d^{\#3}k. \quad (1.10)$$

We also define the bilinear form

$$\pi^{\#}_\varkappa(x) = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} i\left(Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k}) + a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{\frac{1}{2}} d^{\#3}k =$$

$$Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}^{\#3}_c} i\theta(\|\boldsymbol{k}\|, \varkappa)\left(Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k}) + a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{\frac{1}{2}} d^{\#3}k, \quad (1.11)$$

the conjugate momentum to $\varphi^{\#}_\varkappa(x)$. Since the kernels $b(\boldsymbol{k}) = \theta(\|\boldsymbol{k}\|, \varkappa)\left(Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}$ in $L^{\#}_2$ the bilinear forms (1.10)-(111) define operator-valued functions $\varphi^{\#}_\varkappa(\boldsymbol{x})$: ${}^*\widetilde{\mathbb{R}}^{\#\boldsymbol{3}}_c \to L(\mathcal{F}^{\#})$ and $\pi^{\#}_\varkappa(\boldsymbol{x})$: ${}^*\widetilde{\mathbb{R}}^{\#\boldsymbol{3}}_c \to L(\mathcal{F}^{\#})$. For real $f(\boldsymbol{x}), g(\boldsymbol{x})$ such that $\theta(\|\boldsymbol{k}\|, \varkappa)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}\hat{f}(x) \in L^{\#}_2$ and $\theta(\|\boldsymbol{k}\|, \varkappa)[\mu(\boldsymbol{k})]^{\frac{1}{2}}\hat{g}(x) \in L^{\#}_2$, , the bilinear forms $\varphi^{\#}_\varkappa(f)$ and $\pi^{\#}_\varkappa(g)$ define operators whose #-closures on $D\left(N_\varkappa^{\frac{1}{2}}\right)$ are self-#-adjoint. They satisfy the canonical commutation relations

$$Ext\text{-}\exp\left(i\pi^{\#}_\varkappa(g)\right) Ext\text{-}\exp\left(i\varphi^{\#}_\varkappa(g)\right) =$$

$$Ext\text{-}\exp(i\langle f, g\rangle_{\#})\left\{Ext\text{-}\exp\left(i\varphi^{\#}_\varkappa(g)\right) Ext\text{-}\exp\left(i\pi^{\#}_\varkappa(g)\right)\right\}. \quad (1.12)$$

It is furthermore possible to define polynomial functions of the field $\varphi^{\#}_\varkappa(x)$, the Wick polynomials : $\varphi^{\#}_\varkappa(x)$: (see chapter I for a definition of the Wick dots : :). Explicitly, as a bilinear form on $D\left(N_\varkappa^{n/2}\right) \times D\left(N_\varkappa^{n/2}\right)$,

$$:\varphi^{\#n}_\varkappa(x) := \sum_{j=0}^{n} \binom{n}{j} b_x(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n)\, a^*(\boldsymbol{k}_1) \cdots a^*(\boldsymbol{k}_j) a(-\boldsymbol{k}_{j+1}) \cdots a(-\boldsymbol{k}_n), \quad (1.13)$$

where

$$b_x(\boldsymbol{k}_1, \dots, \boldsymbol{k}_n) = \prod_{j=1}^{n} \theta(\|\boldsymbol{k}_j\|, \varkappa)[\mu(\boldsymbol{k}_j)]^{-1/2}\, Ext\text{-}\exp\left(-i\langle \sum_{j=1}^{n} \boldsymbol{k}_j\,, \boldsymbol{x}\rangle\right).$$

Thus for real $f(x) \in S^{\#}\left({}^{*}\widetilde{\mathbb{R}}_c^{\#3}\right)$, the bilinear form

$$:\varphi_{\varkappa}^{\#n}(f) := Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_c^{\#3}} :\varphi_{\varkappa}^{\#n}(\boldsymbol{x}) f(\boldsymbol{x}) : d^{\#3}x$$

has a kernel proportional to $\prod_{j=1}^{n} \theta(\|\boldsymbol{k}_j\|, \varkappa)[\mu(\boldsymbol{k}_j)]^{-1/2} \hat{f}\left(\sum_{j=1}^{n} \boldsymbol{k}_j\right)$. Thus from (1.9) we conclude that $:\varphi_{\varkappa}^{\#n}(f):$ defines a symmetric operator on the domain $D\left(N_{\varkappa}^{n/2}\right)$. It was shown in chapter I sect. 15 that $:\varphi_{\varkappa}^{\#n}(f):$ is essentially self-#-adjoint on this domain.

2. The periodic hyperfinite approximation in configuration space. The cut-off Hamiltonian $H_{\varkappa}(g)$.

The cut-off Hamiltonian $H_{\varkappa}(g)$ acts on $\mathcal{F}_V^{\#}$ and can be written in terms of the field operator $\varphi_{\varkappa}^{\#}(\boldsymbol{x}), \boldsymbol{x} = (x_1, x_2, x_3)$ as

$$H_{\varkappa}(g) = H_{0,\varkappa} + Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_c^{\#3}} \varphi_{\varkappa}^{\#4}(x)\, d^{\#3}x = H_{0,\varkappa} + H_{I,\varkappa,g} \tag{2.1}$$

where $H_{0,\varkappa} = H_{\varkappa}(0)$ is the free hamiltonian, and $0 \le g$. Let

$$\mathrm{C}^{*\infty}(H_{0,\varkappa}) = \bigcap_{n=0}^{*\infty} D(H_{0,\varkappa}^n)$$

be the set of $\mathrm{C}^{*\infty}$ vectors for $H_{0,\varkappa}$. It was shown in [16]-[17] that $H_{\varkappa}(g)$ and $H_{I,\varkappa,g}$ are essentially self-#-adjoint on $\mathrm{C}^{*\infty}(H_{0,\varkappa})$, that

$$D\left(H_{\varkappa}(g)\right) = D\left(H_{0,\varkappa}\right) \cap D\left(H_{I,\varkappa,g}\right) \tag{2.2}$$

and that there are finite or hyperfinite $a, b = b(g)$ such that

$$\|H_{0,\varkappa}\psi\|_{\#} + \|H_{I,\varkappa,g}\psi\|_{\#} \le \|(H_{\varkappa}(g) + b)\psi\|_{\#} \tag{2.3}$$

for all $\psi \in D\left(H_{\varkappa}(g)\right)$.

Note that it is convenient to introduce a periodic hyperfinite approximation in configuration space. Under this approximation, the momentum space variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^{*}\widetilde{\mathbb{R}}_c^{\#3}$ is replaced by a discrete variable $\boldsymbol{k} \in \Gamma_V^3$

$$\Gamma_V^3 = \left\{\boldsymbol{k} = (k_1, k_2, k_3) | k_i = \frac{2\pi n_i}{V}, n_i \in {}^{*}\mathbb{Z}; i = 1,2,3\right\}$$

with $V \in {}^{*}\widetilde{\mathbb{R}}_{c+}^{\#} \backslash {}^{*}\widetilde{\mathbb{R}}_{\mathrm{fin}+}^{\#}$. Thus we define $\mathcal{F}_V^{\#}$, the Fock space for volume V^3 as

$$\mathcal{F}_V^{\#} = \mathfrak{C}\left(l_2^{\#}(\Gamma_V^3)\right) = {}^{*}\mathbb{C}^{\#} \oplus l_2^{\#}(\Gamma_V^3) \oplus \{l_2^{\#}(\Gamma_V^3) \otimes_s l_2^{\#}(\Gamma_V^3)\} \cdots$$

We identify $\mathcal{F}_V^{\#}$ with the subspace of $\mathcal{F}^{\#}$ consisting of piecewise constant functions which are constant on each cube of volume $(2\pi/V)^{3j}$ cantered about a lattice point

$$\{\boldsymbol{k}_1, \dots \boldsymbol{k}_j\} \in \Gamma_V^3 \times \Gamma_V^3 \times \cdots \times \Gamma_V^3 = \Gamma_V^{3j}.$$

The periodic annihilation and creation operators $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ can be extended from $\mathcal{F}_V^{\#}$ to $\mathcal{F}^{\#}$ by the formulas

$$a_V(\boldsymbol{k}) = \left(\frac{V}{2\pi}\right)^{3/2}\left[Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_1\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_2\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_3\, a(\boldsymbol{k}+\boldsymbol{l})\right], \tag{2.4}$$

$$a^{*}_{V}(\boldsymbol{k}) = \left(\frac{V}{2\pi}\right)^{3/2}\left[Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_1\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_2\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_3\, a^*(\boldsymbol{k}+\boldsymbol{l})\right]. \tag{2.5}$$

Therefore the periodic field $\varphi^{\#}_{\varkappa,V}(\boldsymbol{x})$ and the periodic Hamiltonian $H_{\varkappa,V}(g)$ can be extended to act on $\mathcal{F}^{\#}$ by the formulas

$$\varphi^{\#}_{\varkappa,V}(\boldsymbol{x}) = (2V)^{-3/2} Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma^3_V,|\boldsymbol{k}|\leq\varkappa} Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)[a^*(\boldsymbol{k})+a(-\boldsymbol{k})]\big(\mu(\boldsymbol{k})\big)^{-1/2}, \tag{2.6}$$

$$H_{\varkappa,V} = H_{0,\varkappa,V} + H_{I,\varkappa,V}, \tag{2.7}$$

$$H_{I,\varkappa,V} = Ext\text{-}\int_{-V/2}^{V/2} Ext\text{-}\int_{-V/2}^{V/2} Ext\text{-}\int_{-V/2}^{V/2} \varphi^{\#4}_{\varkappa,V}(\boldsymbol{x})\, d^{\#3}x, \tag{2.8}$$

$$H_{0,\varkappa,V} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} a^*(\boldsymbol{k})\, a(\boldsymbol{k})\mu(\boldsymbol{k}_V) d^{\#3}k \tag{2.9}$$

with $\boldsymbol{k_V}$, a lattice point infinite close to $\boldsymbol{k}$,

$$\boldsymbol{k_V} \in \Gamma^3_V, \|\boldsymbol{k}-\boldsymbol{k_V}\| \leq \frac{\pi}{V} \approx 0. \tag{2.10}$$

Remark 2.1 Note the absence of a V in the $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ in (2.1.9). On $\mathcal{F}^{\#}_V$, this definition of $H_{0,\varkappa,V}$ agrees with the standard definition

$$Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma^3_{V,|\boldsymbol{k}|\leq\varkappa}} a^{*}_{V}(\boldsymbol{k}) a_V(\boldsymbol{k})\mu(\boldsymbol{k}\,).$$

The operators $H_{I,\varkappa,V}$ and $H_{\varkappa,V}$ are essentially self adjoint on $\mathrm{C}^{*\infty}\big(H_{0,\varkappa,V}\big)$, and

$$D\big(\, H_{\varkappa,V}\,\big) = D\big(H_{0,\varkappa,V}\big) \cap D\big(H_{I,\varkappa,V}\big). \tag{2.11}$$

For all $\psi \in D\big(\, H_{\varkappa,V}\,\big)$,

$$\big\|H_{0,\varkappa,V}\psi\big\|_{\#} + \big\|H_{I,\varkappa,V}\psi\big\|_{\#} \leq a\big\|\big(\, H_{\varkappa,V}+b\big)\psi\big\|_{\#}, \tag{2.12}$$

where b depend on V. On $\mathcal{F}^{\#}_V$, the operator $H_{I,\varkappa,V}$ has a #-compact resolvent. We want to approximate $H_{I,\varkappa}(g)$ by operators with #-compact resolvents on $\mathcal{F}^{\#}_V$, so we define

$$H_{\varkappa}(g,V) = H_{0,\varkappa,V} + Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} : \varphi^{\#4}_{\varkappa,V}(x) g(\boldsymbol{x}) d^{\#3}x = \tag{2.13}$$

$$= H_{0,\varkappa,V} + H_{I,\varkappa}(g,V).$$

As in chapter I sect. we can show that $H_{\varkappa}(g,V)$, and $H_{I,\varkappa}(g,V)$ are essentially self-#-adjoint on $\mathrm{C}^{*\infty}\big(H_{0,\varkappa,V}\big)$, and that

$$D(H_{\varkappa}(g,V)\,) = D\big(H_{0,\varkappa,V}\big) \cap D\left(H_{I,\varkappa}(g,V)\right). \tag{2.14}$$

Furthermore, for all $\psi \in D(H_{\varkappa}(g,V)\,)$,

$$\big\|H_{0,\varkappa,V}\psi\big\|_{\#} + \big\|H_{I,\varkappa}(g,V)\psi\big\|_{\#} \leq a\|(H_{\varkappa}(g,V)+b)\psi\|_{\#}. \tag{2.15}$$

In this case both g and V serve as volume cutoffs, and the constant $b = b(g,V)$ can be chosen independently of V for fixed g. On the space $\mathcal{F}_V^{\#}$, the operator $H_\varkappa(g,V)$ has a #-compact resolvent. Our hamiltonians are semi-bounded and for each $\varepsilon > 0$, there is a constant b such that

$$0 \leq \varepsilon H_{0,\varkappa} + H_{I,\varkappa}(g) + b, \tag{2.16}$$

$$0 \leq \varepsilon H_{0,\varkappa,V} + H_{I,\varkappa,V} + b, \tag{2.17}$$

$$0 \leq \varepsilon H_{0,\varkappa,V} + H_{I,\varkappa}(g,V) + b, \tag{2.18}$$

see chapter I sect.18. In (2.18), the b can be chosen to be independent of V. Taking $\varepsilon = 1/2$, we have

$$\frac{1}{2} H_{0,\varkappa} \leq H_{I,\varkappa}(g) + b,$$

which implies that for all $\psi \in D\left(\left(H_\varkappa(g)\right)^{\frac{1}{2}}\right)$,

$$\left\| H_{0,\varkappa}^{1/2}\psi \right\|_{\#} \leq \sqrt{2}\left\| \left(H_\varkappa(g) + b(\varkappa)\right)^{1/2}\psi \right\|_{\#}. \tag{2.19}$$

Here we must choose $b(\varkappa)$ at least$|E_\varkappa(2g)|$, where $E_\varkappa(2g)$ is the vacuum energy for the cut-off $2g$.

3. The existence of a vacuum vector $\boldsymbol{\Omega_{\varkappa,g}}$ for $\boldsymbol{H_\varkappa(g)}$

In this section we prove the existence of a vacuum vector $\Omega_{\varkappa,g}$ for $H_\varkappa(g)$. Since the Hamiltonian $H_\varkappa(g)$ is bounded from below, we can define the vacuum energy $E_{\varkappa,g} \triangleq E(\varkappa,g)$ to be the infimum of the spectrum of $H_\varkappa(g)$ and we also refer to $E_{\varkappa,g}$ as the *lower bound* of $H_\varkappa(g)$. We show that $E_{\varkappa,g}$ is an isolated point in the spectrum. In a relativistic theory, the gap between the ground state and the first excited state is the mass of the interacting particle. For this reason we say that $H_\varkappa(g)$ has a mass gap. A vacuum vector $\Omega_{\varkappa,g}$ is defined as a normalized eigenvector of $H_\varkappa(g)$ corresponding to the eigenvalue $E_{\varkappa,g}$.

$$H_\varkappa(g)\Omega_{\varkappa,g} = E_{\varkappa,g}\Omega_{\varkappa,g}, \left\|\Omega_{\varkappa,g}\right\|_{\#} = 1. \tag{3.1}$$

Theorem 3.1 There is exists a vacuum vector $\Omega_{\varkappa,g}$ for Hamiltonian $H_\varkappa(g)$. For any $\varepsilon > 0, \varepsilon \approx 0$ the operator $H_\varkappa(g)$, restricted to the spectral interval $\left[E_{\varkappa,g}, E_{\varkappa,g} + m_0 - \varepsilon\right]$ is #-compact.
Theorem 3.2 The approximate Hamiltonian $H_{\varkappa,V}(g)$, has a vacuum vector $\boldsymbol{\Omega_{\varkappa,g,V}}$. Any hyperinfinite sequence of volumes $\boldsymbol{V_j}$ tending to hyperinfinity ${}^*\infty$ has a hyperinfinite subsequence $V_l, l \in {}^*\mathbb{N}$ such that #-limit

.
$$\Omega_{\varkappa,g} = \#\text{-}\lim_{l \to {}^*\infty} \Omega_{\varkappa,g,V_l} \tag{3.2}$$

exists and satisfies (3.1).
Remark 3.1 Let $E_{\varkappa,g,V}$ be the lower bound of $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$. Since $H_{\varkappa,V}(g)$ has a #-compact resolvent on $\mathcal{F}_V^{\#}$, there is a vacuum vector $\Omega_{\varkappa,g,V}$ for $H_{\varkappa,V}(g) \restriction \mathcal{F}_V^{\#}$. We now see that $E_{\varkappa,g,V}$ is the lower bound for $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$, so that $\Omega_{\varkappa,g,V}$ is a vacuum vector for $H_{\varkappa,V}(g)$.
Remark 3.2 Let $\mathcal{F}_V^{\#\perp}$ be the orthogonal complement of $\mathcal{F}_V^{\#}$. Since $H_{\varkappa,V}(g)$ leaves $\mathcal{F}_V^{\#}$ invariant and is self-#-adjoint, $H_{\varkappa,V}(g)$ also leaves $\mathcal{F}_V^{\#\perp}$ invariant.
Theorem 3.3 The lower bound of $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$ is $E_{\varkappa,g,V} + m_0$, where m_0, is the rest mass of the Fock space bosons.
Remark 3.3 Theorem 3.3 shows that $\Omega_{\varkappa,g,V}$ is a vacuum for $H_{\varkappa,V}(g)$.
Proof We have an orthogonal decomposition in the single particle space

$$\mathcal{F}_1^{\#} = L_2^{\#}\left({}^*\mathbb{R}_c^{\#3}\right) = \mathcal{F}_{1V}^{\#} \oplus \mathcal{F}_{1V}^{\#\perp}. \tag{3.3}$$

Here $\mathcal{F}_{1V}^{\#} = \mathcal{F}_1^{\#} \cap \mathcal{F}_V^{\#}$ consists of functions piecewise constant on intervals cantered at lattice points. Thus we may write

$$\mathcal{F}^{\#} = Ext\text{-}\oplus_{j=0}^{^*\infty} \mathcal{F}^{\#(j)},\ \mathcal{F}_V^{\#\perp} = Ext\text{-}\oplus_{j=1}^{^*\infty} \mathcal{F}^{\#(j)}, \tag{3.4}$$

where $\mathcal{F}^{\#(j)}$ consists of vectors with exactly j particles from $\mathcal{F}_{1V}^{\#\perp}$ and

$$\mathcal{F}^{\#(j)} = (Ext\text{-}\ \mathcal{F}_{1V}^{\#\perp} \otimes_s \cdots \otimes_s \mathcal{F}_{1V}^{\#\perp}) \otimes_s \mathcal{F}_V^{\#} \tag{3.5}$$

In this tensor product decomposition there are j factors $\mathcal{F}_{1V}^{\#\perp}$. The Hamiltonian $H_{\varkappa,V}(g)$ leaves each subspace $\mathcal{F}^{\#(j)}$ invariant, and on $\mathcal{F}^{\#(j)}$ we have $H_{\varkappa,V}(g) = I \otimes A + B \otimes I$, where $A = H_{\varkappa,V}(g) \upharpoonright \mathcal{F}_V^{\#}$ and B is a sum of j copies of $H_{0,\varkappa,V}$ each acting on a single factor $\mathcal{F}_{1V}^{\#\perp}$. Since

$$jm_0 \leq B, \tag{3.6}$$

the Theorem follows from this decomposition.

Theorem 3.4 For $V \leq {}^*\infty$, and for b sufficiently large we have

$$D(H_{0,\varkappa}) \subset D\left(H_{0,\varkappa}^{\frac{1}{2}}\right) \cap D(N_\varkappa) \subset D\big(H_{\varkappa,V}(g) + b\big), \tag{3.7}$$

$$D(H_{0,\varkappa}) \subset D\left(\left[(N_\varkappa + I)^{-1}\big(H_{\varkappa,V}(g) + b\big)\right]^{\#-}\right). \tag{3.8}$$

Here we denote by $A^{\#-}$ #-closure of the operator A.

Proof We take b large enough so that $H_{\varkappa,V}(g) + b$ is positive, see (2.1.18). By (1.9) and (2.14) we get

$$D(H_{0,\varkappa}) \cap D(N_\varkappa^2) \subset D(H_{0,\varkappa}) \cap D\left(H_{I,\varkappa,V}(g)\right) = D\left(H_{\varkappa,V}(g)\right) \subset D\left(\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}\right).$$

Thus for all $\psi \in D(H_{0,\varkappa}) \cap D(N_\varkappa^2)$,

$$\left\|\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}\psi\right\|_{\#}^2 = \langle\psi, \big(H_{\varkappa,V}(g) + b\big)\psi\rangle_{\#} \leq \langle\psi, \big(H_{\varkappa,V} + b\big)\psi\rangle_{\#} +$$

$$+\left\|(N_\varkappa + I)^{-1} H_{I,\varkappa,V} (N_\varkappa + I)^{-1}\right\|_{\#} \|(N_\varkappa + I)\psi\|_{\#}^2.$$

Since $\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}$ is a #-closed operator, we can extend this inequality by #- continuity. As $N_\varkappa$ and $H_{0,\varkappa,V}$ commute, the inequality extends by #-continuity to all $\psi \in D\left(H_{0,\varkappa}^{1/2}\right) \cap D(N_\varkappa) \supset D(H_{0,\varkappa})$. The proof of (3.8) is similar.

Theorem 3.5 Let z be non-real or real and sufficiently negative. Then as V tends to hyper infinity ${}^*\infty$,

$$\left\|\big(H_{\varkappa,V}(g) - zI\big)^{-1} - \big(H_\varkappa(g) - zI\big)^{-1}\right\|_{\#} = O(V^{-1}). \tag{3.9}$$

Proof Let us fix g and z and suppress g when possible. In chapter I sect 16 we have shown that $H_\varkappa(g)$ is essentially self-#-adjoint on $C^{^*\infty}(H_{0,\varkappa})$. Thus vectors of the form $\chi = (H_\varkappa - zI)\psi, \psi \in C^{^*\infty}(H_{0,\varkappa})$, are #-dense in $\mathcal{F}^{\#}$. On these vectors

$$\left\{\big(H_{\varkappa,V} - zI\big)^{-1} - (H_\varkappa - zI)^{-1}\right\}\chi = \big(H_{\varkappa,V} - zI\big)^{-1}\left\{(H_\varkappa - zI)\psi - \big(H_{\varkappa,V} - zI\big)\psi\right\} =$$

$$= \big(H_{\varkappa,V} - zI\big)^{-1}\big(H_\varkappa - H_{\varkappa,V}\big)(H_\varkappa - zI)^{-1}\chi =$$

$$= \left(H_{\varkappa,V} - zI\right)^{-1} (N_\varkappa + I)(N_\varkappa + I)^{-1} \left(H_\varkappa - H_{\varkappa,V}\right)(N_\varkappa + I)^{-1}(N_\varkappa + I)(H_\varkappa - zI)^{-1} \chi.$$

For $\theta \in \mathcal{F}^{\#}$,

$$\left|\langle \theta, \left\{ \left(H_{\varkappa,V} - zI\right)^{-1} - (H_\varkappa - zI)^{-1} \right\} \chi \rangle_{\#}\right| \leq \tag{3.10}$$

$$\leq \left\| (N_\varkappa + I)\left(H_{\varkappa,V} - \bar{z}I\right)^{-1} \right\|_{\#} \cdot \|\theta\|_{\#} \cdot \left\| (N_\varkappa + I)^{-1}\left(H_\varkappa - H_{\varkappa,V}\right)(N_\varkappa + I)^{-1}\right\|_{\#} \times$$

$$\times \|(N_\varkappa + I)(H_\varkappa - zI)^{-1}\|_{\#} \|\chi\|_{\#} \cdot$$

Using (2.15), we find that $\left\| (N_\varkappa + I)\left(H_{\varkappa,V} - \bar{z}I\right)^{-1} \right\|_{\#}$ is bounded uniformly in V, since

$$\left\| (N_\varkappa + I)\left(H_{\varkappa,V} - \bar{z}I\right)^{-1}\psi \right\|_{\#} \leq \mathrm{const} \cdot \left\| \left(H_{0,\varkappa,V} + I\right)\left(H_{\varkappa,V} - \bar{z}I\right)^{-1}\psi \right\|_{\#} \leq$$

$$\leq \mathrm{const} \cdot \left\| H_{\varkappa,V}\left(H_{\varkappa,V} - \bar{z}I\right)^{-1}\psi \right\|_{\#} + \mathrm{const} \cdot \left\| \left(H_{\varkappa,V} - \bar{z}I\right)^{-1}\psi \right\|_{\#},$$

where the constants can be chosen independently of V. By a similar consideration, the orthogonal decomposition (3.3) shows that $(N_\varkappa + I)\left(H_{\varkappa,V} - zI\right)^{-1}$ is a bounded operator. Thus from (3.10), and the fact that the χ are #-dense, we infer

$$\left\| \left(H_{\varkappa,V} - zI\right)^{-1} - (H_\varkappa - zI)^{-1} \right\|_{\#} \leq \mathrm{const} \cdot \left\| (N_\varkappa + I)^{-1}\left(H_\varkappa - H_{\varkappa,V}\right)(N_\varkappa + I)^{-1}\right\|_{\#} \tag{3.11}$$

with a constant independent of V. The difference $H_\varkappa - H_{\varkappa,V} = \left(H_{0,\varkappa} - H_{0,\varkappa,V}\right) + \left(H_{I,\varkappa}(g) - H_{I,\varkappa,V}(g)\right)$ and for infinite large V,

$$\left\| (N_\varkappa + I)^{-1/2}\left(H_{0,\varkappa} - H_{0,\varkappa,V}\right)(N_\varkappa + I)^{-1}\right\|_{\#} = O(V^{-1}). \tag{3.12}$$

This is a simple direct computation, using $|\mu(\boldsymbol{k}_V) - \mu(\boldsymbol{k})| = O(V^{-1})$. For the interaction terms, we use (1.10) to estimate

$$\left\| (N_\varkappa + I)^{-1/2} \left(H_{I,\varkappa}(g) - H_{I,\varkappa,V}(g)\right) (N_\varkappa + I)^{-1}\right\|_{\#} = O(V^{-1}). \tag{3.13}$$

The kernel $b\,(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4)$ corresponding to a monomial in $H_{I,\varkappa}(g)$ is

$b\,(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) = \binom{4}{j} \prod_{j=1}^{4} \theta(\|\boldsymbol{k}_j\|, \varkappa)[\mu(\boldsymbol{k}_j)]^{-1/2} \hat{g}\left(k_1^{(1)} + k_2^{(1)} + k_3^{(1)} + k_4^{(1)}, \dots, k_1^{(3)} + k_2^{(3)} + k_3^{(3)} + k_4^{(3)}\right),$

$0 \leq j \leq 4$. The kernel $b_V(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4)$) for the corresponding monomial in $H_{I,\varkappa,V}(g)$ is obtained by replacing the factor $\prod_{j=1}^{4} \theta(\|\boldsymbol{k}_j\|, \varkappa)[\mu(\boldsymbol{k}_j)]^{-1/2}$ by the factor $\prod_{j=1}^{4} \theta(\|\boldsymbol{k}_{jV}\|, \varkappa)[\mu(\boldsymbol{k}_{jV})]^{-1/2}$. Inspection of the difference $b\,(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) - b_V(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4)$ shows that $\|b\,(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) - b_V(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4)\|_{L_2^{\#}} = O(V^{-1})$.as $V \to {}^*\infty$, from which we conclude that (3.13) is $O(V^{-1})$. The #-convergence of the resolvents follows from (3.11)-(3.13). The #-limit

$$E_{\varkappa,g,V} \to_{\#} E_{\varkappa,g}$$

follows from the #-convergence of the resolvents, since for large positive b,

$$\left(E_{\varkappa,g,V} + b\right)^{-1} = \left\| \left(H_{\varkappa,V}(g) + b\right)^{-1} \right\|_{\#}.$$

Proof of the theorems 3.1 and 3.2 Let $f(x)$ be a #-smooth positive function with support in the interval $[-\varepsilon, m_0-\varepsilon]$. Then $f\big(H_{\varkappa,V}(g)-E_{\varkappa,g,V}\big)\restriction \mathcal{F}_V^{\#}$ is #-compact, since the resolvent of $H_{\varkappa,V}(g)\restriction \mathcal{F}_V^{\#}$is #-compact on $\mathcal{F}_V^{\#}$. By Theorem 3.3, $f\big(H_{\varkappa,V}(g)-E_{\varkappa,g,V}\big)\restriction \mathcal{F}_V^{\#\perp}=0$ and therefore #-compact on the full Fock space $\mathcal{F}_V^{\#}$. By Theorem 3.5, the resolvent $\big(H_{\varkappa,V}(g)-E_{\varkappa,g,V}-z\big)^{-1}$ #-converge in #-norm as $V\to{}^*\infty$, and therefore

$$\big\|f\big(H_{\varkappa,V}(g)-E_{\varkappa,g,V}\big)-f\big(H_{\varkappa}(g)-E_{\varkappa,g}\big)\big\|_{\#}\to_{\#} 0,$$

since $f\big(H_{\varkappa}(g)-E_{\varkappa,g}\big)$ is a bounded function of $\big(H_{\varkappa}(g)-E_{\varkappa,g}-z\big)^{-1}$ which vanishes at hyper infinity. Since the uniform #-limit of #-compact operators is #-compact, $H_{\varkappa}(g)$ restricted to the spectral interval $[-\varepsilon, m_0-\varepsilon]$ is #-compact. This means furthermore that only a finite or hyperfinite number of eigenvalues of $H_{\varkappa,V}(g)$ #-converge to $E_{\varkappa,g}$. Theorem 3.6 shows that the projection onto the corresponding set of eigenvectors of $H_{\varkappa,V}(g)$ #-converge as $V\to{}^*\infty$. Since $\Omega_{\varkappa,g,V}$ is an eigenvector of $f\big(H_{\varkappa,V}(g)-E_{\varkappa,g,V}\big)$ a hyperinfinite subsequence of the $\Omega_{\varkappa,g,V}$ #-converge to a #-limit as $V\to{}^*\infty$. For this #-limit

$$\big(E_{\varkappa,g}+b\big)^{-1}\Omega_{\varkappa,g}=\#\text{-}\lim_{l\to{}^*\infty}\big(E_{\varkappa,g,V_l}+b\big)^{-1}\Omega_{\varkappa,g,V_l}=$$

$$=\#\text{-}\lim_{l\to{}^*\infty}\big(H_{\varkappa,V_l}(g)+b\big)^{-1}\Omega_{\varkappa,g,V_l}=\big(H_{\varkappa}(g)+b\big)^{-1}\Omega_{\varkappa,g}$$

by Theorem 3.5. Hence $\Omega_{\varkappa,g}\in D\big(H_{\varkappa}(g)\big)$, $H_{\varkappa}(g)\Omega_{\varkappa,g}=E_{\varkappa,g}\Omega_{\varkappa,g}$ and $\Omega_{\varkappa,g}$ is a vacuum vector for $H_{\varkappa}(g)$. In the following section we will see that $\Omega_{\varkappa,g}$ and $\Omega_{\varkappa,g,V}$ are unique except for an arbitrary phase multiple $Ext\text{-}\exp(i\theta)$, and that there is a natural choice for this arbitrary phase. With this choice, we then will prove that the $\Omega_{\varkappa,g,V}$ #-converge to $\Omega_{\varkappa,g}$ as $V\to{}^*\infty$.

4. Uniqueness of the vacuum.

In this subsection we prove the uniqueness of a vacuum vector $\Omega_{\varkappa,g}$ for $H_{\varkappa}(g)$.

Theorem 4. 1 The vacuum vector $\Omega_{\varkappa,g,V}$ for $H_{\varkappa}(g)$ is unique.

Remark 4.1 In other words $E_{\varkappa,g}$, the lower bound of $H_{\varkappa}(g)$ is a simple eigenvalue.

Definition 4.1 Let $\mathcal{H}^{\#}=L_2^{\#}(Q,d^{\#}\mu)$ be a non-Archimedean Hilbert space. We say that a bounded operator $A:\mathcal{H}^{\#}\to\mathcal{H}^{\#}$ has a strictly positive kernel provided that

$$\langle\psi,A\chi\rangle_{\#}>0 \tag{4.1}$$

whenever ψ and χ are non-negative $L_2^{\#}$ functions with non-zero #-norms. Such an operator transforms a function $\chi\geq 0$, $\|\chi\|_{\#}\neq 0$ into a function $A\chi$ which is strictly positive #-almost everywhere.

Definition 4.2 Let $\mathcal{H}^{\#}=L_2^{\#}(Q,d^{\#}\mu)$ be a non-Archimedean Hilbert space. We say that a bounded operator $A:\mathcal{H}^{\#}\to\mathcal{H}^{\#}$ has a positive, ergodic kernel if for each ψ,χ as above $\langle\psi.A\chi\rangle\geq 0$ and

$$\langle\psi,A^j\chi\rangle_{\#}>0 \tag{4.2}$$

for some j, depending on ψ and χ. Clearly every A with a strictly positive kernel has a positive, ergodic kernel.

Theorem 4.2 Let A have a positive ergodic kernel, and suppose that $\|A\|_{\#}$ is an eigenvalue of A. Then $\|A\|_{\#}$ is a simple eigenvalue and the corresponding eigenvector can be chosen to be a strictly positive function.

Proof Since A maps positive functions into positive functions it also maps real functions into real functions. If $\psi\in\mathcal{H}^{\#}$ satisfies $A\psi=\|A\|_{\#}\cdot\psi$, then so do $\mathrm{Re}\psi$ and $\mathrm{Im}\psi$. Therefore without loss of generality we may assume that ψ is real. Since $\|A^j\|_{\#}=\|A\|_{\#}^j$, and $A^j\psi=\|A\|_{\#}^j\cdot\psi$, we infer that

$$\|A^j\|_{\#}\cdot\|\psi\|_{\#}^2=\langle\psi.A^j\psi\rangle_{\#}\leq\langle|\psi|,A^j|\psi|\rangle_{\#}\leq\|\boldsymbol{A}\|_{\#}^{\boldsymbol{j}}\cdot\|\psi\|_{\#}^2,$$

$$\langle\psi.A^j\psi\rangle_{\#}=\langle|\psi|,A^j|\psi|\rangle_{\#}.$$

Writing now $\psi = \psi^+ - \psi^-$, where ψ^+ and ψ^- are the positive and negative parts of ψ,

$$\langle \psi^+, A^j\psi^+\rangle_\# - \langle \psi^+, A^j\psi^-\rangle_\# - \langle \psi^-, A^j\psi^+\rangle_\# + \langle \psi^-, A^j\psi^-\rangle_\# =$$

$$= \langle \psi^+, A^j\psi^+\rangle_\# + \langle \psi^+, A^j\psi^-\rangle_\# + \langle \psi^-, A^j\psi^+\rangle_\# + \langle \psi^-, A^j\psi^-\rangle_\#$$

or

$$\langle \psi^+, A^j\psi^-\rangle_\# + \langle \psi^-, A^j\psi^+\rangle_\# = 0. \tag{4.3}$$

Unless $\psi^+ = 0$ or $\psi^- = 0$, each term of (4.3) could be made strictly positive by choosing an appropriate j. Thus either ψ^+ or ψ^- must vanish, and we may choose the eigenvector ψ to be non-negative. If $\chi \geq 0, \|\chi\|_\# \neq 0$, then for some integer j, $0 < \langle \chi, A^j\psi\rangle_\# = \|A\|_\#^j \cdot \langle \chi, \psi\rangle_\#$. This proves that $\chi\psi$ is not zero almost everywhere, and that ψ is strictly positive #-almost everywhere. Finally, if ψ and χ were linearly independent eigenvectors of A with the eigenvalue $\|A\|_\#$, then we could repeat the above argument with the component of χ orthogonal to ψ. This would yield two positive, orthogonal eigenvectors, which would be impossible, and the proof is complete.

Remark 4.2 Let $\varphi_\varkappa^\#(h) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_\varkappa^\#(x)h(x)\,d^{\#3}x$ denote the smeared, time zero free field operators. The spectral projections of the $\varphi_\varkappa^\#(h)$, or the functions $Ext\text{-}\exp\left(i\varphi_\varkappa^\#(h)\right)$ generate a maximal abelian algebra $\mathcal{M}^\#$ of bounded operators on $\mathcal{F}^\#$. Let Q be the spectrum of the algebra $\mathcal{M}^\#$. The no particle vector $\Omega_0 \in \mathcal{F}^\#$ is a cyclic vector for 'DR, namely $\mathcal{F}^\# = \#\text{-}\overline{(\mathcal{M}^\#\Omega_0)}$. Therefore we may introduce a #-measure $d^\#\mu$ on Q so that $\mathcal{F}^\#$ is unitarily equivalent to $L_2^\#(Q, d^\#\mu)$ and so that the equivalence carries $\mathcal{M}^\#$ into $L_{^*\infty}^\#$ and takes Ω_0 into the function 1.

Theorem 4.3 With $\mathcal{F}^\#$ represented as $L_2^\#(Q, d^\#\mu)$, $Ext\text{-}\exp\left(-H_{0,\varkappa}\right)$ has a positive, ergodic kernel.

Proof Let ψ and χ be non-negative. Write $\psi = \psi_1 + \psi_2$, where ψ_1 is the component of ψ along Ω_0. Thus the $L_1^\#$ #-norm of ψ is given by $\|\psi\|_{\#1} = \langle \psi, \Omega_0\rangle_\# = \langle \psi_1, \Omega_0\rangle_\#$. Note $\|\psi\|_{\#1} \neq 0$ whenever ψ is non-zero, and $\left\|Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\psi_2\right\|_{\#1} \leq \left(Ext\text{-}\exp(-tm_0)\right)\|\psi_2\|_{\#1}$, where m_0 is the boson mass. Thus

$$\langle \psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_\# \geq \|\psi\|_{\#1} \cdot \|\chi\|_{\#1} - \|\psi_2\|_{\#1} \cdot \|\chi_2\|_{\#1}\left(Ext\text{-}\exp(-tm_0)\right). \tag{4.4}$$

By choosing t sufficiently large, (4.4) is positive, which proves (4.2). If the following inequality holds

$$Ext\text{-}\exp(-tm_0) < \frac{1}{2}\frac{\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{\|\psi_2\|_{\#1} \cdot \|\chi_2\|_{\#1}} = \frac{1}{2}\frac{\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{\left(\|\psi\|_\#^2 - \|\psi\|_{\#1}^2\right)^{1/2}\left(\|\chi\|_\#^2 - \|\chi\|_{\#1}^2\right)^{1/2}} \tag{4.5}$$

then

$$\langle \psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_\# \geq \frac{1}{2}\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}. \tag{4.6}$$

We need to show that $\langle \psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_\# > 0$ for all finite t. In fact, it is sufficient to prove this for a #-dense set of non-negative ψ and χ. Let us consider an approximate free energy operator

$$H_{0,\varkappa,V} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} a^*(\boldsymbol{k})a(\boldsymbol{k})\mu(\boldsymbol{k}_V)d^{\#3}k. \tag{4.7}$$

For vectors $\psi \in C^{*\infty}\left(H_{0,\varkappa}\right)$, as $V \to {}^*\infty$. $\left\|H_{0,\varkappa,V}\psi - H_{0,\varkappa}\psi\right\|_\# \to_\# 0$. Since $H_{0,\varkappa}$ is essentially self-#-adjoint on $C^{*\infty}\left(H_{0,\varkappa}\right)$, the resolvents of $H_{0,\varkappa,V}$ converge strongly **[18, p. 429]**. Thus the generalized semigroup #-convergence theorem **[18, p. 502]** ensures that for all $\psi \in \mathcal{F}^\#$

$$\left\|Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\psi - Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\psi\right\|_\# \to_\# 0$$

as $V \to {}^*\infty$, and the #-convergence is uniform on #-compact sets of t. Therefore we need only show that for a #-dense set of non-negative ψ and χ $\langle \psi, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi\rangle_\# \geq 0$. Let $F(x_1, \dots, x_n)$ be a non- negative, hyper infinitely #-differentiable function with #-compact support, and let

$$\psi = F\left(\varphi_{\varkappa}^{\#}(f_1), \dots, \varphi_{\varkappa}^{\#}(f_n)\right)\Omega_0, \tag{4.8}$$

where $f_1, \dots, f_n$ are real. The set of all such vectors are #-dense in $\mathcal{F}^{\#+}$, the non-negative vectors in $\mathcal{F}^{\#}$. Furthermore, we define

$$\psi_{\varkappa,V} = F\left(\varphi_{\varkappa,V}^{\#}(f_1), \dots, \varphi_{\varkappa,V}^{\#}(f_n)\right)\Omega_0, \tag{4.9}$$

where $\varphi_{\varkappa,V}^{\#}(f_1)$ is defined by restricting the sum in (2.6) to those

$$\boldsymbol{k} \in \Gamma_{\varkappa,V}^{3} = \Gamma_{V}^{3} \cap \{\boldsymbol{k} | |\boldsymbol{k}| \leq \varkappa\}.$$

Then $\psi_{\varkappa,V} \in \mathcal{F}_{\varkappa,V}^{\#+} \subset \mathcal{F}^{\#+}$ where $\mathcal{F}_{\varkappa,V}^{\#+}$ is the Fock space corresponding to the modes $\boldsymbol{k} \in \Gamma_{\varkappa,V}^{3}$. For any vector $\chi \in C^{*\infty}\left(H_{0,\varkappa}\right)$

$$\left\|\varphi_{\varkappa,V}^{\#}(f)\chi - \varphi_{\varkappa}^{\#}(f)\chi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^{*}\infty,$$

and as $C^{*\infty}\left(H_{0,\varkappa}\right)$ is a #-core for $\varphi_{\varkappa}^{\#}(f)$, the resolvents of $\varphi_{\varkappa,V}^{\#}(f)$ #-converge strongly to the resolvent of $\varphi_{\varkappa}^{\#}(f)$. **[18, p. 429]**. Thus the generalized semigroup #-convergence theorem [19] ensures that for each $\chi \in \mathcal{F}^{\#}$, s real

$$\left\|Ext\text{-}\exp\left(is\varphi_{\varkappa,V}^{\#}(f)\right)\psi - Ext\text{-}\exp(is\, \varphi_{\varkappa}^{\#}(f))\psi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^{*}\infty,$$

and the #-convergence is uniform for #-compact sets of s. By (4.9)

$$\psi_{\varkappa,V} = Ext\text{-}\int \hat{F}(s_1, \dots, s_n)\left[i\sum_{j=1}^{n} Ext\text{-}\exp\left(is\varphi_{\varkappa,V}^{\#}\left(f_j\right)\right)\right] d^{\#}s_1 \cdots d^{\#}s_n,$$

and $\hat{F}(s_1, \dots, s_n)$ vanishes rapidly at hyper infinity, so we conclude that

$$\left\|\psi_{\varkappa,V} - \psi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^{*}\infty.$$

Thus for such vectors ψ, χ,

$$\langle\psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_{\#} = \#\text{-}\lim_{V\to{}^{*}\infty}\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#}$$

and we need only show that

$$\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} \geq 0. \tag{4.10}$$

However on $\mathcal{F}_{\varkappa,V}^{\#}$

$$H_{0,\varkappa,V} = Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma_{\varkappa,V}^{3}} a_{V}^{*}(\boldsymbol{k})a_{V}(\boldsymbol{k})\mu(\boldsymbol{k}) = Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma_{\varkappa,V}^{3}} H_{0,\varkappa,V},$$

so $Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right) = Ext\text{-}\prod_{\boldsymbol{k}\in\Gamma_{\varkappa,V}^{3}} \exp\left(-tH_{0,\varkappa,V}\right)$. It easily verify by explicit computation that each operator $\exp\left(-tH_{0,\varkappa,V}\right)$ have a strictly positive kernel, so (4.10) holds and the proof is complete.

Theorem 4.4 With $\mathcal{F}^{\#}$ represented as $L_2^{\#}(Q, d^{\#}\mu)$, the operator $Ext\text{-}\exp\left(-H_{\varkappa}(g)\right)$ has a positive, ergodic kernel.

Remark 4.3 We expect that $Ext\text{-}\exp\left(-H_{0,\varkappa}\right)$ and $Ext\text{-}\exp\left(-H_{\varkappa}(g)\right)$ have strictly positive kernels.

Proof As in Theorem 4.3, formula (4.7), we consider $H_{\varkappa,V}(g) = H_{0,\varkappa} + H_{I,\varkappa,V}(g)$. The approximate interaction $H_{I,\varkappa,V}(g)$ is constructed with $\varphi_{\varkappa,V}^{\#}$ in place of $\varphi_{\varkappa}^{\#}$. Since $C^{*\infty}\left(H_{0,\varkappa}\right)$ is a #-core for $H_{\varkappa}(g)$, we can argue as in the previous theorem that for all $\psi \in \mathcal{F}^{\#}$

$$Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\psi \to_{\#} Ext\text{-}\exp\left(-tH_{\varkappa}(g)\right)\psi, \text{ as } V \to {}^*\infty.$$

Thus we need only prove that for ψ, χ as in Theorem 4.3

$$0 < \varepsilon < \langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_{\#}. \tag{4.11}$$

and that for sufficiently large t, the constant $\varepsilon = \varepsilon(\psi, \chi, \varkappa, V)$ can be chosen independently of $\varkappa$ and V. On $\mathcal{F}^{\#}_{\varkappa,V}$ we have an explicit representation of $Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)$ given by generalized Feynman-Kac integral formula

$$\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_{\#} = \tag{4.12}$$

$$Ext\text{-}\int_{C_{\varkappa,V}} Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t H_{I,g,\varkappa,V}\left(q(s)\right)d^{\#}t\right]\right)\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)D^{\#}q(\cdot).$$

Here $q(s)$ denotes a points in the spectrum of the modes

$$q_V(\boldsymbol{k}) = a_V(\boldsymbol{k}) + a_V(\boldsymbol{k}) + a^*_V(\boldsymbol{k}) + a^*_V(-\boldsymbol{k})$$

$$q'_V(\boldsymbol{k}) = a_V(\boldsymbol{k}) - a_V(-\boldsymbol{k}) + a^*_V(\boldsymbol{k}) - a^*_V(-\boldsymbol{k})$$

for $\boldsymbol{k} \in \Gamma^3_{\varkappa,V} = \{\boldsymbol{k} | \boldsymbol{k} \in \Gamma^3_V \wedge |\boldsymbol{k}| \leq \varkappa\}$, and $C_{\varkappa,V}$ is the path space for these modes. Since $Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)$ has a strictly positive kernel, (4.12) exhibits $Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)$ explicitly as an operator with a strictly positive kernel. Thus (4.11) is valid, and taking the #-limit as $V \to {}^*\infty$ shows that

$$\langle \psi, Ext\text{-}\exp\left(-tH_{\varkappa}(g)\right)\chi\rangle_{\#} \geq 0. \tag{4.13}$$

We now establish a uniform lower bound on ε in (4.11) to prove that for t sufficiently large (4.13) is strictly positive. Given any positive M we can split the integral (4.13) into two parts. Let $C^{(1)}_{\varkappa,V}$ be those paths such that the exponent in the Feynman-Kac formula satisfies $-\left[Ext\text{-}\int_0^t H_{I,g,\varkappa,V}\left(q(s)\right)d^{\#}t\right] \geq -M$, and let $C^{(2)}_{\varkappa,V}$ be the complementary set of paths. Thus

$$\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_{\#} \geq \left(Ext\text{-}\exp(-M)\right)Ext\text{-}\int_{C^{(1)}_{\varkappa,V}}\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)D^{\#}q(\cdot) =$$

$$= \left(Ext\text{-}\exp(-M)\right)\left\{\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} - Ext\text{-}\int_{C^{(2)}_{\varkappa,V}}\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)D^{\#}q(\cdot)\right\}. \tag{4.14}$$

First we choose t by (4.5) so that (4.6) holds. Then for sufficiently infinitely large V (depending on t),

$$\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} \geq \frac{1}{2}\langle \psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_{\#} \geq \frac{1}{4}\|\psi\|_{\#1}\cdot\|\chi\|_{\#1}.$$

Thus (4.14) becomes

$$\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} \geq$$

$$\geq Ext\text{-}\exp(-M)\left\{\frac{1}{4}\|\psi\|_{\#1}\cdot\|\chi\|_{\#1} - Ext\text{-}\int_{C^{(2)}_{\varkappa,V}}\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)D^{\#}q(\cdot)\right\}.$$

Let $\mathbf{Pr}\{\cdot\}$ denote the #-measure on path space, so that by the generalized Holder inequality

$$\left|Ext\text{-}\int_{C^{(2)}_{\varkappa,V}}\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)D^{\#}q(\cdot)\right| \leq \left(\mathbf{Pr}\left\{C^{(2)}_{\varkappa,V}\right\}\right)^{\frac{(r-1)}{r}}\left\|\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)\right\|_{\#r},$$

where $1 < r < 2$. By the smoothing property of $Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)$ for sufficiently large t

$$\left\|\psi_{\varkappa,V}\left(q(0)\right)\chi_{\varkappa,V}\left(q(t)\right)\right\|_{\#r} \leq \left\|\psi_{\varkappa,V}\right\|_{\#2} \times \left\|\chi_{\varkappa,V}\right\|_{\#2}$$

and for V sufficiently infinitely large, this is dominated by $2\|\psi\|_{\#2} \cdot \|\chi\|_{\#2}$. Thus with the choices so far made for V, t, M,

$$\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_{\#} \geq Ext\text{-}\exp(-M)\left\{\frac{1}{4}\|\psi\|_{\#1} \cdot \|\chi\|_{\#1} - 2\|\psi\|_{\#2} \cdot \|\chi\|_{\#2}\left(\mathbf{Pr}\left\{C_{\varkappa,V}^{(2)}\right\}\right)^{\frac{(r-1)}{r}}\right\}$$

$$\geq$$

$$\geq \tfrac{1}{8}\left(Ext\text{-}\exp(-M)\right)\|\psi\|_{\#1} \cdot \|\chi\|_{\#1} > \varepsilon(\psi, \chi, t)$$

provided in addition that

$$\mathbf{Pr}\left\{C_{\varkappa,V}^{(2)}\right\} \leq \left(\frac{\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{16\|\psi\|_{\#2} \cdot \|\chi\|_{\#2}}\right)^{\frac{r}{(r-1)}}. \tag{4.15}$$

We now show that for M sufficiently large, (4.15) is satisfied and therefore theorem is proved.
Note that

$$\mathbf{Pr}\left\{C_{\varkappa,V}^{(2)}\right\} = \mathbf{Pr}\left\{M \leq Ext\text{-}\int_0^t H_{I,\varkappa,g,V}\, q(s) d^{\#}s\right\} = \mathbf{Pr}\left\{1 \leq M^{-1}\left(Ext\text{-}\int_0^t H_{I,\varkappa,g,V}\, q(s) d^{\#}s\right)\right\} \leq$$

$$\leq M^{-2}\left[Ext\text{-}\int_{C_{\varkappa,V}}\left|Ext\text{-}\int_0^t H_{I,\varkappa,g,V}\, q(s) d^{\#}s\right|^2 d^{\#}q(\cdot)\right].$$

Replacing the integral over s by a #-limit of hyperfinite Riemann sums, we obtain a bound in terms of generalized Wiener integrals depending on a hyperfinite number of times.

$$\mathbf{Pr}\left\{C_{\varkappa,V}^{(2)}\right\} \leq \#\text{-}\lim_{n\to^*\infty}\left(\frac{t}{nM}\right)^2\left[Ext\text{-}\sum_{i,j=1}^{n} Ext\text{-}\int_{C_{\varkappa,V}} H_{I,\varkappa,g,V}\, q\left(\frac{ti}{n}\right) H_{I,\varkappa,g,V}\, q\left(\frac{tj}{n}\right) d^{\#}q(\cdot)\right].$$

By the definition of the generalized Wiener integral, this expression can be evaluated in terms of no-particle expectation values, and it equals

$$\#\text{-}\lim_{n\to^*\infty}\left(\frac{t}{nM}\right)^2 Ext\text{-}\sum_{i,j=1}^{n}\langle\Omega_0, \left\{Ext\text{-}\exp\left(-|i-j|tH_{0,\varkappa}/n\right)\right\} \cdot H_{I,\varkappa,g,V}\Omega_0\rangle_{\#}.$$

By the generalized Schwarz inequality

$$\mathbf{Pr}\left\{C_{\varkappa,V}^{(2)}\right\} \leq \left(\frac{t}{M}\right)^2\left\|H_{I,\varkappa,g,V}\Omega_0\right\|_{\#}^2 \leq \left(\frac{tD_{\varkappa}}{M}\right)^2$$

for some constant $D_{\varkappa}$ independent of V. Thus we choose

$$M \geq Dt\left(\frac{16\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{16\|\psi\|_{\#2} \cdot \|\chi\|_{\#2}}\right)^{\frac{r}{(r-1)}}.$$

Combining Theorem 4.4 with Theorem 4.2 yields a proof of Theorem 4.1. Clearly the same proof applies to $H_{\varkappa,V}(g)$, to show that its vacuum is unique.

Corollary 4.5. Let $\Omega_{\varkappa,g,V}$ be the vacuum for $H_{\varkappa,g,V}$, with its phase determined by the requirement

$$\langle\Omega_0, \Omega_{\varkappa,g,V}\rangle_{\#} > 0. \tag{4.16}$$

Then $\#\text{-}\lim_{V\to^*\infty}\Omega_{\varkappa,g,V} = \Omega_{\varkappa,g}$ exists, $\Omega_{\varkappa,g}$ is the vacuum for $H_{\varkappa,g}$, and

$$\langle \Omega_0, \Omega_{\varkappa,g} \rangle_{\#} > 0. \tag{4.17}$$

Proof A hyper infinite sequence $\Omega_{\varkappa,g,V_j}$ with $V_j \to {}^*\infty$ has a #-convergent hyper infinite subsequence by Theorem 4.2, #-converging to a vacuum for $H_{\varkappa,g}$. The phase (4.16) fixes the phase (4.17) so every #-convergent hyper infinite subsequence has the same #-limit $\Omega_{\varkappa,g}$. Thus the $\Omega_{\varkappa,g,V}$ #-converge to $\Omega_{\varkappa,g}$, as required.

Corollary 4.6 The vacuum $\Omega_{\varkappa,g}$ is a cyclic vector for $\mathcal{M}$.

Proof The function $\Omega_{\varkappa,g}$ is positive for #-almost all $q \in Q^{\#}$, and $\mathcal{M} = L^{\#}_{{}^*\infty}(Q^{\#})$ in the $L^{\#}_2(Q^{\#}, d^{\#}f)$ representation of $\mathcal{F}^{\#}$.

5. The Heisenberg picture field operators

In the Heisenberg picture operators have the time dependence

$$A(t) = Ext\text{-}\exp\big(it\, H_{\varkappa}(g)\big)A(0)Ext\text{-}\exp\big(-i\, tH_{\varkappa}(g)\big) \tag{5.1}$$

This definition of the dynamics contains the cut-off function $g(\boldsymbol{x})$ explicitly. For an important class of operators $A(0)$, however, $A(t)$ is independent of $g(\boldsymbol{x})$ provided that $g(\boldsymbol{x}) = \lambda$, the coupling constant, on a suitably large set. For example, we take $A(0)$ to be an observable representing a measurement performed in some 3-dimensional region $B \subset {}^*\mathbb{R}^{\#3}_c$ of space (at time $t = 0$). Then $A(t)$ represents the same measurement performed at time t. A Hamiltonian with a hyperfinite ultraviolet cut-off $\varkappa \in {}^*\mathbb{R}^{\#}_{c+} \backslash {}^*\mathbb{R}^{\#}_{\text{fin}+}$, such as $H_{\varkappa}(g)$, propagates information with at most the speed of light. Therefore if $g(\boldsymbol{x}) = \lambda$ on a region containing B, and t is sufficiently small, the fact that $g(\boldsymbol{x})$ does not equal λ everywhere will never be recorded by a measurement $A(t)$. For each localized observable $A(0)$ and each t, we make an appropriate choice for $g(\boldsymbol{x})$. Therefore (5.1) provides the correct dynamics for the $(\varphi^4)_4$ quantum field theory with the cut quantum field theory with the cut-off removed. In this section we consider the quantum field operators $\varphi^{\#}_{\varkappa}(\boldsymbol{x}, t)$ or ${}^*\mathbb{R}^{\#4}_c$

$$\varphi^{\#}_{\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} \varphi^{\#}_{\varkappa}(\boldsymbol{x}, t)\, d^{\#3}x d^{\#}t. \tag{5.2}$$

We see that integration helps in (4.2) because $\varphi^{\#}_{\varkappa}(f)$ is an operator while $\varphi^{\#}_{\varkappa}(\boldsymbol{x}, t)$ is a bilinear form. Actually the time integration is not required and for real f,

$$A(t) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \varphi^{\#}_{\varkappa}(\boldsymbol{x}, t)\, d^{\#3}x \tag{5.3}$$

is also a self-#-adjoint operator depending #-continuously on t. We expect that this is a special feature of the two dimensional model we are considering and that sharp time fields will not be operators in four dimensions. For this reason, basic physical concepts have been formulated in terms of the time averaged fields (5.2) rather than the sharp time fields (5.3). For example, Wightman's axioms for a quantum field theory are expressed in terms of the opera tors (5.2), and we will show that many of his axioms are satisfied for our model. The time integration in (5.2) presents some new difficulties (for example in the proof of self-#-adjointness or of locality) which would not occur if we considered only the sharp time operators (5.3). An advantage of the time averaged field is that products $\varphi(f_1) \cdots \varphi(f_n)$ can be defined on vectors with finite energy (Corollary 6.5). In fact we will construct a dense domain which the localized field operator $\varphi(f)$ leaves invariant, and on which p(f) is essentially self-#-adjoint for real f.

6. An invariant domain for localized quantum fields.

In this section we study the Heisenberg picture field localized in a 4-dimensional region of space time $\mathcal{B}$. We find that $\varphi^{\#}_{\varkappa}(\boldsymbol{x}, t)$ is a bilinear form and that for real f, $\varphi^{\#}_{\varkappa}(f)$ is a #-densely defined symmetric operator. We start with the region B, a bounded open subset of space time. We require that $H_{\varkappa}(g)$ be a Hamiltonian for $\mathcal{B}$. This means that the spatial cut-off $g(\boldsymbol{x})$ equals the coupling constant λ on a sufficiently large interval to contain the domain of dependence of $\mathcal{B}$. In other words, assuming that the velocity of light is one, for every point $(\boldsymbol{y}, t) \in \mathcal{B}$,

$$g(\boldsymbol{x}) = \lambda, \text{ if } \|\boldsymbol{x} - \boldsymbol{y}\| < t. \tag{6.1}$$

It is convenient to deal with the field

$$\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t) = Ext\text{-}\exp\big(it\ H_{\varkappa}(g)\big)\varphi^{\#}_{\varkappa}(\boldsymbol{x})Ext\text{-}\exp\big(-i\ tH_{\varkappa}(g)\big)$$

and its time #-derivative

$$\pi^{\#}_{\varkappa,g}(\boldsymbol{x},t) = Ext\text{-}\exp\big(it\ H_{\varkappa}(g)\big)\pi^{\#}_{\varkappa}(\boldsymbol{x})Ext\text{-}\exp\big(-i\ tH_{\varkappa}(g)\big) = \partial^{\#}\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)/\partial^{\#}t.$$

The time zero fields $\varphi^{\#}_{\varkappa}(\boldsymbol{x})$ and its conjugate momentum $\pi^{\#}_{\varkappa}(\boldsymbol{x})$ were defined in chapter I. We shall see that for $(\boldsymbol{x},t) \in \mathcal{B}$, $\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)$ is independent of g, and equals the field $\varphi^{\#}_{\varkappa}(\boldsymbol{x},t)$. Thus all the cut-offs have been removed in the definition of $\varphi^{\#}_{\varkappa}(\boldsymbol{x},t)$. For each $C^{{}^*\infty}$-function $f(\boldsymbol{x},t)$ with support in $\mathcal{B}$, we show that

$$\varphi^{\#}_{\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c}\varphi^{\#}_{\varkappa}(\boldsymbol{x},t)f(\boldsymbol{x},t)\,d^{\#3}xd^{\#}t \tag{6.2}$$

is an operator whose domain contains

$$D^{\#}_{\varkappa,g} = C^{{}^*\infty}\big(\ H_{\varkappa}(g)\big) = \cap_{n=0}^{{}^*\infty}D\big(H^{n}_{\varkappa}(g)\big), \tag{6.3}$$

In fact $D^{\#}_{\varkappa,g}$ is an invariant domain, i.e.

$$\varphi^{\#}_{\varkappa}(f)D^{\#}_{\varkappa,g} \subset D^{\#}_{\varkappa,g}, \tag{6.4}$$

so that $D^{\#}_{\varkappa,g} \subset C^{{}^*\infty}\big(\varphi^{\#}_{\varkappa}(f)\big)$. We note that this invariant domain may depend on the region $\mathcal{B}$ in which the field $\varphi^{\#}_{\varkappa}(f)$ is localized. For $\psi \in D^{\#}_{\varkappa,g}$ the expectation values

$$\langle\psi,\varphi^{\#}_{\varkappa}(\boldsymbol{x}_1,t_1)\cdots\varphi^{\#}_{\varkappa}(\boldsymbol{x}_n,t_n)\psi\rangle_{\#} \tag{6.5}$$

is ${}^*\mathbb{C}^{\#4}_c$- valued Schwartz distribution in $\mathcal{D}^{\#\prime}(\mathcal{B}\times\cdots\times\mathcal{B})$. If $f(\boldsymbol{x},t)$ is a function in $S^{\#}({}^*\mathbb{R}^{\#4}_c)$, then $\varphi^{\#}_{\varkappa,g}(f)$ still is defined on $D^{\#}_{\varkappa,g}$and leaves it invariant. The expectation values (6.5) of $\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)$ are tempered distributions in $S^{\#\prime}({}^*\mathbb{R}^{\#4}_c)$. However, the fields $\varphi^{\#}_{\varkappa,g}(f)$ may depend on g.

Lemma 6.1. The field $\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)$ is a bilinear form on $D\big((\ H_{\varkappa}(g)+b)^{1/2}\big)\times D\big((\ H_{\varkappa}(g)+b)^{1/2}\big)$ #-continuous in $\boldsymbol{x}$ and t. Namely for $\psi \in D\left((\ H_{\varkappa}(g)+b)^{\frac{1}{2}}\right)$, $\langle\psi,\varphi^{\#}_{\varkappa}(\boldsymbol{x}\,,t\,)\psi\rangle_{\#}$ is a #-continuous function. Furthermore

$$\left|Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c}\langle\psi,\varphi^{\#}_{\varkappa},g(\boldsymbol{x}\,,t\,)\psi\rangle_{\#}\frac{\partial^{\#}}{\partial^{\#}x_i}f(\boldsymbol{x})\right| \leq \mathrm{const}\cdot\|f\|_{\#2}\langle\psi,(\ H_{\varkappa}(g)+b)\psi\rangle_{\#}, i=1,2,3. \tag{6.6}$$

Proof The free field $\varphi^{\#}_{\varkappa}(\boldsymbol{x}\,,0\,)$ is the sum of two expressions of the form (1.8). The kernels $\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})$ are in $L^{\#}_2$. Furthermore we have $\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2} \in L^{\#}_2$. The estimate (1. 9) has been generalized to cover such kernels, giving us

$$\left\|\big(\ H_{0,\varkappa}+I\big)^{-1/2}\varphi^{\#}_{\varkappa,g}(\boldsymbol{x}\,,0\,)\big(\ H_{0,\varkappa}+I\big)^{-1/2}\right\|_{\#} \leq \mathrm{const}\cdot\left\|\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}\right\|_{\#} < {}^*\infty\,. \tag{6.7}$$

Thus for $\psi \in D\left((\ H_{\varkappa}(g)+b)^{\frac{1}{2}}\right)$, $Ext\text{-}\exp\big(-it\ H_{\varkappa}(g)\big)\psi \in D\left((\ H_{\varkappa}(g)+b)^{\frac{1}{2}}\right) \subset D\big(H^{1/2}_{0,\varkappa}\big)$, by (2.19) and therefore $\langle\psi,\varphi^{\#}_{\varkappa,g}(\boldsymbol{x}\,,t\,)\psi\rangle_{\#} = \langle Ext\text{-}\exp\big(-it\ H_{\varkappa}(g)\big)\psi,\varphi^{\#}_{\varkappa,g}(\boldsymbol{x}\,,0\,)Ext\text{-}\exp\big(-it\ H_{\varkappa}(g)\big)\psi\rangle_{\#}$ is defined and

$$\left|\langle\psi,\varphi^{\#}_{\varkappa,g}(\boldsymbol{x}\,,t\,)\psi\rangle_{\#}\right| \leq \mathrm{const}\cdot\left\|\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}\right\|_{\#}\cdot\langle\psi,(\ H_{\varkappa}(g)+b)\psi\rangle_{\#}$$

Since $\left\|\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}|\boldsymbol{k}|^2\hat{f}(\boldsymbol{k})\right\|_{\#} \leq \left\|\theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}|\boldsymbol{k}|^2\right\|_{{}^*\infty}\cdot\|f\|_{\#2} \leq \mathrm{const}\cdot\|f\|_{\#2}$ the inequality (4.2.6) holds. Let us write $b_{x_i}, i=1,2,3$ for b to denote the dependence of b on x_i. Then

$\left\|\left(b_{x_i}-b_{y_i}\right)\theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{-1/2}\right\|_{\#2}$ is a function of (x_i-y_i) only and it #-tends to zero as $|\boldsymbol{x}-\boldsymbol{y}|\ \to_{\#} 0$. Since

$$\left|\langle\psi,\left(\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)-\varphi^{\#}_{\varkappa,g}(\boldsymbol{y},t)\right)\psi\rangle_{\#}\right|\leq \text{const}\cdot\left\|\left(I+H_{0,\varkappa}\right)^{1/2}\psi\right\|^2_{\#}\cdot\left\|\left(b_{x_i}-b_{y_i}\right)\theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}\right\|_{\#2}\leq$$

$$\leq \text{const}\cdot\left\|\left(H_{\varkappa}(g)+b\right)^{1/2}\psi\right\|^2_{\#}\cdot\left\|\left(b_{x_i}-b_{y_i}\right)\theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}\right\|_{\#2} \qquad (6.8)$$

we have continuity with respect to $\boldsymbol{x}$. Also

$$\left\|\left(H_{\varkappa}(g)+b\right)^{\frac{1}{2}}\left(Ext\text{-}\exp\left(-it\,H_{\varkappa}(g)\right)-Ext\text{-}\exp\left(-is\,H_{\varkappa}(g)\right)\right)\right\|_{\#}\to_{\#} 0$$

as $|t-s|\ \to_{\#} 0$. Thus

$$\left|\langle\psi,\left(\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)-\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},s)\right)\psi\rangle_{\#}\right|\leq$$

$$\leq \text{const}\cdot\left\|\left(I+H_{0,\varkappa}\right)^{1/2}\left(Ext\text{-}\exp\left(-it\,H_{\varkappa}(g)\right)-Ext\text{-}\exp\left(-is\,H_{\varkappa}(g)\right)\right)\psi\right\|_{\#}\cdot\left\|\left(b_{x_i}\right)\theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}\right\|_{\#2}\times$$

$$\times\left\{\left\|\left(I+H_{0,\varkappa}\right)^{1/2}\left(Ext\text{-}\exp\left(-it\,H_{\varkappa}(g)\right)\right)\psi\right\|_{\#}+\left\|\left(I+H_{0,\varkappa}\right)^{1/2}\left(Ext\text{-}\exp\left(-is\,H_{\varkappa}(g)\right)\right)\psi\right\|_{\#}\right\}\leq$$

$$\leq$$

$$\text{const}\cdot\left\|\left(H_{\varkappa}(g)+b\right)^{1/2}\left(Ext\text{-}\exp\left(-it\,H_{\varkappa}(g)\right)-Ext\text{-}\exp\left(-is\,H_{\varkappa}(g)\right)\right)\psi\right\|_{\#}\cdot\left\|\left(b_{x_i}\right)\theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{-\frac{1}{2}}\right\|_{\#2}\times$$

$$\times\left\|\left(H_{\varkappa}(g)+b\right)^{1/2}\psi\right\|_{\#}\to_{\#} 0 \qquad (6.9)$$

as $|t-s|\ \to_{\#} 0$.

From (6.8)-(6.9) we see that $\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)$ is jointly #-continuous in x and t. Probably $\pi^{\#}_{\varkappa,g}(\boldsymbol{x},t)$ is a bilinear form on $D\left(\left(H_{\varkappa}(g)+b\right)^{3/2}\right)\times D\left(\left(H_{\varkappa}(g)+b\right)^{3/2}\right)$ #-continuous in $\boldsymbol{x}$ and t, but our estimates are not strong enough to prove this. The functions $f(\boldsymbol{x},t)$ in $S^{\#}_{\text{fin}}\ ({}^{*}\mathbb{R}^{\#4})$ determine bounded #-measures $d^{\#}v\ =\ f(\boldsymbol{x},t)d^{\#3}xd^{\#}t$, so $\varphi^{\#}_{\varkappa,g}(f)=Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\,f(\boldsymbol{x},t)d^{\#3}xd^{\#}t$ is a bilinear form. If $d^{\#}v_n\to_{\#} d^{\#}v$ in the weak topology for #-measures, then $Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\ d^{\#}v_n\to_{\#} Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\ d^{\#}v$ in the weak sense that for $\psi\in D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$

$$\langle\psi,Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\ d^{\#}v_n\,\psi\rangle_{\#}\to_{\#}\langle\psi,Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\ d^{\#}v\ \psi\rangle_{\#}. \qquad (6.10)$$

We define also the sharp time field

$$A^{\#}_{\varkappa,g}(t)=Ext\text{-}\int\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\,f(\boldsymbol{x},t)d^{\#3}x \qquad (6.11)$$

and

$$B^{\#}_{\varkappa,g}(t)=Ext\text{-}\int\pi^{\#}_{\varkappa,g}(\boldsymbol{x},t)\,f(\boldsymbol{x},t)d^{\#3}x. \qquad (6.12)$$

Lemma 6.2 Let function $f(\boldsymbol{x},t)$ in $S^{\#}_{\text{fin}}\ ({}^{*}\mathbb{R}^{\#4})$ be real. Then $A^{\#}_{\varkappa,g}(t)$ and $B^{\#}_{\varkappa,g}(t)$ define self- #-adjoint operators, and their domains include $D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$. With a constant c independent of g,

$$\left\|A^{\#}_{\varkappa,g}(t)\psi\right\|_{\#}+\left\|B^{\#}_{\varkappa,g}(t)\psi\right\|_{\#}\leq c\{\|f(\cdot,t)\|_{\#2}+\|D^{\#}_{x}f(\cdot,t)\|_{\#2}\}\left\|\left(H_{\varkappa}(g)+b\right)^{\frac{1}{2}}\psi\right\|_{\#}, \qquad (6.13)$$

for all $\psi\in D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$.

Proof It is sufficient to consider $\varphi^{\#}_{\varkappa}(f_t)=Ext\text{-}\int\varphi^{\#}_{\varkappa}(\boldsymbol{x})\,f(\boldsymbol{x},t)d^{\#3}x$ in place **o**f $A^{\#}_{\varkappa,g}(t)$ and $\pi^{\#}_{\varkappa}(f_t)=$

$Ext\text{-}\int \pi^{\#}_{\varkappa}(\boldsymbol{x})\, f(\boldsymbol{x},t) d^{\#3}x$ in place **o**f $B^{\#}_{\varkappa,g}(t)$, as they are unitarily equivalent by the unitary operator $Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)$, and this unitary leaves $D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$ invariant. The same is true for $\pi^{\#}_{\varkappa}(f_t)$ By (2. 19) we have

$$\left\|(I+N_{\varkappa})^{1/2}\psi\right\|^2_{\#} \le m_0^{-1}\left\|\left(I+H_{0,\varkappa}\right)^{1/2}\psi\right\|^2_{\#} \le 2m_0^{-1}\left\|\left(H_{\varkappa}(g)+b\right)^{1/2}\psi\right\|^2_{\#}$$

so we need only prove that

$$\|\varphi^{\#}_{\varkappa}(f_t)\psi\|_{\#} + \|\pi^{\#}_{\varkappa}(f_t)\psi\|_{\#} \le c\{\|f(\cdot,t)\|_{\#2} + \|D^{\#}_x f(\cdot,t)\|_{\#2}\}\left\|(I+N_{\varkappa})^{\frac{1}{2}}\psi\right\|_{\#}. \tag{6.14}$$

The lemma now follows from (1. 9). For example, $\pi^{\#}_{\varkappa}(f_t)$ is the sum of two operators of the form (1. 8) with kernels $b(\boldsymbol{k}) = \theta(\boldsymbol{k},\varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{\frac{1}{2}}\left(Ext\text{-}\int\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right) f(\boldsymbol{x},t)d^{\#3}x\right)$, satisfying the inequality

$$\|b\|_{\#2} = \left\|(-D^{\#2}_x + m_0^2)^{1/4} f(\cdot,t)\right\|_{\#2} \le \text{const}\cdot(\|f(\cdot,t)\|_{\#2} + \|D^{\#}_x f(\cdot,t)\|_{\#2}).$$

The kernel for $\varphi^{\#}_{\varkappa}(f_t)$ can be bounded by the $\|f(\cdot,t)\|_{\#2}$ #-norm alone. The estimate (6.13) now follows from (6.14). The self-#-adjointness of $A^{\#}_{\varkappa,g}(t)$ and $B^{\#}_{\varkappa,g}(t)$ can be proven by showing that $\varphi^{\#}_{\varkappa}(f_t)$ and $\pi^{\#}_{\varkappa}(f_t)$ are self-#-adjoint. But (4.14) ensures that every vector in $\mathcal{F}^{\#}$ with a finite or hyperfinite number of particles is #-analytic for $\varphi^{\#}_{\varkappa}(f_t)$ and for $\pi^{\#}_{\varkappa}(f_t)$, so these operators are essentially self-#-adjoint on the domain of vectors with a finite or hyperfinite number of particles. Hence they are uniquely determined by their definition on that domain.

We now explain the sense in which the integral (6.12) #-converges, since we did not show that $\pi^{\#}_{\varkappa,g}(f_t)$ was a bilinear form. If $\psi \in D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$, then $\psi \in D\left(\left(N_{\varkappa}+I\right)^{1/2}\right)$ and

$$\langle\psi, Ext\text{-}\hat{\pi}^{\#}_{\varkappa}(\boldsymbol{k})\psi\rangle_{\#} = \theta(\boldsymbol{k},\varkappa)[\mu(\boldsymbol{k})]^{\frac{1}{2}}\{\langle a(\boldsymbol{k})\psi,\psi\rangle_{\#} + \langle\psi, a(-\boldsymbol{k})\psi\rangle_{\#}\} \tag{6.15}$$

is a slowly increasing, locally summable function, and hence a tempered distribution in $S^{\#\prime}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$. Thus $\langle\psi, \pi^{\#}_{\varkappa}(\boldsymbol{x})\psi\rangle_{\#}$ is by definition the distribution Fourier transform of (6.15), and hence a tempered distribution in $S^{\#\prime}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$. Finally (6.12) is the weak integral

$$\langle\psi, B^{\#}_{\varkappa,g}(t)\psi\rangle_{\#} = Ext\text{-}\int d^{\#3}x f(\boldsymbol{x},t)\left\langle\left(Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right)\psi, \pi^{\#}_{\varkappa}(\boldsymbol{x})\left(Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right)\psi\right\rangle_{\#}.$$

Theorem 6.3 Let $\psi \in D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$, and let $f(\boldsymbol{x},t)$ be a real function in $S^{\#}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$. Then the vectors $A^{\#}_{\varkappa,g}(t)\psi$ and $B^{\#}_{\varkappa,g}(t)\psi$ are strongly #-continuous and are rapidly decreasing functions of t. The integrals $Ext\text{-}\int A^{\#}_{\varkappa,g}(t)\psi d^{\#}t = \varphi^{\#}_{\varkappa,g}(f)\psi$ and $Ext\text{-}\int B^{\#}_{\varkappa,g}(t)\psi d^{\#}t = \pi^{\#}_{\varkappa,g}(f)\psi$ exist and define $\varphi^{\#}_{\varkappa,g}(f)$ and $\pi^{\#}_{\varkappa,g}(f)$ as #-closed symmetric operators with domains containing $D\left(\left(H_{\varkappa}(g)+b\right)^{1/2}\right)$. We have the estimate

$$\left\|\varphi^{\#}_{\varkappa,g}(f)\psi\right\|_{\#} + \left\|\pi^{\#}_{\varkappa,g}(f)\psi\right\|_{\#} \le c\left(Ext\text{-}\int\{\|f(\cdot,t)\|_{\#2} + \|D^{\#}_x f(\cdot,t)\|_{\#2}\}d^{\#}t\right)\left(H_{\varkappa}(g)+b\right)^{1/2}\psi \tag{6.16}$$

with a constant c independent of f and g.

Proof We write

$$|f(\cdot,t)|_1 = c\|f(\cdot,t)\|_{\#2} + c\|D^{\#}_x f(\cdot,t)\|_{\#2} \tag{6.17}$$

and

$$A^{\#}_{\varkappa,g}(s)\psi - A^{\#}_{\varkappa,g}(t)\psi = \left(I - Ext\text{-}\exp\left(-i(t-s)\, H_{\varkappa}(g)\right)\right) A^{\#}_{\varkappa,g}(s)\psi +$$

$$+\left[Ext\text{-}\exp\left(it\, H_{\varkappa}(g)\right)\right]\left\{Ext\text{-}\int \varphi^{\#}_{\varkappa}(\boldsymbol{x})\left(f(\boldsymbol{x},s) - f(\boldsymbol{x},t)\right)d^{\#3}x\right\}\left[Ext\text{-}\exp\left(-is\, H_{\varkappa}(g)\right)\right]\psi +$$

$$A^{\#}_{\varkappa,g}(s)\left[Ext\text{-}\exp\left(-i(t-s)\,H_{\varkappa}(g)\right)\right]\psi$$

Thus by (6.13),

$$\|\,A^{\#}_{\varkappa,g}(s)\psi - A^{\#}_{\varkappa,g}(t)\psi\|_{\#} \leq \left\|\left(I - Ext\text{-}\exp\left(-i(t-s)\,H_{\varkappa}(g)\right)\right)\,A^{\#}_{\varkappa,g}(s)\psi\right\|_{\#} +$$

$$+|f(\cdot,s)-f(\cdot,t)|_1\left\|(\,H_{\varkappa}(g)+b)^{\frac{1}{2}}\psi\,\right\|_{\#}+$$

$$+|f(\cdot,t)|_1\left\|\left(Ext\text{-}\exp\left(-i(t-s)\,H_{\varkappa}(g)\right)-I\right)(\,H_{\varkappa}(g)+b)^{1/2}\psi\right\|_{\#}\to_{\#} 0$$

as $t\ \to_{\#} s$. This proves the #-continuity. The rapid decrease is ensured by (6.13) and the fact that $f\in S^{\#}_{\text{fin}}\left({}^{*}\overline{\mathbb{R}}^{\#4}\right)$. A similar argument works for $B^{\#}_{\varkappa,g}(t)\psi$. The integrals defining $\varphi^{\#}_{\varkappa,g}(f)$ and $\pi^{\#}_{\varkappa,g}(f)$ now exist; (6.16) follows from integrating (6.13). Since $A^{\#}_{\varkappa,g}(t)$ and $B^{\#}_{\varkappa,g}(t)$ are self -#-adjoint, for $\psi\in D\left((\,H_{\varkappa}(g)+b)^{1/2}\right)\subset$

$$\langle\psi,\varphi^{\#}_{\varkappa,g}(f)\psi\rangle_{\#} = Ext\text{-}\int\langle\psi,\,A^{\#}_{\varkappa,g}(t)\psi\rangle_{\#}d^{\#}t = Ext\text{-}\int\langle\,A^{\#}_{\varkappa,g}(t)\psi,\psi\rangle_{\#}d^{\#}t$$

is a real, and similarly for $\pi^{\#}_{\varkappa,g}(f)$. Symmetric operators are #-closable and we now define $\varphi^{\#}_{\varkappa,g}(f)$ and $\pi^{\#}_{\varkappa,g}(f)$ as the #-closure of the above operators on the domain

$$D\left((\,H_{\varkappa}(g)+b)^{1/2}\right). \tag{6.18}$$

Remark 6.1 (a) The integrals defining $\varphi^{\#}_{\varkappa,g}(f)\psi$ and $\pi^{\#}_{\varkappa,g}(f)\psi$ are strong Riemann integrals, $\varphi^{\#}_{\varkappa,g}(f)$ is a strong #-limit of operators of the form

$$Ext\text{-}\textstyle\sum_{i=1}^{n}\,A^{\#}_{\varkappa,g}(t_i), n\in{}^{*}\mathbb{N}_{\infty}. \tag{6.19}$$

Conversely using the #-continuity of $A^{\#}_{\varkappa,g}(t)\psi$, we see that an operator of the form (6.19) is a strong #-limit of a hyper infinite sequence $\varphi^{\#}_{\varkappa,g}\left(f_j\right), j\in{}^{*}\mathbb{N}_{\infty}$ and the f_j can be chosen with the #-norm

$$|f|_1 = c\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}}\left\{\|f(\cdot,t)\|_{\#2}+\textstyle\sum_{i=1}^{3}\left\|\partial^{\#}_{x_i}f(\cdot,t)\right\|_{\#2}\right\}d^{\#}t\right) \tag{6.20}$$

uniformly bounded. For both #-limits the #-convergence occurs on the domain (6.18) and similar considerations apply to $\pi^{\#}_{\varkappa,g}(f)$. Furthermore $\varphi^{\#}_{\varkappa,g}(f)$ and $\pi^{\#}_{\varkappa,g}(f)$ can be defined whenever $|f|_{\#1}<{}^{*}\infty$.
(b) Using (2.19) in order to estimate $H_{0,\varkappa}$, we have from (6.16),

$$\left\|\varphi^{\#}_{\varkappa,g}(f)\Omega_{\varkappa,g}\right\|_{\#}+\left\|\pi^{\#}_{\varkappa,g}(f)\Omega_{\varkappa,g}\right\|_{\#}\leq|f|_1\left(\left|E_{\varkappa,2g}-E_{\varkappa,g}\right|+1\right)^{\frac{1}{2}}, \tag{6.21}$$

but the bound on the right grows in the diameter of the support of g.

Theorem 6.4 [18] Let $|f|_{\#1}$ be the #-norm $|f|_{\#1} = c\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}}\left\{\|f(\cdot,t)\|_{\#2}+\sum_{i=1}^{3}\left\|\partial^{\#}_{x_i}f(\cdot,t)\right\|_{\#2}\right\}d^{\#}t\right)$.
Let $|f|_{\#1}$ is finite. Then on the domain $D\left(\left(H_{\varkappa,g}+b\right)^{\frac{3}{2}}\right)$,), the field $\varphi^{\#}_{\varkappa}(f)$ satisfies the following equation

$$\left(\partial^{\#}_{t}\varphi^{\#}_{\varkappa}\right)(f) = -\varphi^{\#}_{\varkappa}\left(\partial^{\#}_{t}f\right) = \pi^{\#}_{\varkappa}(f) = \left[iH_{\varkappa,g},\varphi^{\#}_{\varkappa}(f)\right]. \tag{6.22}$$

Proof Note that the first equality in (6.22) is the definition of a distribution #-derivative. The out the difference quotient $\Delta_{\varepsilon}f(x,t)$ to #-derivative $\partial^{\#}_{t}f$ reads $\Delta_{\varepsilon}f(x,t)=\frac{[f(x+\varepsilon,t)-f(x,t)]}{\varepsilon}, \varepsilon\approx 0,$ note that

#-$\lim_{\varepsilon\to_\# 0}\Delta_\varepsilon f(x,t)=\partial_t^\# f(x,t)$. Note that for any vector ψ such that $\psi\in D\left(\left(H_{\varkappa,g}+b\right)^{\frac{1}{2}}\right)$ by canonical consideration we get

$$\#\text{-}\lim_{\varepsilon\to_\# 0}\left\|\varphi_\varkappa^\#(\partial_t^\# f)\psi-\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi\right\|_\#=0.$$

We have for $\psi\in D\left(\left(H_{\varkappa,g}+b\right)^{\frac{3}{2}}\right)$ that

$$\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi=\varepsilon^{-1}(I-Ext\text{-}\exp[i\varepsilon H])\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,t-\varepsilon)f(x,t)d^{\#3}x\psi d^\# t\right\}+$$

$$+\varepsilon^{-1}\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}A_\varkappa(f,t)\left(Ext\text{-}\exp\left[i\varepsilon H_{\varkappa,g}\right]-I\right)\psi d^\# t\right\}. \quad (6.23)$$

Here the last term #-converges as $\varepsilon\to_\# 0$ and it #-limit is: $i\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}A_\varkappa(f,t)H_{\varkappa,g}\psi d^\# t\right)$.
Since $\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi$ #-converges as $\varepsilon\to_\# 0$, the remaining term in expression for $\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi$ #-converges also to a #-limit ψ_1. For $\chi\in D\left(H_{\varkappa,g}\right)$ we obtain that

$$\langle\chi,\psi_1\rangle=\#\text{-}\lim_{\varepsilon\to_\# 0}\langle\chi,\varepsilon^{-1}\left(I-Ext\text{-}\exp\left[i\varepsilon H_{\varkappa,g}\right]\right)\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,t-\varepsilon)f(x,t)d^{\#3}x\psi d^\# t\right\}\rangle=$$
$$\langle iH_{\varkappa,g}\chi,\varphi_\varkappa^\#(f)\psi\rangle.$$

Since $H_{\varkappa,g}=H_{\varkappa,g}^*$, it follows that $\varphi_{\varkappa,g}^\#(f)\psi\in D\left(H_{\varkappa,g}\right)$ and $\psi_1=iH_{\varkappa,g}\varphi_{\varkappa,g}^\#(f)\psi$ and therefore:

$$-\varphi_\varkappa^\#(\partial_t^\# f)\psi=\left[iH_{\varkappa,g},\varphi_\varkappa^\#(f)\right]\psi.$$

From the above equation we obtain

$$i\langle\psi,\varphi_\varkappa^\#(\partial_t^\# f)\psi\rangle=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}}\langle H_\varkappa\psi(t),Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\psi(t)\ \rangle\,d^\# t-$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}}\langle Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\psi(t),\ H_{\varkappa,g}\psi(t)\rangle\,d^\# t. \quad (6.24)$$

Here $\psi(t)=Ext\text{-}\exp\left[itH_{\varkappa,g}\right]\psi$. Note that $\psi(t)\in D\left(H_{0,\varkappa}\right)\cap D\left(H_{I,\varkappa,g}\right)$, and

$$\left\|H_{I,\varkappa,g}\left(\psi(t)-\psi(s)\right)\right\|_\#\leq a\left\|\left(H_{\varkappa,g}+b\right)\left(\psi(t)-\psi(s)\right)\right\|_\#\to_\# 0,$$

as $|t-s|\to_\# 0$. Therefore we may substitute $H_{0\varkappa}+H_{I,\varkappa}$ for $H_{\varkappa,g}$ and consider each term separately. Note that the operators $H_{I,\varkappa}$ and $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,0)f(x,t)d^{\#3}x$ commute and therefore $H_{I,\varkappa}$ contribute zero to equality above. The following identity by canonical computation holds for any $\psi\in D(H_{0\varkappa})$, in particular for $\psi(t)=Ext\text{-}\exp[itH]\psi\in D(H_{0\varkappa})$

$$\langle H_{0\varkappa}\psi,Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\psi\ \rangle-\langle\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\ \right]\psi,H_{0\varkappa}\psi\rangle=$$

$$\langle\psi,-i\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\pi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\ \right]\psi\rangle.$$

Therefore finally we get

$$i\langle\psi,\varphi_\varkappa^\#(\partial_t^\# f)\psi\rangle=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}}\langle\psi(t),-iExt\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\pi_\varkappa^\#(x,0)f(x,t)d^{\#3}x\psi\ \rangle\,d^\# t=\langle\psi,-i\pi_\varkappa^\#(f)\psi\rangle.$$

This equality finalized the proof.
Remark 6.2 (a) in exactly the same fashion one proves that

$$\partial_{tt}^{\#2}\varphi_{\varkappa,g}^\#(f)=\partial_t^\#\pi_{\varkappa,g}^\#(f)=-\pi_{\varkappa,g}^\#\left(\partial_t^\# f\right)=\left[iH_{\varkappa,g},\pi_{\varkappa,g}^\#(f)\right]$$

if $\left|\partial_{tt}^{\#2} f\right|_1$ is also finite or hyperfinite. The commutator is a bilinear form on $D_{\varkappa,g}^{\#} \times D_{\varkappa,g}^{\#},\ D_{\varkappa,g}^{\#} = C^{*\infty}(H_{\varkappa,g})$, namely

$$\partial_{tt}^{\#2} \varphi_{\varkappa,g}^{\#}(f) = \sum_{i=1}^{i=3} \partial_{x_i x_i}^{\#2} \varphi_{\varkappa,g}^{\#}(f) - m_0^2 \varphi_{\varkappa,g}^{\#}(f) - 4Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \vdots\, \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x},t) \,\vdots\, f(\boldsymbol{x},t) g(\boldsymbol{x}) d^{\#3}x\, d^{\#}t. \tag{6.25}$$

Here we define the $\vdots\, \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x},t) \,\vdots$ product by

$$\vdots\, \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x},t) := \left(Ext\text{-}\exp\left[itH_{\varkappa,g}\right]\right) \colon \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x}) \colon \left(Ext\text{-}\exp\left[-itH_{\varkappa,g}\right]\right), \tag{6.26}$$

which we now prove is an operator valued non - Archimedean distribution. First we note that

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \vdots\, \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x},t) \,\vdots\, f(\boldsymbol{x},t) d^{\#3}x$$

is a sum of monomials in creation and annihilation operators with kernels in $L_2^{\#}$, and their $L_2^{\#}$ #-norms are #-continuous in t. Thus by (2.9)

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \vdots\, \varphi_{\varkappa}^{\#3}(\boldsymbol{x},t) \,\vdots\, f(\boldsymbol{x},t) g(\boldsymbol{x}) d^{\#3}x$$

is a bilinear form on $D_{\varkappa,g}^{\#} \times D_{\varkappa,g}^{\#}$. By (2. 3),

$$(H+b)^{-1} \left\{ Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \vdots\, \varphi_{\varkappa}^{\#3}(\boldsymbol{x},t) \,\vdots\, f(\boldsymbol{x},t) g(\boldsymbol{x}) d^{\#3}x \right\} (H+b)^{-1}$$

is a bounded operator, #-norm #-continuous in t. Thus on $D(H_{\varkappa,g}) \times D(H_{\varkappa,g})$,

$$\vdots\, \varphi_{\varkappa,g}^{\#3}(f) := Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \vdots\, \varphi_{\varkappa,g}^{\#3}(\boldsymbol{x},t) \,\vdots\, f(\boldsymbol{x},t) g(\boldsymbol{x}) d^{\#3}x\, d^{\#}t$$

is defined as a bilinear form. Hence (6.25) holds as an equation for bilinear forms on $D(H_{\varkappa,g}) \times D(H_{\varkappa,g})$. But each term except the last is an operator defined on $D\left((H+b)^{\frac{1}{2}}\right)$. Thus $\vdots\, \varphi_{\varkappa,g}^{\#3}(f) \,\vdots$ is actually an operator on $D\left((H+b)^{\frac{1}{2}}\right)$, and in fact for real f(x, t) it is essentially self #-adjoint. Furthermore, on $D\left((H+b)^{\frac{1}{2}}\right) \times D\left((H+b)^{\frac{1}{2}}\right)$ each term in (6.25) except the last is a bilinear form which is a distribution of order two. Thus the same is true for $\langle \psi, \vdots\, \varphi_{\varkappa}^{\#3}(\boldsymbol{x},t) \,\vdots\, \psi \rangle_{\#}$. We have used (6.26) to define the cube of the interacting field. It would be interesting to determine whether this definition agrees with conventional notions involving the separation of points. We shall see in this section and the following one that the $g(x)$ in equation (6.25) can be removed if $f(\boldsymbol{x},t)$ has #-compact support and $g(\boldsymbol{x}) = \lambda$ on a sufficiently large set. Then (6.25) becomes

$$\left(\frac{\partial^{\#2}}{\partial_t^{\#2}} + \sum_{i=1}^{i=3} \frac{\partial^{\#2}}{\partial_{x_i}^{\#2}} + m_0^2 \right) \varphi_{\varkappa}^{\#3}(f) = -4\lambda \,\vdots\, \varphi_{\varkappa}^{\#3}(f) \,\vdots,$$

which is a non-linear equation for a self-#-adjoint operator valued distribution.

(b) The identity $\pi_{\varkappa}^{\#}(f) = \left[iH_{\varkappa,g}, \varphi_{\varkappa}^{\#}(f)\right]$ implies

$$B_{\varkappa,g}^{\#}(t) = \left[iH_{\varkappa,g},\ A_{\varkappa,g}^{\#}(t)\right] \tag{6.27}$$

provided that the right and left sides of (6.27) make sense and are #-continuous in t. They are certainly defined and are #-continuous as bilinear forms on $D(H_{\varkappa,g}) \times D(H_{\varkappa,g})$. To see that (6.27) makes sense as operators on $D\left(\left(H_{\varkappa,g}+b\right)^{\frac{3}{2}}\right)$ we need only show that $A_{\varkappa,g}^{\#}(t)$ maps $D\left(\left(H_{\varkappa,g}+b\right)^{\frac{3}{2}}\right)$ into $D(H_{\varkappa,g})$. We choose a hyper infinite sequence $f_j(\boldsymbol{x},t) = f(\boldsymbol{x},t_0)\delta_j(t-t_0), j \in {}^*\mathbb{N}$ where $\delta_j(t-t_0)$ is a hyper infinite sequence of #-smooth functions #-converging to $\delta^{\#}(t-t_0)$ in the w^* #-topology on #-measures and with the #-norms $\left|f_j\right|_{\#1}$, uniformly bounded. Then the bilinear forms #-converge, which means that the inner products

$$\langle\theta, iH^{\,m}_{\varkappa,g}\,\varphi^{\#}_{\varkappa,g}(f)\psi\rangle_{\#} = \langle\theta, i\varphi^{\#}_{\varkappa,g}(f)H^{\,m}_{\varkappa,g}\psi\rangle_{\#} + \langle\theta, \pi^{\#}_{\varkappa,g}(f)\psi\rangle_{\#}$$

#-converge for $\theta \in D(H_{\varkappa,g})$. However the #-norms

$$\left\|H^{\,m}_{\varkappa,g}\,\varphi^{\#}_{\varkappa,g}(f)\psi\right\|_{\#} \leq \left\|i\varphi^{\#}_{\varkappa,g}(f)H^{\,m}_{\varkappa,g}\psi\right\|_{\#} + \left\|\pi^{\#}_{\varkappa,g}(f)\psi\right\|_{\#} + |f|_1\left\|(H_{\varkappa,g}+b)^{\frac{3}{2}}\psi\right\|_{\#}$$

are uniformly bounded, and so the inner products #-converge for all $\theta \in \mathcal{F}^{\#}$. Thus the #-limit $A^{\#}_{\varkappa}(t)$ = weak #-lim $\varphi^{\#}_{\varkappa}(f_j)\psi$ is in $D(H^{*}_{\varkappa,g}) = D(H_{\varkappa,g})$ which proves (6.27) on the domain $D\left((H_{\varkappa,g}+b)^{\frac{3}{2}}\right)$.

Corollary 6.5 Let $f \in S^{\#}_{\text{fin}}\left({}^{*}\widetilde{\mathbb{R}}^{\#4}\right)$. Then $D^{\#}_{\varkappa,g} = C^{*\infty}(H_{\varkappa,g}) \subset C^{*\infty}\left(\varphi^{\#}_{\varkappa,g}(f)\right)$, and $\varphi^{\#}_{\varkappa,g}(f)\,D^{\#}_{\varkappa,g} \subset D^{\#}_{\varkappa,g}$.

Proof Using Theorem 6.4, we prove by hyper infinite induction on $m \in {}^{*}\mathbb{N}$ that $\varphi^{\#}_{\varkappa,g}(f)\,D^{\#}_{\varkappa,g} \subset D\left(H^{\,m}_{\varkappa,g}\right)$ and that for $\psi \in D^{\#}_{\varkappa,g}$,

$$H^{\,m}_{\varkappa,g}\,\varphi^{\#}_{\varkappa,g}(f)\psi = \varphi^{\#}_{\varkappa,g}(f)H^{\,m}_{\varkappa,g}\psi + Ext\text{-}\sum_{j=1}^{m}\binom{m}{j}\,i^j\,\varphi^{\#}_{\varkappa,g}\left(\partial^{\#j}_t f\right)H^{\,m-j}_{\varkappa,g}\psi.$$

This formula is a special case of the identity $A^mB = Ext\text{-}\sum_{\boldsymbol{j=1}}^{\boldsymbol{m}}\binom{\boldsymbol{m}}{\boldsymbol{j}}\left[(\mathrm{ad}A)^{\boldsymbol{j}}B\right]A^{m-j}$, $m \in {}^{*}\mathbb{N}$ Thus we obtain

$$\left\|H^{\,m}_{\varkappa,g}\,\varphi^{\#}_{\varkappa,g}(f)\psi\right\|_{\#} \leq Ext\text{-}\sum_{j=1}^{m}\binom{m}{j}\left|\partial^{\#j}_t f\right|_{\#1}\left\|(H_{\varkappa,g}+b)^{m-j+\frac{1}{2}}\psi\right\|_{\#}. \tag{6.28}$$

Theorem 6.5. Let $f \in C^{*\infty}_0(\mathcal{B}^{\#}_4)$, that is f is $C^{*\infty}_0$ with support in the #-open region of space time $\mathcal{B}^{\#}_4$. Let $H_{\varkappa,g}$ be a Hamiltonian for $\mathcal{B}^{\#}_4$, so that $g(\boldsymbol{x}) = \lambda$ on a large set. Then

$$\varphi^{\#}_{\varkappa,g}(f) = \varphi^{\#}_{\varkappa}(f) \text{ and } \pi^{\#}_{\varkappa,g}(f) = \pi^{\#}_{\varkappa}(f) \text{ are independent of } g/\lambda.$$

Proof The spectral projections $E^{t}_{\#}(\lambda,\varkappa)$ of the sharp time field

$$A_{0,\varkappa}(t) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_c}\varphi^{\#}_{\varkappa,g}(\boldsymbol{x},t)f(\boldsymbol{x},t_0)\,d^{\#3}x = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#}_c}\lambda d^{\#}\,E^{t}_{\#}(\lambda,\varkappa)$$

are given by the formula

$$E^{t}_{\#}(\lambda,\varkappa) = \left(Ext\text{-}\exp\left[itH_{\varkappa,g}\right]\right)E^{0}_{\#}(\lambda,\varkappa)\left(Ext\text{-}\exp\left[-itH_{\varkappa,g}\right]\right)$$

and are independent of g. Thus $A_{0,\varkappa}(t)$ is independent of g and so is $A_{\varkappa}(t)$. By (3.1.7), for all g,

$$D_{0,\varkappa} = D\left(H^{\,1/2}_{0,\varkappa}\right) \cap D(N_{\varkappa}) \subset D\left((H_{\varkappa,g}+b)^{\frac{1}{2}}\right) \subset D\left(\varphi^{\#}_{\varkappa,g}(f)\right)$$

so that $\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D_{0,\varkappa}$ is independent of g. Thus to complete the proof, we only need to show that the domain of $\varphi^{\#}_{\varkappa,g}(f) = \#\text{-}\overline{\left(\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D\left((H_{\varkappa,g}+b)^{\frac{1}{2}}\right)\right)}$ is independent of g. Since $H_{\varkappa,g}$ is essentially self-#-adjoint on the domain $C^{*\infty}(H_{0,\varkappa}) \subset D_{0,\varkappa}$, so is $(H_{\varkappa,g}+b)^{\frac{1}{2}}$ Thus by (6.16) $\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D\left((H_{\varkappa,g}+b)^{\frac{1}{2}}\right) \subset \#\text{-}\overline{\left(\varphi^{\#}_{\varkappa}(f) \upharpoonright D_{0,\varkappa}\right)}$. Therefore $\#\text{-}\overline{\left(\varphi^{\#}_{\varkappa}(f) \upharpoonright D_{0,\varkappa}\right)} = \varphi^{\#}_{\varkappa,g}(f)$, so $\varphi^{\#}_{\varkappa,g}(f) = \varphi^{\#}_{\varkappa}(f)$ is independent of g. Similarly $\pi^{\#}_{\varkappa,g}(f) = \pi^{\#}_{\varkappa}(f)$ is independent of g.

Theorem 6.7 Let $\psi \in D^{\#}_{\varkappa,g}$, with $H_{\varkappa,g}$ a Hamiltonian for $\mathcal{B}^{\#}_4$. Then

$$\langle\psi, \varphi^{\#}_{\varkappa}(\boldsymbol{x}_1,t_1)\cdots\varphi^{\#}_{\varkappa}(\boldsymbol{x}_n,t_n)\psi\rangle_{\#}$$

is a distribution in $D^{\#\prime}(\mathcal{B}\times\cdots\times\mathcal{B})$

Proof This follows directly from our previous estimates (6.16) and (6.28).

7. Essential self-#-adjointness.

The main result of this section is the proof that for real test functions $f = f(x,t)$ with #-compact support, the field $\varphi_{\varkappa}^{\#}(f)$ is self-#-adjoint, and essentially self-#-adjoint on $D_{0,\varkappa} = D\left(H_{0,\varkappa}^{1/2}\right) \cap D(N_{\varkappa})$, or on any $D_{\varkappa,g}$ where $H_{\varkappa}(g)$ is a Hamiltonian for the support of f. We see furthermore that if f is real and $|f|_{\#1}$ defined by (6.20) is finite or hyperfinite, then, $\varphi_{\varkappa,g}^{\#}(f)$ is self-#-adjoint and essentially self-#-adjoint on $D_{\varkappa,g}$. The proof has three main steps. First, we assume that f is a regular function of t; in that case we use an analytic vector argument to show that $\varphi_{\varkappa,g}^{\#}(f)$ is essentially self-#-adjoint on $D_{\varkappa,g}$. As a second step, we take #-limits in the resolvents $(\varphi_{\varkappa,g}^{\#}(f) - z)^{-1}$ as f tends to a more general function. In this way, we obtain a self-#-adjoint operator $\varphi_{\varkappa,g}^{\#}(f)$. As a third step, we show that $\varphi_{\varkappa,g}^{\#}(f)$ is essentially self-adjoint on$D\left(H_{\varkappa}(g)\right)$. The regularity we impose on f is the requirement that its Fourier transform be a #-smooth function with #-compact support, or more generally that for the #-norm $|f|_{\#1}$, of (6.20) there exist constants $\alpha = \alpha(f)$ and $\beta = \beta(f)$

$$|D_t^r f| \leq \alpha\beta^r, r \in {}^*\mathbb{N}. \tag{7.1}$$

For a vector ψ, we consider the conditions

$$\|(H_{\varkappa}(g) + b)^r\psi\|_{\#} \leq ad^r \ , r \in {}^*\mathbb{N}. \tag{7.2}$$

Lemma 7.1 Assume (7.1) and (7.2). Then

$$\left\|(H_{\varkappa}(g) + b)^r \varphi_{\varkappa,g}^{\#}(f)\psi\right\|_{\#} \leq \alpha a d c(b+\beta)^r \tag{7.3}$$

for some constant c independent of r, ψ, f.

Proof . By (6.28) and (7.1)-(7.2)

$$\left\|(H_{\varkappa}(g) + b)^r \varphi_{\varkappa,g}^{\#}(f)\psi\right\|_{\#} \leq Ext\text{-}\sum_{j=0}^{r}\binom{r}{j} c\alpha\beta^j \left\|(H_{\varkappa}(g) + b)^{r-j+1}\varphi_{\varkappa,g}^{\#}(f)\psi\right\|_{\#} \leq$$

$$\leq Ext\text{-}\sum_{j=0}^{r}\binom{r}{j} c\alpha\beta^j a d^{r-j+1} \leq \alpha a d c(b+\beta)^r.$$

Lemma 7.2 Assume (7.1) and (7.2). Then ψ is an #-analytic vector for $\varphi_{\varkappa,g}^{\#}(f)$. In particular for real $f(\boldsymbol{x},t)$, $\varphi_{\varkappa,g}^{\#}(f)$ is essentially self-#-adjoint on $D_{\varkappa,g}$.

Proof We applying the preceding lemma successively. We see that multiplication by $\varphi_{\varkappa,g}^{\#}(f)$ changes the constants a and d of (7.2) as follows: $\to\ \alpha a d c \to d + \beta$. Thus

$$\left\|(H_{\varkappa}(g) + b)^r \left[\varphi_{\varkappa,g}^{\#}(f)\right]^k\psi\right\|_{\#} \leq a(\alpha c)^k\left[Ext\text{-}\prod_{j=0}^{k-1}(d+j\beta)\right](d+k\beta)^r,$$

and

$$\left\|\left[\varphi_{\varkappa,g}^{\#}(f)\right]^k\psi\right\|_{\#} \leq aB^k k!^{\#}$$

for some constant B, which proves that ψ is #-analytic for $\varphi_{\varkappa,g}^{\#}(f)$. The essential self-#-adjointness of $\varphi_{\varkappa,g}^{\#}(f)$ follows from generalized Nelson's analytic vector theorem, see ref.[17]-[18]. We can draw more information from (7.3). If we write $H_{\varkappa}(g) + \ b = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \lambda d^{\#} E_{\#}(\lambda,\varkappa)$, then (7.2) is equivalent to $\psi \in \mathrm{Range}\{E_{\#}(d,\varkappa)\}$ and (7.3) gives that

$$\varphi_{\varkappa,g}^{\#}(f)\mathrm{Range}\{E_{\#}(d,\varkappa)\} \subset \mathrm{Range}\{E_{\#}(d+b,\varkappa)\}. \tag{7.4}$$

Because $\varphi_{\varkappa,g}^{\#}(f)$ is self-#-adjoint we have

$$\varphi_{\varkappa,g}^{\#}(f)\mathrm{Range}\left\{\left(I - E_{\#}(d,\varkappa)\right)\right\} \subset \mathrm{Range}\left\{\left(I - E_{\#}(d-b,\varkappa)\right)\right\}. \tag{7.5}$$

These two inclusions have simple physical interpretations. We imagine that $\varphi^{\#}_{\varkappa,g}(f)$ is written as a sum of two operators, one creating physical wave packets associated with $H_{\varkappa}(g)$, and the other annihilating them. Because of (7.1) the wave packets have energy at most β, and so $\varphi^{\#}_{\varkappa,g}(f)$ can increase or decrease the total energy $H_{\varkappa}(g)$, by at most β.

We note that $\varphi^{\#}_{\varkappa,g}(f)$ is essentially self-#-adjoint on the domain $\bigcup_l \operatorname{Range}\{E_{\#}(l,\varkappa)\}$, by the proof of Lemma 7.2 and the remarks above. Our next step is to take #-limits with respect to f in the resolvents $R = R(f,z) = (\varphi^{\#}_{\varkappa,g}(f) - z)^{-1}$. As preparation, we now prove that R preserves #-regularity, which means

$$\left\|(H_{\varkappa}(g)+b)^{1/2}R(f,z)\psi\right\|_{\#} \leq M\left\|(H_{\varkappa}(g)+b)^{\frac{1}{2}}\psi\right\|_{\#}. \tag{7.6}$$

Lemma 7.3. Let f be real and satisfy (7.1). Then the estimate (7.6) holds for $\operatorname{Im} z \neq 0$. The constants M and b depend only on z, g, and $|f|_1$.

Proof To prove this lemma, we obtain uniform estimates on approximating operators R_n. If (7.6) holds for R_n, with M independent of $n \in {}^*\mathbb{N}$, and

$$R = \text{strong-\#-lim}\, R_n, \tag{7.7}$$

then (7.6) also holds for R. In fact $\left\|(H_{\varkappa}(g)+b)^{\frac{1}{2}}\psi\right\|_{\#}$ defines a #-norm on the domain $D\left((H_{\varkappa}(g)+b)^{\frac{1}{2}}\right) = \mathcal{H}_1$, which makes it into a non-Archimedean Hilbert space. The inequality (7.6) in equivalent to R_n, being a bounded operator on $\mathcal{H}_1$, and the #-norm $\|R_n\|_{\#1,1}$, of R_n, as an operator from $\mathcal{H}_1$ to itself is defined by

$$\|R_n\|_{\#1,1} = \left\|(H_{\varkappa}(g)+b)^{\frac{1}{2}}R_n(H_{\varkappa}(g)+b)^{-\frac{1}{2}}\right\|_{\#} \leq M. \tag{7.8}$$

From the strong #-convergence (7.7) on $\mathcal{F}^{\#}$, we conclude that on a #-dense set of vectors in $\mathcal{H}_1$, R_n #-converges weakly to R. Since the operators $R_n, n \in {}^*\mathbb{N}$ are uniformly bounded on $\mathcal{H}_1$, $R_n \to_{\#} R$ in weak operator #-convergence on $\mathcal{H}_1$. Thus the #-norm $\|R_n\|_{\#1,1}$ is bounded by the #-lim sup of the $\|R_n\|_{\#1,1}$ and (7.6) holds for R. Let

$$H_{\varkappa}(g)+b = Ext\text{-}\int_{{}^*\mathbb{R}^{\#}_c} \lambda d^{\#} E_{\#}(\lambda,\varkappa). \tag{7.9}$$

We approximate $\varphi^{\#}_{\varkappa,g}(f)$ by the bounded self-#-adjoint operator $C_n = E_{\#}(n,\varkappa)\varphi^{\#}_{\varkappa,g}(f)E_{\#}(n,\varkappa), n \in {}^*\mathbb{N}$. From (7.4) it is clear that $C_n \to_{\#} \varphi^{\#}_{\varkappa,g}(f)$ on vectors with #-compact support in the energy. Since $\varphi^{\#}_{\varkappa,g}(f)$ is essentially self-#-adjoint on this domain, the resolvents also #-converge strongly [18]

$$\#\text{-}\lim_{n\to{}^*\infty} R_n(z) = \#\text{-}\lim_{n\to{}^*\infty}(C_n - z)^{-1} = R(z),$$

proving (7.7). We now show that (7.8) holds and it is sufficient to prove

$$\left\|(H_{\varkappa}(g)+b)^{\frac{1}{2}}R_n(H_{\varkappa}(g)+b)^{-\frac{1}{2}}\psi\right\|_{\#} \leq M\|\psi\|_{\#} \tag{7.10}$$

for ψ in the #-dense set $D_{\varkappa,g}$. Since $(H_{\varkappa}(g)+b)^{-\frac{1}{2}}$ and R_n both map $D_{\varkappa,g}$ onto $D_{\varkappa,g}$ we need only prove that on the domain $D_{\varkappa,g} \times D_{\varkappa,g}$,

$$H_{\varkappa}(g)+b \leq M^2(C_n-\bar{z})(H_{\varkappa}(g)+b)(C_n-z) = \tag{7.11}$$

$$= M^2(C_n-x)(H_{\varkappa}(g)+b)(C_n-x)+(My)^2(H_{\varkappa}(g)+b)+iM^2y[H_{\varkappa}(g),C_n],$$

where $z = x + iy$. As the first term is positive, it is sufficient to show that

$$0 \leq [(My)^2-1](H_{\varkappa}(g)+b)+M^2y[iH_{\varkappa}(g),C_n]. \tag{7.12}$$

But

$$[\,iH_\varkappa(g),C_n] = E_\#(n,\varkappa)\left[\,iH_\varkappa(g),\varphi^\#_{\varkappa,g}(f)\right]E_\#(n,\varkappa) = E_\#(n,\varkappa)\pi^\#_{\varkappa,g}(f)E_\#(n,\varkappa).$$

By Theorem 6.3,

$$|\langle\psi,[\,iH_\varkappa(g),C_n]\psi\rangle_\#| = \left|\langle E_\#(n,\varkappa)\psi,\pi^\#_{\varkappa,g}(f)E_\#(n,\varkappa)\psi\rangle_\#\right| \le \|\psi\|_\#\cdot\left\|\pi^\#_{\varkappa,g}(f)E_\#(n,\varkappa)\psi\right\|_\# \le$$

$$\le \|\psi\|_\#\cdot\ |f|_{\#1}\cdot\left\|(\,H_\varkappa(g)+b)^{\frac{1}{2}}E_\#(n,\varkappa)\psi\right\|_\# \le \frac{1}{2}\ |f|_{\#1}\cdot\left\{\varepsilon\,\langle\psi,(\,H_\varkappa(g)+b)^{\frac{1}{2}}\psi\rangle_\# + \varepsilon^{-1}\langle\psi,\psi\rangle_\#\right\}$$

for any $\varepsilon > 0$. Furthermore, the #-norm $|f|_{\#1}$ of (6.20) can be chosen independent of b for large b, since for $b_1 \le b_2$ we have that $H_\varkappa(g)+b_1 \le H_\varkappa(g)+b_2$. Therefore (7.12) is valid as long as

$$0 \le \left\{(My)^2 - 1 - \frac{1}{2}M^2y\varepsilon\,|f|_{\#1}\right\}(\,H_\varkappa(g)+b) - \frac{M^2y}{2\varepsilon}\ |f|_{\#1}.$$

For each $|f|_{\#1}, y \ne 0$, we can pic M large so that $(My)^2 > 3$, ε small enough so that $\frac{1}{2}M^2y\varepsilon\,|f|_{\#1} < 1$, and b large enough so that the inequality is valid. This completes the proof.
We now show that the resolvents of approximate field operators #-converge. We use the spectral projections $E_\#(n,\varkappa), n \in {}^*\mathbb{N}$ defined by (7.9) to cut-off the field. If $\varphi^\#_{\varkappa,g}(f)$ is a #-closed symmetric field operator, then $E_\#(n,\varkappa)\varphi^\#_{\varkappa,g}(f)E_\#(n,\varkappa)$ is a bounded, self-#-adjoint approximation to $\varphi^\#_{\varkappa,g}(f)$.
Lemma 7.4. Let $f_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of real functions satisfying (7.1) with β depending on n. If the graphs $\mathrm{G}\left(\varphi^\#_{\varkappa,g}(f_n)\right)$ #-converge to the graph of a #-densely defined operator, if

$$\left\|(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\{\varphi^\#_{\varkappa,g}(f_n)-\varphi^\#_{\varkappa,g}(f_m)\}(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\# \to_\# 0 \tag{7.13}$$

and if the #-norms $|f_n|_{\#1}$, are uniformly bounded, then the resolvents

$$R_n(z) = (C_n - z)^{-1} \tag{7.14}$$

of

$$C_n = E_\#(n,\varkappa)\varphi^\#_{\varkappa,g}(f_n)E_\#(n,\varkappa). \tag{7.15}$$

#-converge strongly to the resolvent of a self-#-adjoint operator C.
Proof This result is a special case of [ref.[19], Th. 5 and Cor. 6]. See that paper for notation. Recall that #-measure $E_\#(n,\varkappa)$is defined by (4.3.9). Note that

$$\left\|(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\left(C_n-\varphi^\#_{\varkappa,g}(f_n)\right)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\# \le \tag{7.16}$$

$$\left\|(E_\#(n,\varkappa)-I)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\#\cdot\left\{\left\|\varphi^\#_{\varkappa,g}(f_n)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\# + \left\|(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\varphi^\#_{\varkappa,g}(f_n)\right\|_\#\right\} \le$$

$$\le n^{-1/2}\left\{\left\|\varphi^\#_{\varkappa,g}(f_n)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\# + \left\|(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\varphi^\#_{\varkappa,g}(f_n)\right\|_\#\right\}.$$

By Theorem 6.3 the operator $\varphi^\#_{\varkappa,g}(f_n)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}$ and its #-adjoint are bounded with

$$\left\|\varphi^\#_{\varkappa,g}(f_n)(\,H_\varkappa(g)+b)^{-\frac{1}{2}}\right\|_\# \le c\,|f_n|_{\#1},$$

which is bounded uniformly in $n, \in {}^*\mathbb{N}$. Then

$$\left\| (H_\varkappa(g) + b)^{-\frac{1}{2}} \left(C_n - \varphi^{\#}_{\varkappa,g}(f_n) \right) (H_\varkappa(g) + b)^{-\frac{1}{2}} \right\|_{\#} = O\left(n^{-1/2}\right) \quad (7.17)$$

and so by (7.13),

$$\left\| (H_\varkappa(g) + b)^{-\frac{1}{2}} (C_n - C_m) (H_\varkappa(g) + b)^{-\frac{1}{2}} \right\|_{\#} \to_{\#} 0$$

as $n, m \to {}^*\infty$The required uniform boundedness of the resolvents $\|(C_n - z)^{-1}\|_{\#1,1} <$ const follows from Lemma 7.3.

We now discuss when the hypotheses of Lemma 7.4 are satisfied. If the $\varphi^{\#}_{\varkappa,g}(f_n)$ #-converge strongly on a #-dense domain, then the graphs #-converge. The $\varphi^{\#}_{\varkappa,g}(f_n)$ will #-converge on $D_{\varkappa,g}$ if $f_n \to_{\#} f$ as $n \to {}^*\infty$ in the #-norm $|\cdot|_{\#1}$; they will also #-converge for some hyper infinite sequence $f_n, n \in {}^*\mathbb{N}$

$$f_n \to_{\#} Ext\text{-}\sum_{i=1}^{i=n} f(\cdot, t_i)\delta^{\#}(t - t_i) \quad (7.18)$$

with $f(\cdot, t_i) \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4})$. We can choose $f(\boldsymbol{x}, t)$ to have the form

$$f(\boldsymbol{x}, t) = Ext\text{-}\sum_{i=1}^{i=n} f(\cdot, t_i)\delta_n(t - t_i),$$

where $\delta_n(t) \geq 0$ has support in $|t| < n^{-1}$, and $Ext\text{-}\int \delta_n(t)\, d^{\#}t$. For such a sequence $|f_n|_{\#1}$ is uniformly bounded in n, $n \in {}^*\mathbb{N}$. From (6.9) we see that w^* #-convergence of the $\delta_n(t)$ as bounded #-measures implies (7.13). Thus the hypotheses are satisfied for the sequence (7.18). They are also satisfied if the f_n #-converge in the #-norm $|\cdot|_{\#1}$, and every f_n with finite $|f_n|_{\#1}$ is the #-limit of such a hyper infinite sequence.

Theorem 7.5. Let f be real and $|f|_{\#1}$ finite. Then the operator $\varphi^{\#}_{\varkappa,g}(f)$ is self-#-adjoint and essentially self-#-adjoint on $D_{\varkappa,g}$. A real linear combination of sharp time fields with real test functions in $S^{\#}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#3}\right)$, $Ext\text{-}\sum_{i=1}^{i=r} A_{\varkappa,g}(t_i)$, is also essentially self-#-adjoint on $D_{\varkappa,g}$.

Proof The two cases are similar and we only consider $\varphi^{\#}_{\varkappa,g}(f)$. We first prove that the operator C of Lemma 7.4 extends $\varphi^{\#}_{\varkappa,g}(f)$

$$\varphi^{\#}_{\varkappa,g}(f) \subset C. \quad (7.19)$$

As in the proof of Theorem 6.6, we have

$$\varphi^{\#}_{\varkappa,g}(f) = \left(\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D_\varkappa \right)^{\#-} = \left(\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D_0 \right)^{\#-} \quad (7.20)$$

where $(A)^{\#-}$ is #-closure of the operator A and $D_0 = D\left(H_\varkappa^{1/2}\right) \cap D(N_\varkappa)$. Let $R_n(z)$ be defined by (7.14), where f_n approximates f and satisfies the hypotheses of Lemma 7.4. Thus $R(z) = \#-\lim_{n \to {}^*\infty} R_n(z)$ exists and is the resolvent of a self-#-adjoint operator C. For $\psi \in D_{\varkappa,g}, C_n\psi = E(n)\, \varphi^{\#}_{\varkappa,g}(f)E(n)\psi \to_{\#} \varphi^{\#}_{\varkappa,g}(f)\psi$, and #-convergence can be shown on $D\left((H(g) + b)^{3/2}\right)$. For $\chi = \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)\psi$,

$$R(z)\chi = \#-\lim_{n \to {}^*\infty} R_n(z)\,\chi = \#-\lim_{n \to {}^*\infty} R_n(z)\,(C_n - z)\psi = \psi.$$

Thus we obtain that $(C - z)^{-1}\left(\varphi^{\#}_{\varkappa,g}(f) \upharpoonright D_g \right) = I \upharpoonright D_g$. And therefore by (7.20), (7.19) is valid.

We now show that $\varphi^{\#}_{\varkappa,g}(f)$ is equal to C, which completes the proof. We need only show that if $\psi \in D(C)$, then $\psi \in D\left(\varphi^{\#}_{\varkappa,g}(f) \right)$. We first notice that

$$R(z)D\left((H_\varkappa(g) + b)^{\frac{1}{2}} \right) \subset D\left((H_\varkappa(g) + b)^{\frac{1}{2}} \right) \quad (7.21)$$

and that (7.8) is valid for $R(z)$. The argument for this is the same as the proof of Lemma 7.3, but the approximate operator $C_n = E_{\#}(n,\varkappa)\varphi^{\#}_{\varkappa,g}(f_n)E_{\#}(n,\varkappa)$ replaces the C_n. of the former proof. The remaining calculation is the same since the $|f_n|_{\#1}, n \in {}^*\mathbb{N}$, are assumed uniformly bounded. We now introduce the smoothing operator

$$P_j = \left(1 + \tfrac{1}{j}(\,H_\varkappa(g) + b)^2\right)^{-1} \tag{7.22}$$

with the properties $\|P_j\|_{\#} \leq 1$

$$\text{strong }\#\text{-}\lim_{n\to{}^*\infty} P_j = I, \tag{7.23}$$

and for $r < 2$,

$$\|P_j(\,H_\varkappa(g) + b)^r\|_{\#} = \|(\,H_\varkappa(g) + b)^r P_j\|_{\#} \leq j^{\frac{r}{2}}. \tag{7.24}$$

Let $\psi \in D(C)$ and $\psi = R(z)\chi$. Then $P_j\psi = \psi_j \to_{\#} \psi$, as $j \to {}^*\infty$ and $\psi_j \in D(\,H_\varkappa(g)) \subset D\left(\varphi^{\#}_{\varkappa,g}(f)\right)$. If $\varphi^{\#}_{\varkappa,g}(f)\psi_j$ #-converges and then ψ is in the domain of the #-closed operator $\varphi^{\#}_{\varkappa,g}(f)$, so we prove this

$$\left(\varphi^{\#}_{\varkappa,g}(f) - z\right)\psi_j = \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)P_j\psi = \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)P_jR(z)\chi = \tag{7.25}$$

$$= \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)R(z)\,P_j\chi + \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)[\,P_j, R(z)]\chi.$$

The last equality is valid since $P_j\chi \in D(\,H_\varkappa(g)^2) \subset D\left((\,H_\varkappa(g) + b)^{1/2}\right)$ and by (7.21)

$$R(z)\,P_j\chi \in D\left((\,H_\varkappa(g) + b)^{\frac{1}{2}}\right) \subset D\left(\varphi^{\#}_{\varkappa,g}(f)\right).$$

Since C extends $\varphi^{\#}_{\varkappa,g}(f)$, $\left(\varphi^{\#}_{\varkappa,g}(f) - z\right)\psi_j = P_j\chi + \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)[\,P_j, R(z)]\chi$. As $P_j\chi \to_{\#} \chi = (C - z)\psi$ to conclude that $\psi \in D\left(\varphi^{\#}_{\varkappa,g}(f)\right)$, we need to show that

$$\Lambda_j = \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)[\,P_j, R(z)]\chi \to_{\#} 0. \tag{7.26}$$

We now claim that

$$\Lambda_j = \#\text{-}\lim_{n\to{}^*\infty}(C_n - z)[\,P_j, R_n(z)]\chi, \tag{7.27}$$

where C_n and R_n are defined in (7.14-(7.15). Since $(C_n - z)R_n(z) = (C - z)R\,(z) = I$, we need only prove the existence of the limit (7.27) with the commutator removed. As observed in the first part of the proof, for $\psi \in D\left((\,H_\varkappa(g) + b)^{\frac{3}{2}}\right)$, $\|C_n\psi - C\,\psi\|_{\#} = \|C_n\psi - \varphi^{\#}_{\varkappa,g}(f)\,\psi\|_{\#} \to_{\#} 0$. Since $P_jR(z)\chi \in D(\,H_\varkappa(g)^2) \subset D\left((\,H_\varkappa(g) + b)^{3/2}\right)$, as $n \to {}^*\infty$, $(C_n - z)\,P_jR\,(z)\chi \to_{\#} \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)P_jR\,(z)\chi$. Also $R_n(z)\chi \to_{\#} R\,(z)\chi$, and by Theorem 6.3 and (7.24), $\|(C_n - z)\,P_j\|_{\#} \leq \text{const}\cdot\,|f_n|_{\#1}\cdot j^{\frac{1}{4}}$, which is bounded uniformly in $n \in {}^*\mathbb{N}$. Therefore

$$(C_n - z)\,P_jR_n\,(z)\chi \to_{\#} \left(\varphi^{\#}_{\varkappa,g}(f) - z\right)P_jR\,(z)\chi,$$

and (7.27) is established. Thus $\Lambda_j = \#\text{-}\lim\limits_{n\to{}^*\infty}\Lambda_{j,n}$, where

$$\Lambda_{j,n} = (C_n - z)[\,P_j, R_n(z)]\chi = (C_n - z)R_n(z)\,P_j\left[P_j^{-1}, (C_n - z)\right]P_jR\,(z)\chi =$$

$$= j^{-1}\,P_j[(\,H_\varkappa(g) + b)^2, (C_n - z)]P_jR\,(z)\chi =$$

$$= j^{-1}\, P_j\{(\,H_\varkappa(g)+b)[\,H_\varkappa(g),C_n]+[\,H_\varkappa(g),C_n](\,H_\varkappa(g)+b)\}\,P_jR\,(z)\chi =$$

$$= -ij^{-1}\,P_j\big\{(\,H_\varkappa(g)+b)E_\#(n,\varkappa)\pi^\#_{\varkappa,g}\big(f_n\,\big)E_\#(n,\varkappa)+E_\#(n,\varkappa)\pi^\#_{\varkappa,g}\big(f_n\,\big)E_\#(n,\varkappa)(\,H_\varkappa(g)+b)\big\}P_jR\,(z)\chi.$$

Now by (7.24) we obtain

$$\big\|P_j(\,H_\varkappa(g)+b)\big\|_\# = \big\|(\,H_\varkappa(g)+b)P_j\big\|_\# \le\, j^{-1/2}$$

and

$$\big\|E_\#(n,\varkappa)\pi^\#_{\varkappa,g}\big(f_n\,\big)E_\#(n,\varkappa)\,P_j\big\|_\# + \big\|\,P_jE_\#(n,\varkappa)\pi^\#_{\varkappa,g}\big(f_n\,\big)E_\#(n,\varkappa)\big\|_\# \le \mathrm{const}\cdot\, |f_n|_{\#1}\cdot\, j^{1/4} \le \mathrm{const}\cdot\, j^{1/4}$$

as the $|f_n|_{\#1}$ are assumed uniformly bounded. The constant is independent of j and n. Therefore $\|\,\Lambda_{j,n}\|_\# \le \mathrm{const}\cdot\, j^{-1/4}$, and

$$\#\text{-}\lim_{j\to{}^*\infty}\|\,\Lambda_j\|_\# \le \#\text{-}\lim_{j\to{}^*\infty}\Big(\#\text{-}\lim_{n\to{}^*\infty}\|\,\Lambda_{j,n}\|_\#\Big) = 0.$$

Thus (4.3.26) is established and the proof is complete.

8. The field as a tempered distribution in $S^{\#\prime}_{\mathrm{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$.

In the previous sections we studied the quantum field $\varphi^\#_{\varkappa,g}(f)$ corresponding to the Hamiltonian $H_\varkappa(g)$. We found that if $\varphi^\#_{\varkappa,g}(f)$ is localized, namely if f has finitely bounded #-compact support in B and $H_\varkappa(g)$ is a Hamiltonian for B, then $\varphi^\#_{\varkappa,g}(f) = \varphi^\#_\varkappa(f)$ is independent of the spatial cut-off g. In this section we show that there is a cut-off independent field $\varphi^\#_\varkappa(f)$ defined for all $f \in S^{\#\prime}_{\mathrm{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$, and $\varphi^\#_\varkappa(f)$ agrees with the previous one when f has #-compact support. The domain of $\varphi^\#_\varkappa(f)$ includes $D_{0,\varkappa} = D\big(H^{1/2}_{0,\varkappa}\big)\cap D\big(N_{0,\varkappa}\big)$, and on this domain $\varphi^\#_\varkappa(f)$ is a tempered distribution in $S^{\#\prime}_{\mathrm{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$.

Lemma 8.1. Let $D_{0,\varkappa} = D\big(H^{1/2}_{0,\varkappa}\big)\cap D\big(N_{0,\varkappa}\big)$. For $\psi \in D_{0,\varkappa}$, $\langle\psi,\, \varphi^\#_\varkappa(\boldsymbol{x},t\,)\psi\rangle_\#$ is a #-continuous, polynomially bounded function and

$$\Big|Ext\text{-}\textstyle\int_{{}^*\mathbb{R}^{\#3}}\langle\psi,\, \varphi^\#_\varkappa(\boldsymbol{x},t\,)\psi\rangle_\# D^{\#\prime}_{\boldsymbol{x}}f(\boldsymbol{x},t\,)d^{\#3}x\Big| \le O(t)\|f(\cdot,t\,)\|_{\#2}\langle\psi,\big(H_{0,\varkappa}+N^2_{0,\varkappa}+I\big)\psi\rangle_\#. \quad (8.1)$$

Proof We divide space time into a number of similar regions with a partition of unity. Let $\xi(\boldsymbol{x},t\,), \boldsymbol{x} = (x_1,x_2,x_3)$, be a $C^{^*\infty}$ function satisfying

$$0 \le \xi(\boldsymbol{x},t\,) \le 1, \quad (8.2)$$

$$\mathrm{supp}(\xi) \subset \{(\boldsymbol{x},t\,)||\boldsymbol{x}| \le 1, |t| \le\}, \quad (8.3)$$

and such that

$$Ext\text{-}\textstyle\sum_{ij}\xi_{ij}\,(\boldsymbol{x},t\,) = Ext\text{-}\sum_{ij}\xi(x_1-i,x_2-i,x_3-i,t-j) = 1. \quad (8.4)$$

Thus if $f(\boldsymbol{x},t\,) \in S^\#_{\mathrm{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$,

$$f\, = Ext\text{-}\textstyle\sum_{ij}f_{ij}\,(\boldsymbol{x},t\,) = Ext\text{-}\sum_{ij}f(\boldsymbol{x},t\,)\xi_{ij}\,(\boldsymbol{x},t\,) \quad (8.5)$$

with $f_{ij}(\boldsymbol{x},t\,)$ a $C^{^*\infty}$ function with support in the cube

$$B_{ij} = \{(\boldsymbol{x},t\,)||x_1-i| \le 1, |x_2-i| \le 1, |x_3-i| \le 1, |t-j| \le 1\}. \quad (8.6)$$

We also pick a $C^{*\infty}$ function $g_0(\boldsymbol{x})$ such that

$$g_0(\boldsymbol{x}) = \lambda, \text{ if } |\boldsymbol{x}| \leq 2, \tag{8.7}$$

and

$$g_0(\boldsymbol{x}) = 0, \text{ if } |\boldsymbol{x}| \geq 3. \tag{8.8}$$

Thus $H_{\varkappa}(g_{ij})$ is a Hamiltonian for B_{ij} when

$$g_{ij}(\boldsymbol{x}) = g_0\left(\frac{x_1 - i}{1+|j|}, \frac{x_2 - i}{1+|j|}, \frac{x_3 - i}{1+|j|}\right). \tag{8.9}$$

Furthermore

$$\left\| \left(I + N_{0,\varkappa}\right)^{-1} H_{I,\varkappa,g_{ij}} \left(I + N_{0,\varkappa}\right)^{-1} \right\|_{\#} = O(j) \tag{8.10}$$

as the kernels of operators contributing to $H_{I,\varkappa,g_{ij}}$ have $L_2^{\#}$ #-norms with are $O(j)$. For $(\boldsymbol{x}, t) \in B_{ij}$ and $\psi \in D_{0,\varkappa}$, we have by Lemma 6.1, that $\langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t)\psi \rangle_{\#}$ is #-continuous and

$$|\langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t)\psi \rangle_{\#}| \leq \text{const} \cdot \langle \psi, \left(H_{\varkappa}(g_{ij}) + \tilde{b}(\varkappa) \right) \psi \rangle_{\#} \tag{8.11}$$

where the constant is independent of $\boldsymbol{x}, t, i$, and j. Here $\tilde{b}(\varkappa)$ is hyperfinite constant proportional to the lower bound $b(\varkappa)$ of $H_{\varkappa}(g_{ij})$, see (2.19). Note that that the lower bound of $H_{\varkappa}(g_{ij})$, is proportional to the diameter of the support of g_{ij}, namely $O(j)$). Thus (8.11) gives the bound for $(\boldsymbol{x}, t) \in B_{ij}, \psi \in D_{0,\varkappa}$

$$|\langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t)\psi \rangle_{\#}| \leq \tag{8.12}$$

$$\text{const} \cdot \left\{ \left\| H_{0,\varkappa}^{1/2} \psi \right\|_{\#}^2 + \left\| \left(I + N_{0,\varkappa}\right) \right\|_{\#}^2 \cdot \left\| \left(I + N_{0,\varkappa}\right)^{-1} H_{I,\varkappa,g_{ij}} \left(I + N_{0,\varkappa}\right)^{-1} \right\|_{\#} + \tilde{b}(\varkappa) \, \|\psi\|_{\#}^2 \right\} \leq$$

$$\leq \text{const} \cdot \|f(\cdot, t)\|_{\#2} \langle \psi, \left(H_{0,\varkappa} + N_{0,\varkappa}^2 + I\right)\psi \rangle_{\#} \cdot b(\varkappa) \cdot O(j),$$

by (8.10) and the above discussion of $\tilde{b}(\varkappa)$. Since $O(j) = O(|t|)$, we have proved polynomial boundedness. Thus, as in Lemma 6.1,

$$\left| Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}} \langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t)\psi \rangle_{\#} D_x^{\#} f(\boldsymbol{x}, t) d^{\#3}x \right| \leq O(t) \|f(\cdot, t)\|_{\#2} \langle \psi, \left(H_{0,\varkappa} + N_{0,\varkappa}^2 + I\right)\psi \rangle_{\#},$$

which yields (8.1). We now define the sharp time fields

$$A_{\varkappa}(f, t) = Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}^{\#3}} \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t) f(\boldsymbol{x}, t) d^{\#3}x \tag{8.13}$$

and

$$B_{\varkappa}(f, t) = Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}^{\#3}} \pi_{\varkappa}^{\#}(\boldsymbol{x}, t) f(\boldsymbol{x}, t) d^{\#3}x. \tag{8.14}$$

Lemma 8.2 Let $f(\boldsymbol{x}, t) \in S_{\text{fin}}^{\#}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$ be real. Then $A_{\varkappa}(f, t)$ and $B_{\varkappa}(f, t)$ define self-#-adjoint operators, and their domain includes $D_{0,\varkappa}$. For $\psi \in D_{0,\varkappa}$, $\|A_{\varkappa}(f, t)\psi\|_{\#} + \|B_{\varkappa}(f, t)\psi\|_{\#} \leq |f(\cdot, t)|_{\#2} \cdot \|F_{\varkappa}\psi\|_{\#}$, where

$$F_{\varkappa} = \left(H_{0,\varkappa} + N_{0,\varkappa}^2 + I\right)^{1/2},$$

and $\quad |f(\cdot, t)|_{\#2} = c(1 + |t|)\left\{ \|f(\cdot, t)\|_{\#2} + \left\| D_x^{\#'} f(\cdot, t) \right\|_{\#2} \right\}$.

Proof The proof is similar to that of Lemmas 6.2 and 8.1.

Theorem 8.3 Let $f(\boldsymbol{x}, t) \in S_{\text{fin}}^{\#}\left({}^*\widetilde{\mathbb{R}}^{\#4}\right)$ be a real function in $S_{\text{fin}}^{\#}$. The vectors $A_{\varkappa}(t)\psi$ and $B_{\varkappa}(t)\psi$, where

$\psi \in D_{0,\varkappa}$, are #-continuous and rapidly decreasing in t. Their integrals over t exist and define #-closed symmetric operators $\varphi_{\varkappa}^{\#}(f)$ and $\pi_{\varkappa}^{\#}(f)$ with domains containing $D_{0,\varkappa}$. The fields $\varphi_{\varkappa}^{\#}(f), \pi_{\varkappa}^{\#}(f)$, $A_{\varkappa}(f,t)$ and $B_{\varkappa}(f,t)$ are all independent of $g(\boldsymbol{x})$. For any vector $\psi \in D_{0,\varkappa}$ we have

$$\| \varphi_{\varkappa}^{\#}(f)\psi\|_{\#} + \| \pi_{\varkappa}^{\#}(f)\psi\|_{\#} \leq |f|_{\#2} \cdot \|F_{\varkappa}\psi\|_{\#},$$

where $|f|_{\#2} = Ext\text{-}\int |f(\cdot,t)|_{\#2} d^{\#}t$ and $F_{\varkappa} = \left(H_{0,\varkappa} + N_{0,\varkappa}^{2} + I\right)^{1/2}$.

Proof This proof is based on the proofs of Lemma 8.1, Theorem 6.3, and Theorem 6.6. The fields $\varphi_{\varkappa}^{\#}(f)$ and $\pi_{\varkappa}^{\#}(f)$ are defined as their #-closures on $D_{0,\varkappa}$.

9. Locality

In this section we derive locality of the field operators. Locality means that two field operators $\varphi_{\varkappa}^{\#}(f)$ and $\varphi_{\varkappa}^{\#}(h)$ commute provided the supports of f and h are spacelike separated. In other words, whenever

$$(\boldsymbol{x},t) \in \operatorname{supp}(f) \text{ and } (\boldsymbol{y},s) \in \operatorname{supp}(h),$$

we have that

$$|\boldsymbol{x} - \boldsymbol{y}| > |t - s|.$$

Under this hypothesis a signal originating in $\operatorname{supp}(f)$ (caused, for example, by the process of performing the measurement of $\varphi_{\varkappa}^{\#}(f)$ cannot be recorded by the measurement of $\varphi_{\varkappa}^{\#}(h)$. Thus one expects that the measurement of $\varphi_{\varkappa}^{\#}(f)$ does not interfere with the measurement of $\varphi_{\varkappa}^{\#}(h)$, and that the joint measurement of $\varphi_{\varkappa}^{\#}(f)$ and $\varphi_{\varkappa}^{\#}(h)$ can be performed in either order. The rigorous mathematical statement that the measurements can be performed in either order is that $\varphi_{\varkappa}^{\#}(f)$ and $\varphi_{\varkappa}^{\#}(h)$ commutes. For any #-closed operator A, a #-core $D^{\#}(A)$ of A is defined to be a #-dense domain contained in $D(A)$ such that $A = \#\text{-}\overline{(A \restriction D)}$.
Self-#-adjoint operators A and B commute if and only if for any spectral projection E of B, and #-core D of A, $ED \subset D(A)$ and for $\psi \in D$, $EA\,\psi = A\,E\psi$.

Definition 9.1 (i) Let $\psi \in D^{\#}(A)$, we say that vector ψ is a near standard vector if $\|\psi\|_{\#} \in {}^{*}\widetilde{\mathbb{R}}_{\text{fin}}^{\#}$.
(ii) A near standard #-core $D_{\text{fin}}^{\#}(A)$ of A is defined to be a subdomain $D_{\text{fin}}^{\#}(A) \subset D^{\#}(A)$ which contains all near standard vectors ψ such that: (a) $\psi \in D^{\#}(A)$ and (b) vector $A\psi$ is a near standard vector.
(iii) A near standard domain $D_{\text{fin}}(A)$ of A is defined to be a subdomain $D_{\text{fin}}(A) \subset D(A)$ which contains all near standard vectors ψ such that: (a) $\psi \in D(A)$ and (b) vector $A\psi$ is a near standard vector.

Definition 9.2 Self-#-adjoint operators A and B $\approx$ -commute on domain $D_{\text{fin}}^{\#}(A) \cap D_{\text{fin}}^{\#}(B)$ if for any near standard vector $\psi \in D_{\text{fin}}^{\#}(A) \cap D_{\text{fin}}^{\#}(B)$ the following condition holds $AB\psi \approx BA\psi$.

Lemma 9.1 Self-#-adjoint operators A and B $\approx$ -commute on domain $D_{\text{fin}}^{\#}(A) \cap D_{\text{fin}}^{\#}(B)$ if and only if for any spectral projection E_B of B, and near standard #-core $D_{\text{fin}}^{\#}(A)$ of A, $E_B D_{\text{fin}}^{\#}(A) \subset D_{\text{fin}}(A)$ and for all $\psi \in D_{\text{fin}}^{\#}(A)$, $E_B A\psi \approx AE_B\psi$.

Theorem 9.1 If $\operatorname{supp}(f)$ and $\operatorname{supp}(h)$ are spacelike separated, $\varphi_{\varkappa}^{\#}(f)$ and $\varphi_{\varkappa}^{\#}(h)$ $\approx$ -commute.

Proof Let

$$A_{\varkappa,g}(f,s) = Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}^{\#3}} \varphi_{\varkappa}^{\#}(\boldsymbol{x},t) f(\boldsymbol{x},t) d^{\#3}x$$

and

$$B_{\varkappa,g}(h,t) = Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}^{\#3}} \pi_{\varkappa}^{\#}(\boldsymbol{x},t) f(\boldsymbol{x},t) d^{\#3}x$$

be the sharp time fields obtained from the test functions f and h correspondingly. First we prove that $A_{\varkappa,g}(f,t)$ and $B_{\varkappa,g}(h,t)$ commute. For any #-open set $\mathfrak{J}$ in space, we define the algebra $\mathfrak{C}^{\#}(\mathfrak{J})$ as the weak #-closure of the finitely bounded functions of the $t = 0$ fields

$$Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}^{\#3}} \varphi_{\varkappa}^{\#}(\boldsymbol{x},0) f_0(\boldsymbol{x}) d^{\#3}x \text{ and } Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}^{\#3}} \pi_{\varkappa}^{\#}(\boldsymbol{x},0) f_0(\boldsymbol{x}) d^{\#3}x$$

as f_0 runs over the $C^{*\infty}$- functions with support in $\mathfrak{I}$. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are disjoint #-open sets, then elements of $\mathfrak{C}^{\#}(\mathfrak{I}_1)$ and $\mathfrak{C}^{\#}(\mathfrak{I}_2)$ commute, and it was shown in [17] that

$$\left[Ext\text{-}\exp\left(-i\sigma\, H_{\varkappa}(g)\right)\right]\mathfrak{C}^{\#}(\mathfrak{I})\left[Ext\text{-}\exp\left(-i\sigma\, H_{\varkappa}(g)\right)\right] \subset \mathfrak{C}^{\#}(\mathfrak{I}_{\sigma}), \tag{9.1}$$

where $\mathfrak{I}_{\sigma}$ is the set of all points in space with distance less than $|\sigma|$ from $\mathfrak{I}$. The proof in [17] is valid whether or not $g(\boldsymbol{x}) = \text{const}$ on the set $\mathfrak{I}$. If $\mathfrak{I}_1$ is a small neighbourhood of supp $(f) \cap \{\text{time} = s\}$ and $\mathfrak{I}_2$ is similarly defined with respect to h at time t, then $\mathfrak{I}_1$ and $(\mathfrak{I}_2)_{t-s}$ are disjoint. Since the finitely bounded functions of $A_{\varkappa,g}(f,s)$ belong to $\left[Ext\text{-}\exp\left(-is\, H_{\varkappa}(g)\right)\right]\mathfrak{C}^{\#}(\mathfrak{I}_1)\left[Ext\text{-}\exp\left(-is\, H_{\varkappa}(g)\right)\right]$

and the bounded functions of $B_{\varkappa,g}(h,t)$ belong to

$$\left[Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right]\mathfrak{C}^{\#}(\mathfrak{I}_2)\left[Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right] \subset$$

$$\subset \left[Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right]\mathfrak{C}^{\#}((\mathfrak{I}_2)_{s-t})\left[Ext\text{-}\exp\left(-it\, H_{\varkappa}(g)\right)\right],$$

$A_{\varkappa,g}(f,s)$ and $B_{\varkappa,g}(h,t) \approx$ -commute. Let E be a spectral projection of $A_{\varkappa,g}(f,s)$ and let $\psi \in D_{\varkappa,g,\text{fin}}$ a near standard #-core for $\varphi^{\#}_{\varkappa,g}(h)$. Then $E\psi \in D_{\text{fin}}\left(B_{\varkappa,g}(h,t)\right)$ for all t and

$$\langle\, \varphi^{\#}_{\varkappa,g}(h)\theta, E\psi\rangle_{\#} \approx Ext\text{-}\int\langle\theta, B_{\varkappa,g}(h,t)E\psi\rangle_{\#} d^{\#}t \approx \langle\theta, E\left[Ext\text{-}\int B_{\varkappa,g}(h,t)\psi d^{\#}t\right]\rangle_{\#} \approx \langle\theta, E\varphi^{\#}_{\varkappa,g}(h)\rangle_{\#} \tag{9.2}$$

for all $\theta \in D_{\varkappa,g,\text{fin}}$. Thus

$$E\psi \in D_{\text{fin}}\left(\varphi^{\#}_{\varkappa,g}(h)^{*}\right) = D_{\text{fin}}\left(\varphi^{\#}_{\varkappa,g}(h)\right),$$

$$\varphi^{\#}_{\varkappa,g}(h)E\psi \approx E\varphi^{\#}_{\varkappa,g}(h)\psi,$$

and $A_{\varkappa,g}(f,s) \approx$ -commutes with $\varphi^{\#}_{\varkappa,g}(h)$. Now let F be a spectral projection for $\varphi^{\#}_{\varkappa,g}(h)$.

Then $F\psi \in D_{\text{fin}}\left(A_{\varkappa,g}(f,s)\right)$ for all s and

$$\langle\, \varphi^{\#}_{\varkappa,g}(f)\theta, E\psi\rangle_{\#} \approx Ext\text{-}\int\langle\theta, A_{\varkappa,g}(f,t)E\psi\rangle_{\#} d^{\#}t \approx \langle\theta, F\left[Ext\text{-}\int A_{\varkappa,g}(h,t)\psi d^{\#}t\right]\rangle_{\#} \tag{9.3}$$

as before in (4.5.2), so that $E\psi \in D_{\text{fin}}\left(\varphi^{\#}_{\varkappa,g}(f)^{*}\right) = D_{\text{fin}}\left(\varphi^{\#}_{\varkappa,g}(f)\right)$ and $\varphi^{\#}_{\varkappa,g}(f)E\psi \approx E\varphi^{\#}_{\varkappa,g}(f)\psi$. Therefore, $\varphi^{\#}_{\varkappa}(f)$ and $\varphi^{\#}_{\varkappa}(h) \approx$ -commute.

10. Space time covariance

Space time covariance means that the field transforms in the expected fashion under the space time translation $\boldsymbol{x}' = (x'_1, x'_2, x'_3), t'$,

$$\varphi^{\#}_{\varkappa}(\boldsymbol{x},t) \to \varphi^{\#}_{\varkappa}(\boldsymbol{x}+\boldsymbol{x}', t+t') \tag{10.1}$$

By its canonical definition the field transforms correctly under time translation. Let $U(\boldsymbol{x}')$ be the unitary operator on $\mathcal{F}^{\#}$ which implements the free field space translation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{x}'$. By definition, $U(\boldsymbol{x}')$ acts on each vector $\theta_j\left(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_j\right)$ in the j particle subspace $\mathcal{F}^{\#}_j$ by

$$U(\boldsymbol{x}')\theta\left(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_j\right) = \left(Ext\text{-}\exp\left[i\langle\boldsymbol{x}', \textstyle\sum_{\boldsymbol{i=1}}^{\boldsymbol{j}}\boldsymbol{k}_i\rangle\right]\right)\theta\left(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_j\right). \tag{10.2}$$

We use the convention that $U(\boldsymbol{x}')$ is the Schrödinger picture operator. On a suitable domain,

$$U(-\boldsymbol{x}')a^{*}(\boldsymbol{k})U(\boldsymbol{x}') = (Ext\text{-}\exp[i\langle\boldsymbol{k}, \boldsymbol{x}'\rangle])a^{*}(\boldsymbol{k}), \tag{10.3}$$

$$U(-\boldsymbol{x}')a\,(\boldsymbol{k})U(\boldsymbol{x}') = (Ext\text{-}\exp[i\langle\boldsymbol{k}, \boldsymbol{x}'\rangle])a\,(\boldsymbol{k}), \tag{10.4}$$

and from the definition (2.10) we have

$$U(-\boldsymbol{x}')\varphi_{\varkappa}^{\#}(\boldsymbol{x})U(\boldsymbol{x}') = \varphi_{\varkappa}^{\#}(\boldsymbol{x}+\boldsymbol{x}'). \tag{10.5}$$

Now $U(\boldsymbol{x}')$ does not commute with $H_{\varkappa}(g)$, but in fact

$$U(-\boldsymbol{x}')\,H_{\varkappa}(g)U(\boldsymbol{x}') = H_{\varkappa}(g'),$$

where $g'(\boldsymbol{x}) = g(\boldsymbol{x}-\boldsymbol{x}')$. But $\varphi_{\varkappa}^{\#}(\boldsymbol{x},t)$ is independent of the space cutoff function $g(\boldsymbol{y})$ provided that $g(\boldsymbol{y}) = \lambda$ for $|\boldsymbol{x}-\boldsymbol{y}| > |t|$. Thus if $g(\boldsymbol{y}) = \lambda$ for $|\boldsymbol{x}+\boldsymbol{x}'-\boldsymbol{y}| > |t|$,

$$U(-\boldsymbol{x}')\varphi_{\varkappa}^{\#}(\boldsymbol{x},t)U(\boldsymbol{x}') = U(-\boldsymbol{x}')\left[Ext\text{-}\exp\left(it\,H_{\varkappa}(g)\right)\right]\varphi_{\varkappa}^{\#}(\boldsymbol{x})\left[Ext\text{-}\exp\left(-it\,H_{\varkappa}(g)\right)\right]U(\boldsymbol{x}') =$$

$$= \left[Ext\text{-}\exp\left(it\,H_{\varkappa}(\,g')\right)\right]\varphi_{\varkappa}^{\#}(\boldsymbol{x}+\boldsymbol{x}')\left[Ext\text{-}\exp\left(-it\,H_{\varkappa}(\,g')\right)\right] = \varphi_{\varkappa}^{\#}(\boldsymbol{x}+\boldsymbol{x}',t) \tag{10.6}$$

on a suitable domain, for example on $D_{0,\varkappa}\times D_{0,\varkappa}$. Thus $U(\boldsymbol{x}')$ implements the space translation for fields $\varphi_{\varkappa}^{\#}(f)$.

11.The algebra of local observables

To each #-open region $B \subset {}^{*}\widetilde{\mathbb{R}}_{c}^{\#4}$ of space time, we associate a non-Archimedean $C_{\#}^{*}$ algebra $\mathfrak{C}^{\#}(B)$ in such a way that the self-#-adjoint elements of $\mathfrak{C}^{\#}(B)$ are exactly the operators corresponding to experiments which may be performed in B.

Definition 11.1 Let $A \in \mathfrak{C}^{\#}(B)$, we say that operator is near-standard if $\|A\|_{\#} \in {}^{*}\widetilde{\mathbb{R}}_{c,\text{fin}}^{\#}$ and $\text{st}(\|A\|_{\#}) \neq 0$. The sub algebra of the all near-standard operators in $\mathfrak{C}^{\#}(B)$ will be denoted by $\mathfrak{C}_{\approx}^{\#}(B)$.

Definition 11.2 The C^{*} algebra of standard local observables $\text{st}(\mathfrak{C}_{\approx}^{\#}(B))$ is defined by

$$\text{st}(\mathfrak{C}_{\approx}^{\#}(B)) = \{\text{st}(A)|A \in \mathfrak{C}_{\approx}^{\#}(B)\}.$$

Remind that the requirements for a local quantum theory are: to each bounded open region B of space time, there is an associated non-Archimedean $C_{\#}^{*}$ algebra $\mathfrak{C}^{\#}(B)$ containing the identity.

(a) Isotony: if $B_1 \supset B_2$, then $\mathfrak{C}^{\#}(B_1) \supset \mathfrak{C}^{\#}(B_2)$.

(b) Locality: B_1 and B_2 are space like separated, then $\text{st}(\mathfrak{C}_{\approx}^{\#}(B_1))$ commutes with $\text{st}(\mathfrak{C}_{\approx}^{\#}(B_2))$.

(c) The algebra of local observables $\mathfrak{C}^{\#}$ is defined as the #-norm #-closure of the union of the $\mathfrak{C}^{\#}(B)$.

(d) The algebra is primitive; in other words, it has a faithful, irreducible representation.

(e) Lorentz covariance: Let $\{a,\Lambda\}$ be an element of the inhomogeneous Lorentz group $L_{+}^{\uparrow}$. Then there is a representation $\sigma_{\{a,\Lambda\}}$ of $L_{+}^{\uparrow}$ by a group of $*$ - automorphisms of $\mathfrak{C}^{\#}$, such that for a bounded region B

$$\sigma_{\{a,\Lambda\}}\mathfrak{C}_{\approx}^{\#}(B) \approx \mathfrak{C}_{\approx}^{\#}(\{a,\Lambda\}B). \tag{11.1}$$

In this section we consider several possible definitions for the non-Archimedean algebra $\mathfrak{C}^{\#}(B)$. The different definitions undoubtedly lead to different $C_{\#}^{*}$ algebras. In order to arrive at a natural and aesthetic definition, we prove that all reasonable candidates for $\mathfrak{C}^{\#}(B)$ have the same weak #-closure; we take this weakly #-closed algebra as the definition of $\mathfrak{C}^{\#}(B)$.

Definition 11.3 $\mathfrak{C}^{\#}(B)$ is the weakly #-closed operator algebra generated by the operators

$$\left\{F[\varphi_{\varkappa}^{\#}(f)]|F \in L_{*\infty}^{\#}, \text{supp}(f) \subset B, |f|_{\#1} \in {}^{*}\widetilde{\mathbb{R}}_{c,\text{fin}}^{\#}\right\}.$$

The definition is unchanged if we replace $L_{*\infty}^{\#}$, by some non- Archimedean $*$ - algebra which is #-dense in the weak operator topology. It is also unchanged if we replace the class of test functions by another (for example $D_{\text{fin}}^{\#}(B)$) having the same #-closure in the $|\cdot|_{\#1}$ #-norm. In fact, if $|f_n - f|_{\#1} \to_{\#} 0$, then $(\varphi_{\varkappa}^{\#}(f_n) -$

$z)^{-1} \rightarrow_{\#} (\varphi_{\varkappa}^{\#}(f) - z)^{-1}$ in the strong operator topology by Lemma 7.4 and by the generalized semigroup convergence theorem $Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f_n)\right) \rightarrow_{\#} Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)$ Thus $Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)$ and $F\left(\varphi_{\varkappa}^{\#}(f)\right)$ belong to the weak #-closure if each $f_n, n \in {}^*\mathbb{N}$ is admitted as a test function in definition of non-Archimedean $C_{\#}^{*}$ algebra $\mathfrak{C}^{\#}(B)$. The same algebra $\mathfrak{C}^{\#}(B)$ is generated by the finitely bounded functions of sharp time fields $A_{\varkappa}(t) = = Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}_c^{\#3}} \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t) f(\boldsymbol{x}, t) d^{\#3}x$, $f \in D_{\text{fin}}^{\#}(B)$. In fact, using a hyper infinite sequence $f_n, n \in {}^*\mathbb{N}$ such as (4.3.18), we have the resolvents #-converging $(\varphi_{\varkappa}^{\#}(f_n) - z)^{-1} \rightarrow_{\#} (A_{\varkappa}(t) - z)^{-1}$, and so $F\left(A_{\varkappa}(t)\right) \in \mathfrak{C}^{\#}(B)$.Thus the sharp time fields generate a smaller algebra. However, if $f \in D_{\text{fin}}^{\#}(B)$, we can approximate $i\varphi_{\varkappa}^{\#}(f_n)$ by the following hyperfinite sum $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$, with strong #-convergence on $D_g^{\#}$. By Lemma 4.3.4 the resolvents #-converge, so $F\left(\varphi_{\varkappa}^{\#}(f)\right)$ belongs to the weakly #-closed algebra generated by the finitely bounded functions of hyperfinite linear combinations of the sharp time fields. We now see that all such $F\left(\varphi_{\varkappa}^{\#}(f)\right)$ belong to the algebra generated by the finitely bounded functions of the sharp time fields themselves. Let F be a finitely bounded operator commuting with $A_{\varkappa}(t_1), A_{\varkappa}(t_2), \ldots$, and $A_{\varkappa}(t_n), n \in {}^*\mathbb{N}$. Then by the generalized spectral theorem, F commutes with $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$ on the domain $D_{\varkappa,g}^{\#}$, which by Theorem 4.3.5 is a #-core for $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$. Thus F commutes with $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$. Thus the commutant of $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$ is larger than that of $Ext\text{-}\sum_{i=1}^{i=n} A_{\varkappa}(t_i)\Delta t_i$, and the double commutant smaller. Therefore, the sharp time fields generate $\mathfrak{C}^{\#}(B)$ as asserted.

Theorem 11.1 With mentioned above definition of $\mathfrak{C}^{\#}(B)$, the axioms (a)-(f) are satisfied

12. Estimates on the interaction Hamiltonian

Let $\mathcal{F}^{\#}$ be the Pock space for a massive, neutral scalar Geld in two-dimensional space-time. The elements of $\mathcal{F}^{\#}$ are sequences of functions on momentum space. Let the annihilation and creation operators be normalized by the relation

$$[a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta^{\#}(\boldsymbol{k} - \boldsymbol{k}'). \tag{12.1}$$

Thus the free-field Hamiltonian is

$$H_{0,\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} a^*(\boldsymbol{k}) a(\boldsymbol{k}) \omega(\boldsymbol{k}) d^{\#3}k. \tag{12.2}$$

The $t = 0$ field with hyperfinite ultraviolet cut-oft $\varkappa$ is

$$\varphi_{\varkappa}^{\#}(x) = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} Ext\text{-}\exp(-i\langle \boldsymbol{k}, \boldsymbol{x} \rangle)[a^*(\boldsymbol{k}) + a(\boldsymbol{k})]\, d^{\#3}k \tag{12.3}$$

The spatially cut-off interaction Hamiltonian reads

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^*\widetilde{\mathbb{R}}_c^{\#3}} : \varphi_{\varkappa}^{\#4}(\boldsymbol{x}) : g(\boldsymbol{x})\, d^{\#3}x = \tag{12.4}$$

$$\sum_{j=0}^{4} \binom{4}{j} \Big\{ Ext\text{-}\int_{|\boldsymbol{k_1}| \leq \varkappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|\boldsymbol{k_m}| \leq \varkappa} d^{\#3}k_m\, a^*(\boldsymbol{k}_1) \cdots a^*(\boldsymbol{k}_j) a(-\boldsymbol{k_{j+1}}) \cdots$$

$$\times\, a(-\boldsymbol{k_4}) \hat{g} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \right) \prod_{i=1}^{4} [\omega(\boldsymbol{k}_{\,\boldsymbol{i}})]^{-1/2} d^{\#3} k_{\,1} \ldots d^{\#3} k_{\,4} \Big\},$$

where we let $\boldsymbol{k}_{\,\boldsymbol{i}} = \left(k_i^{(1)}, k_i^{(2)}, k_i^{(3)}\right), i = 1,2,3.$

The total Hamiltonian reads

$$H_{\varkappa}(g) = H_{0,\varkappa} + H_{I,\varkappa}(g) \tag{12.5}$$

We let

$$N_{\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} a^*(\boldsymbol{k}) a(\boldsymbol{k}) d^{\#3}k, \tag{12.6}$$

and

$$D^{\#}_{0,\varkappa} = \bigcap_{n=0}^{^*\infty} D\left(H^{n}_{0,\varkappa}\right). \tag{12.7}$$

Theorem 12.1 For any $\varepsilon \in {}^*\widetilde{\mathbb{R}}^{\#}_{\text{fin}+}$ and for fixed $g(x) \in S^{\#}_{\text{fin}}\left({}^*\widetilde{\mathbb{R}}^{\#\mathbf{3}}_{c}\right)$ there is a constant b such that as bilinear forms on $D^{\#}_{0,\varkappa} \times D^{\#}_{0,\varkappa}$

$$-\left[H^{\frac{1}{2}}_{0,\varkappa}, \left[H^{\frac{1}{2}}_{0,\varkappa},\ H_{I,\varkappa}(g)\right]\right] \leq \varepsilon H^{2}_{0,\varkappa} + b, \tag{12.8}$$

$$-\left[N_{\varkappa}, \left[N_{\varkappa},\ H_{I,\varkappa}(g)\right]\right] \leq \varepsilon N^{2}_{\varkappa} + b. \tag{12.9}$$

Theorem 12.2 Let $W\colon \mathcal{F}^{\#} \to \mathcal{F}^{\#}$ be an operator of the form

$$W = Ext\text{-}\int_{|\boldsymbol{k_1}|\leq\varkappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|\boldsymbol{k_m}|\leq\varkappa} d^{\#3}k_m\, w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\, a^*(\boldsymbol{k}_{\mathbf{1}}) \cdots a(-\boldsymbol{k_m}), \tag{12.10}$$

where $w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m) \in L^{\#}_{2}\left(\left({}^*\widetilde{\mathbb{R}}^{\#\mathbf{3m}}_{c}\right)\right)$. Then

$$\left\|(N_{\varkappa} + I)^{-j/2} W (N_{\varkappa} + I)^{-(m-j)/2}\right\|_{\#} \leq \text{const}\|w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\|_{L^{\#}_{2}}, \tag{12.11}$$

$$\left\|\left(H_{0,\varkappa} + I\right)^{-1}\left[H^{\frac{1}{2}}_{0,\varkappa}, \left[H^{\frac{1}{2}}_{0,\varkappa}, W\right]\right]\left(H_{0,\varkappa} + I\right)^{-1}(N_{\varkappa} + I)^{-\frac{(m-4)}{2}}\right\|_{\#} \leq$$

$$\leq \text{const}\left\|\omega^{\frac{1}{2}}\left(\textstyle\sum_{i=1}^{4} k^{(1)}_{\boldsymbol{i}},\ \sum_{\boldsymbol{i=1}}^{\mathbf{4}} k^{(2)}_{i},\ \ \sum_{\boldsymbol{i=1}}^{\mathbf{4}} k^{(3)}_{i}\ \right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\right\|_{L^{\#}_{2}}, \tag{12.12}$$

$$\left\|\left[H^{\frac{1}{2}}_{0,\varkappa}, \left[H^{\frac{1}{2}}_{0,\varkappa}, W\right]\right](N_{\varkappa} + I)^{-m/2}\right\|_{\#} \leq \text{const} \times \varkappa^{4}\left\|\textstyle\sum_{i=1}^{m} \omega(\boldsymbol{k}_i) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\right\|_{L^{\#}_{2}}. \tag{12.13}$$

Theorem 12.3 Let the operator W be as above. Then

$$\left\|\left[H^{\frac{1}{2}}_{0,\varkappa}, \left[H^{\frac{1}{2}}_{0,\varkappa}, W\right]\right](N_{\varkappa} + I)^{-m/2}\right\|_{\#} \leq \text{const}\|w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\|_{L^{\#}_{2}}. \tag{12.14}$$

Proof of Theorem 12.1.Introduce the $t = 0$ field $\varphi^{\#}_{\mu}(x)$ with an hyperfinite ultraviolet cut-oft $\mu < \varkappa$:

$$\varphi^{\#}_{\mu}(x) = Ext\text{-}\int_{|\boldsymbol{k}|\leq\mu} Ext\text{-}\exp(-i\langle\boldsymbol{k}, \boldsymbol{x}\rangle)[a^*(\boldsymbol{k}\,) + a(\boldsymbol{k})]\, d^{\#3}k$$

The spatially cut-off interaction Hamiltonian $H_{I,\mu}(g)$ corresponding to the $t = 0$ field $\varphi^{\#}_{\mu}(x)$ reads

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{^*\widetilde{\mathbb{R}}^{\#3}_{c}} :\varphi^{\#4}_{\varkappa}(\boldsymbol{x}):\, g(\boldsymbol{x})\, d^{\#3}x. \tag{12.15}$$

Note that

$$H_{I,\varkappa}(g) = \text{strong } \#\text{-}\lim_{\mu \to_{\#} \varkappa} H_{I,\mu}(g). \tag{12.16}$$

If we write $H_{I,\varkappa}(g)$ as a sum of five operators of the form W in (12.10), then by Theorem 12.2 taken for the case $m = 4$ we get

$$\left\|\left(H_{0,\varkappa} + I\right)^{-1}\left[H^{\frac{1}{2}}_{0,\varkappa}, \left[H^{\frac{1}{2}}_{0,\varkappa}, W\right]\right]\left(H_{0,\varkappa} + I\right)^{-1}\right\|_{\#} \leq$$

$$\leq \text{const} \left\| \omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \right\|_{L_2^{\#}}. \quad (12.17)$$

Since the kernel $w(\boldsymbol{k_1}, \dots, \boldsymbol{k}_4)$ has an over-all factor $\hat{g}\left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \right)$, where $\hat{g}(\boldsymbol{k})$ is the Fourier transform of the spatial cut-off $g(\boldsymbol{x})$, the fast decrease of $\hat{g}(\boldsymbol{k})$) ensures that

$$\omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \in L_2^{\#}$$

Thus the kernel for the corresponding cut-off interaction term w_μ approximates $w_\varkappa$ in the sense that

$$\left\| \omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \right) \left(w_\varkappa(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) - w_\mu(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \right) \right\|_{L_2^{\#}} \to_{\#} 0 \quad (12.18)$$

as $\mu \to_{\#} \varkappa$. This is holds for each W making up $H_{I,\varkappa}(g)$, so we infer that there exists a μ_0 such that for any μ such that: $\mu_0 < \mu < \varkappa$

$$\left\| \left(H_{0,\varkappa} + I \right)^{-1} \left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, \left(H_{I,\varkappa}(g) - H_{I,\mu}(g) \right) \right] \right] \left(H_{0,\varkappa} + I \right)^{-1} \right\|_{\#} \leq \frac{1}{2}\varepsilon. \quad (12.19)$$

13. Self #-adjointness of the interaction Hamiltonian

For a real spatial cut-off $g(x)$ in the Schwartz space $S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#\boldsymbol{3}})$, the interaction part of the Hamiltonian $H_{I,\varkappa}(g)$ is self #-adjoint.

Theorem 13.1 If $g \in S_{\text{fin}}^{\#}\left({}^{*}\widetilde{\mathbb{R}}_c^{\#\boldsymbol{3}}\right)$ is real, then

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_c^{\#\boldsymbol{3}}} : \varphi_\varkappa^{\#4}(\boldsymbol{x}) : g(\boldsymbol{x})\, d^{\#3}x \quad (13.1)$$

is essentially self #-adjoint on $D_{0,\varkappa}^{\#} = \bigcap_{n=0}^{{}^{*}\infty} D\left(H_{0,\varkappa}^{n}\right)$.

Let us introduce a domain $D_{1,\varkappa}^{\#}$ obtained by applying any polynomial of the $t = 0$ fields $\varphi_\varkappa^{\#}(f_i)$, for real $f_i \in S_{\text{fin}}^{\#}\left({}^{*}\widetilde{\mathbb{R}}_c^{\#\boldsymbol{3}}\right)$ the no particle state Ω_0. Clearly $D_{1,\varkappa}^{\#} \subset D_{0,\varkappa}^{\#}$, and any vector Ω in $D_{1,\varkappa}^{\#}$ is an entire vector for $\varphi_\varkappa^{\#}(f)$, which means that the hyperinfinite power series

$$Ext\text{-}\sum_{n=0}^{{}^{*}\infty} \frac{\left\| \varphi_\varkappa^{\#n}(f)\Omega \right\|_{\#}}{n!} z^n \quad (13.2)$$

defines an entire function of s. Since $D_{1,\varkappa}^{\#}$ is #-dense in Fock space, Generalized Nelson's analytic vector theorem (see [19] Chapt.vi, sect.5) shows that for real f, $\varphi_\varkappa^{\#}(f)$ is essentially self #-adjoint on $D_{1,\varkappa}^{\#}$. A similar argument can be made for the canonically conjugate $t = 0$ fields $\pi_\varkappa^{\#}(f)$. Let $\mathcal{M}_\varkappa^{\#}$ denote the von Neumann algebra of operators generated by the spectral projections of all the $t = 0$ field $\varphi_\varkappa^{\#}(f), f \in S_{\text{fin}}^{\#}\left({}^{*}\widetilde{\mathbb{R}}_c^{\#\boldsymbol{3}}\right)$. The algebra $\boldsymbol{\mathcal{M}}_\varkappa^{\#}$ is maximal Abelian. In other words, a bounded operator which commutes with all operators in $\mathcal{M}_\varkappa^{\#}$ is itself in $\mathcal{M}_\varkappa^{\#}$. Let us consider $\varphi_\varkappa^{\#}(f)$ for $\text{supp}(f) \subset \boldsymbol{O} \subset {}^{*}\widetilde{\mathbb{R}}_{\boldsymbol{c}}^{\#\boldsymbol{3}}$, where $\boldsymbol{O}$ is an #-open region of space. (The support of a function is the smallest #-closed set outside of which the function vanishes identically.) Define $\boldsymbol{\mathfrak{C}}_\varkappa^{\#}(\boldsymbol{O})$ as the von Neumann algebra of operators generated by the spectral projections of all the fields $\varphi_\varkappa^{\#}(f)$ and $\pi_\varkappa^{\#}(f)$ with $\text{supp}(f) \subset \boldsymbol{O}$. Since

$$\varphi_\varkappa^{\#}(\boldsymbol{x}, t) = Ext\text{-}\exp\left(itH_{0,\varkappa}\right) \varphi_\varkappa^{\#}(\boldsymbol{x}) Ext\text{-}\exp\left(-itH_{0,\varkappa}\right) = \quad (13.3)$$

$$= Ext\text{-}\int_{{}^{*}\mathbb{R}_{\boldsymbol{c}}^{\#3}} d^{\#3}\boldsymbol{y} \left\{ \Delta_{\#}(\boldsymbol{x} - \boldsymbol{y}, t; m)\, \pi_\varkappa^{\#}(\boldsymbol{y}) - \left[\frac{\partial^{\#}}{\partial^{\#} t} \Delta_{\#}(\boldsymbol{x} - \boldsymbol{y}, t; m) \right] \varphi_\varkappa^{\#}(\boldsymbol{y}) \right\},$$

where $\Delta_{\#}(\boldsymbol{x}, t; m)$ is the solution of the generalized Klein-Gordon equation with Cauchy data $\Delta_{\#}(\boldsymbol{x}, 0; m) = 0$, $\frac{\partial^{\#}}{\partial^{\#} t} \Delta_{\#}(\boldsymbol{x}, 0; m) = \delta^{\#}(x)$. (see [15],Eq.111) and $\Delta_{\#}(\boldsymbol{x}, t; m)$ vanishes outside the light cone, we infer that

$$Ext\text{-}\exp\left(itH_{0,\varkappa}\right)\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O})Ext\text{-}\exp\left(-itH_{0,\varkappa}\right) \subset \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}_t), \tag{13.4}$$

where $\boldsymbol{O}_t$ is the region $\boldsymbol{O}$ expanded by t.

Theorem 13.2 If $g(x) \in S_{\mathrm{fin}}^{\#}\left({}^*\widetilde{\mathbb{R}}_c^{\#3}\right)$ is real and has its support in an #-open rectangular parallelepiped $\boldsymbol{O} \subset {}^*\mathbb{R}_c^{\#3}$, then for the $H_{I,\varkappa}(g)$ of (13.1)

$$Ext\text{-}\exp\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}_{\varkappa}^{\#} \cap \boldsymbol{\mathcal{M}}_{\varkappa}^{\#}.$$

Theorem 13.3 Let T be any operator with domain $D_{1,\varkappa}^{\#}$ such that

$$T\, D_{1,\varkappa}^{\#} \subset D\left(\varphi_{\varkappa}^{\#n}(f)\right), \tag{13.5}$$

$$T\, D_{1,\varkappa}^{\#} \subset D\left(\left(T \upharpoonright D_{1,\varkappa}^{\#}\right)^*\right), \tag{13.6}$$

$$[T,\, \varphi_{\varkappa}^{\#n}(f)]\, D_{1,\varkappa}^{\#} = 0. \tag{13.7}$$

Then

$$\boldsymbol{\mathcal{M}}_{\varkappa}^{\#} D_{1,\varkappa}^{\#} \subset D\left(T \upharpoonright D_{1,\varkappa}^{\#}\right), \tag{13.8}$$

$$[\#\text{-}\overline{T}, \boldsymbol{\mathcal{M}}_{\varkappa}^{\#}]\, D_{1,\varkappa}^{\#} = 0. \tag{13.9}$$

Proof For $\Omega \in D_{1,\varkappa}^{\#}$, from (13.5) and (13.7) we get

$$T\, \varphi_{\varkappa}^{\#n}(f)\Omega = \varphi_{\varkappa}^{\#n}(f)T\Omega.$$

But by (13.6), for real f

$$\|T\, \varphi_{\varkappa}^{\#n}(f)\Omega\|_{\#}^2 = \langle T\Omega,\, \varphi_{\varkappa}^{\#2n}(f)T\Omega\rangle_{\#} = \langle T^*T\Omega,\, \varphi_{\varkappa}^{\#2n}(f)\Omega\rangle_{\#} \leq \|T^*T\Omega\|_{\#}\|\, \varphi_{\varkappa}^{\#2n}(f)\Omega\|_{\#}.$$

Thus the #-convergent power series (3.2) shows that for $\Omega \in D_{1,\varkappa}^{\#}$,

$$\#\text{-}\overline{T}\left(Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)\right)\Omega = Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)T\Omega. \tag{13.10}$$

It is clear that (13.10) is still valid with $Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)$ replaced by strong #-limits of sums of such exponentials, and hence (13.8) and (13.9).

Theorem 13.4 Let $\mathcal{M}$ is a maximal Abelian algebra of bounded operators on a non-Archimedean Hilbert space $\mathcal{H}$ with a cyclic vector Ω_0. Let T be a symmetric operator with domain $\mathcal{M}\Omega_0$, and let T commute with $\mathcal{M}$.Then T is essentially self #-adjoint.

Proof Without loss of generality, $\mathcal{M} = L_{{}^*\infty}^{\#}(X)$ and $\mathcal{H} = L_2^{\#}(X)$ for some #-measure space (X, Σ, μ), and Ω_0 is the function 1. Let $f \in L_2^{\#}(X)$. Then $t \in L_2^{\#}(X)$ and T is multiplication by t, with domain $L_{{}^*\infty}^{\#}(X)$. Let $f \in L_2^{\#}(X)$ and suppose $tf \in L_2^{\#}(X)$ also and let $f_n(x) = f(x)$ if $|f(x)| \leq n, n \in {}^*\mathbb{N}$ and $f_n(x) \equiv 0$ otherwise. Then $f_n \in L_{{}^*\infty}^{\#} = D(T)$ and $f_n \to_{\#} f$, $tf_n \to_{\#} tf$ in $L_2^{\#}$ norm by the bounded #-convergence theorem. Thus $\{f, tf\}$ is in the graph of the #-closure of T**.** Thus the #-closure of T is self #-adjoint, and T is essentially self #-adjoint.

Remark 13.1 Let $T_n, n \in {}^*\mathbb{N}$ be a hyperinfinite sequence of operators with the property of T in the Theorem 13.4. Then $T_n \to_{\#} T$ strongly on the domain $\mathcal{M}\Omega_0$ if and only if $T_n\Omega_0 \to_{\#} T\Omega_0$.

Proof of the Theorems 13.1 and 13.2**.** We apply now the Theorems 13.3 and 13.4 with the case $T = H_{I,\varkappa}(g)$, $\mathcal{M}$ in Theorem 13.4 as in Theorem 13.3, the non-Archimedean Hilbert space Fock space $\mathcal{F}^{\#}$, and Ω_0 the Fock no-particle state. The hypotheses (13.5) and (13.6) can be verified by a direct computation. Thus $H_{I,\varkappa}(g)$ is essentially self #-adjoint on $D_{1,\varkappa}^{\#} \subset D_{0,\varkappa}^{\#}$, and hence $H_{I,\varkappa}(g)$ is essentially self #-adjoint on $D_{0,\varkappa}^{\#}$.

If we assume that $\sup(g) \subset \boldsymbol{O}$, then as $\boldsymbol{O}$ is an #-open region, $\sup(g) \subset \boldsymbol{O}_1$ where $\boldsymbol{O}_1$ is $\boldsymbol{O}$ contracted by some small amount $\varepsilon > 0, \varepsilon \approx 0$. Since $H_{I,\varkappa}(g)$ commutes with $\mathcal{M}$, and $\mathcal{M}$ is maximal Abelian, -exp $\left(itH_{I,\varkappa}(g)\right)$

$\in \mathcal{M}$. Furthermore the argument in the proof of Theorem 13.3, can be repeated to show that $H_{I,\varkappa}(g)$ commutes with $\mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}'_1)$, where $\boldsymbol{O}'_1$ is the complement of the #-closure of $\boldsymbol{O}_1$. Since $\mathfrak{C}^{\#}_{\varkappa}({}^*\widetilde{\mathbb{R}}^{\#3}_c)$ is irreducible and $H_{I,\varkappa}(g)$ commutes with $\mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}'_1)$, $Ext\text{-}\exp\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_2)$ where $\boldsymbol{O}_2$ is $\boldsymbol{O}_1$ expanded by any amount $\varepsilon' > 0$. Taking $\varepsilon' < \varepsilon$, we have $Ext\text{-}\exp\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O})$, which completes the proof.

14. #-Self adjointness of the total Hamiltonian

Theorem 14.1 (a) For real $g(x) \in S^{\#}_{\text{fin}}({}^*\widetilde{\mathbb{R}}^{\#3}_c)$, the total Hamiltonian $H_{\varkappa}(g) = H_{0,\varkappa}(g) + H_{I,\varkappa}(g)$ is self #-adjoint with the domain $D(H_{\varkappa}(g)) = D\left(H_{0,\varkappa}(g)\right) \cap D\left(H_{I,\varkappa}(g)\right)$.
(b) The total Hamiltonian $H_{\varkappa}(g)$ is essentially self #-adjoint on the domain

$$D^{\#}_{0,\varkappa} = \bigcap_{n=0}^{^*\infty} D\left(H^n_{0,\varkappa}\right).$$

Remark 14.1 In order to prove the self #-adjointness of $H_{\varkappa}$, we combine the estimates of section 14, the #-self adjointness of $H_{I,\varkappa}(g)$ proved in section 15, and a singular perturbation theory developed in [19], see also section 21 in this paper. We need the following result which is a special case of Theorem 21.6 from section 21 in this paper.

Theorem 14.2 Under the hypotheses (i)-(iii) below, the operator $H_{\varkappa} = H_{0,\varkappa} + H_{I,\varkappa}$z is self #-adjoint.
(i) Both $H_{0,\varkappa}$ and $H_{I,\varkappa}$ are self #-adjoint. The domain $D^{\#}_{0,\varkappa}$ is contained in the domain of $H_{I,\varkappa}$, and $H_{I,\varkappa}$ is essentially self #-adjoint on $D^{\#}_{0,\varkappa}$.
(ii) Let $N_{\varkappa}$ be a positive self #-adjoint operator, commuting with $H_{0,\varkappa}$, and such that $N_{\varkappa} \leq \text{const } H_{0,\varkappa}$. Suppose that the operators $(N_{\varkappa} + I)^{-1}H_{I,\varkappa}(N_{\varkappa} + I)^{-1}$ and $(N_{\varkappa} + I)^{-1}H_{I,\varkappa}(N_{\varkappa} + I)^{-3}$ are bounded.
(iii) Suppose that for any $\varepsilon > 0$, there exists a number $b \in {}^*\mathbb{R}^{\#}_c$ such that as bilinear forms on $D^{\#}_{0,\varkappa} \times D^{\#}_{0,\varkappa}$,

$$-H_{I,\varkappa} \leq \varepsilon N_{\varkappa} + bI, \tag{14.1}$$

$$-\left[H^{\frac{1}{2}}_{0,\varkappa}\left[H^{\frac{1}{2}}_{0,\varkappa} H_{I,\varkappa}\right]\right] \leq \varepsilon H^2_{0,\varkappa} + bI, \tag{14.2}$$

$$-\left[N_{\varkappa}, \left[N_{\varkappa}, H_{I,\varkappa}\right]\right] \leq \varepsilon N^3_{\varkappa} + bI. \tag{14.3}$$

Proof of Theorem 14.1 In order to prove that $H_{\varkappa}(g)$ is self #-adjoint, we apply Theorem 16.2 in the case that $H_{0,\varkappa}$ is the free Hamiltonian, $N_{\varkappa}$ is the number operator, and $H_{I,\varkappa}$ is the interaction Hamiltonian $H_{I,\varkappa}(g)$. Thus we need to verify (i)-(iii). Condition (i) was dealt with in Theorem 13.1, while condition (ii) is a consequence of (14.11).In refs.[15] and [17] it is shown that for any $\varepsilon > 0$, there is a number $b \in {}^*\mathbb{R}^{\#}_c$ such that

$$-H_{I,\varkappa}(g) \leq \varepsilon H_{0,\varkappa} + bI.$$

By following that proof, but using the smoothing operator $Ext\text{-}\exp(-tN_{\varkappa})$, in place of $Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)$, one arrives at the estimate (16.1) required in (iii). The remaining estimates (14.2) and (14.3) were established in Theorem 14.1.Thus we conclude from Theorem 14.2 that $H_{\varkappa}(g)$ is self #-adjoint on the domain $D\left(H_{0,\varkappa}\right) \cap D\left(H_{I,\varkappa}(g)\right)$. We now show that $H_{\varkappa}(g)$ is essentially self #-adjoint on $D\left(H_{0,\varkappa}\right)$. We first show that $H_{\varkappa}(g)$ is essentially self #-adjoint on $D_2 = D\left(H_{0,\varkappa}\right) \cap D(N^2_{\varkappa})$. By (14.11) it is clear that the domain of $H_{\varkappa}(g)$ contains D_2. For $\psi \in D\left(H_{\varkappa}(g)\right) = D\left(H_{0,\varkappa}\right) \cap D\left(H_{I,\varkappa}(g)\right)$, consider hyperinfinite sequence $\psi_n \in D_2, n \in {}^*\mathbb{N}$ defined by

$$\psi_n = n(nI + N_{\varkappa})^{-1}\psi. \tag{14.4}$$

Thus $\|\psi_n - \psi\|_{\#} + \left\|H_{0,\varkappa}\psi_n - H_{0,\varkappa}\psi\right\|_{\#} \to_{\#} 0$ as $n \to {}^*\infty$.

We need to study the following differences

$$H_{I,\varkappa}\psi_n - H_{I,\varkappa}\psi = -N_\varkappa(nI + N_\varkappa)^{-1}\, H_{I,\varkappa}\psi + n\big[H_{I,\varkappa},(nI + N_\varkappa)^{-1}\big]\psi, n \in {}^*\mathbb{N}. \tag{14.5}$$

Since $N_\varkappa(nI + N_\varkappa)^{-1}, n \in {}^*\mathbb{N}$ is a uniformly bounded hyperinfinite sequence #-converging to zero on the #-dense set $D(N_\varkappa)$, it #-converges to zero and $\|N_\varkappa(nI + N_\varkappa)^{-1}\, H_{I,\varkappa}\psi\|_\#$ as $n \to {}^*\infty$. But for the second term in (14.5) we get

$$n\big[H_{I,\varkappa},(nI + N_\varkappa)^{-1}\big]\psi = \big[H_{I,\varkappa},(nI + N_\varkappa)^{-1}\big](nI + N_\varkappa)n(nI + N_\varkappa)^{-1}\psi = \tag{14.6}$$

$$= (nI + N_\varkappa)^{-1}\big[N_\varkappa, H_{I,\varkappa}\big]n(nI + N_\varkappa)^{-1}\psi =$$

$$= (nI + N_\varkappa)^{-1}(I + N_\varkappa)(I + N_\varkappa)^{-1}\big[N_\varkappa, H_{I,\varkappa}\big] \times$$

$$\times\, (I + N_\varkappa)^{-1}n(nI + N_\varkappa)^{-1}(I + N_\varkappa)\psi.$$

Note that as $n \to {}^*\infty$, hyperinfinite sequence $\delta_n = n(nI + N_\varkappa)^{-1}(I + N_\varkappa)\psi, n \in {}^*\mathbb{N}$ #-converges strongly to $(I + N_\varkappa)\psi$, that by (14.11), $(I + N_\varkappa)^{-1}\big[N_\varkappa, H_{I,\varkappa}\big](I + N_\varkappa)^{-1}$ is bounded, and hyperinfinite sequence $\gamma_n = (nI + N_\varkappa)^{-1}(I + N_\varkappa)\psi, n \in {}^*\mathbb{N}$ #-converges strongly to zero. Thus we get $\big\|\big[H_{I,\varkappa},(nI + N_\varkappa)^{-1}\psi\big]\big\|_\# \to_\# 0$ as $n \to {}^*\infty$, and so $\|H_{I,\varkappa}\psi_n - H_{I,\varkappa}\psi\|_\# \to_\# 0$ as $n \to {}^*\infty$. Thus we can to conclude that $H_\varkappa(g)$ is the #-closure of $H_\varkappa(g)$ restricted to D_2, so $H_\varkappa(g)$ is essentially self #-adjoint on D_2. Let D_2 be a Hilbert space endowed with the #-norm $\|\cdot\|'_\#$ such that

$$(\|\psi\|'_\#)^2 = \|\psi\|^2_\# + \|H_{0,\varkappa}\psi\|^2_\# + \|N_\varkappa\psi\|^2_\#. \tag{14.7}$$

From (14.11) we infer that

$$\|H_\varkappa(g)\,\psi\|_\# \leq \mathrm{const}\|\psi\|'_\#,$$

so that $H_\varkappa(g)$ is essentially self #-adjoint on any subset of D_2 which is #-dense in the Hilbert space D_2. For any $\psi \in D_2, \psi_\lambda = Ext\text{-}\exp\big(-\lambda H_{0,\varkappa}\big)\psi \in D^\#_{0,\varkappa} = \bigcap_{n=0}^{^*\infty} D\big(H^n_{0,\varkappa}\big)$, and $\|\psi - \psi_\lambda\|_{D^\#_{1,\varkappa}} \to_\# 0$ as $\lambda \to_\# 0$. Thus $H_\varkappa(g)$ is essentially self #-adjoint on D_0.

15. Removing the spatial cut-off and locality

For the reader's convenience, we sketch a proof of generalized Segal's theorem that the self #-adjointness of $H_\varkappa(g)$ allows the removal of the spatial cut-off. In fact, if A is a bounded function of the free fields localized in a bounded region of space at $t = 0$, then

$$\sigma_t(A) = Ext\text{-}\exp\big(itH_\varkappa(g)\big)A Ext\text{-}\exp\big(-itH_\varkappa(g)\big)$$

is independent of $g(x)$ provided that $g(\boldsymbol{x}) = \lambda$, the desired coupling constant, on a sufficiently large region, depending on t. Furthermore, if A is localized in the region of space O, then $\sigma_t(A)$ is localized in the region O_t, where O_t is the region O expanded by t. (We have taken the velocity of light to be one.) In other words, the time translation σ_t gives rise to a local theory. If one chooses for the operator A a spectral projection of the $t = 0$ field $\varphi^\#_\varkappa(f)$, one can piece together the time translation operator for the fields themselves. In section 16, we showed that $H_\varkappa = H_{0,\varkappa} + H_{I,\varkappa}$, which is sum of two self #-adjoint operators, is itself self #-adjoint. As a consequence of this fact, the generalized Trotter product formula (see [19] Chapt.6, section 5.10) says that for all $\psi \in \mathcal{F}^\#$

$$Ext\text{-}\exp\big(itH_\varkappa(g)\big)\psi = \#\text{-}\lim_{n\to{}^*\infty}\left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right)\right]\right)\psi.$$

And therefore we obtain

$\sigma_t(A)\psi =$

$\#\text{-}\lim_{n\to^*\infty}\left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right)\right]\right)^n A\left(\left[Ext\text{-}\exp\left(\frac{-itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{-itH_{I,\varkappa}(g)}{n}\right)\right]\right)^n\psi.$

I.et $\boldsymbol{O}$ be the region of space defined by $|\boldsymbol{x}| < M, t = 0$, and let $A \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O})$, where $\mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O})$ is defined above in section 13. Given an arbitrary, positive ε, split $g(\boldsymbol{x})$ into two infinitely #-differentiable parts $g_1(\boldsymbol{x}), g_2(\boldsymbol{x})$ such that

$$g(\boldsymbol{x}) = g_1(\boldsymbol{x}) + g_2(\boldsymbol{x}),$$

where $\operatorname{supp}(g_1(\boldsymbol{x})) \subset \boldsymbol{O}_\varepsilon$ and $\operatorname{supp}(g_2(\boldsymbol{x})) \cap \boldsymbol{O}_{\frac{\varepsilon}{2}} = \emptyset$ is empty. Write now

$$H_{I,\varkappa}(g) = H_{I,\varkappa}(g_1) + H_{I,\varkappa}(g_2),$$

so that as a consequence of theorems 15.1 and 15.2, $H_{I,\varkappa}(g_1)$ and $H_{I,\varkappa}(g_2)$ commute, and

$$Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right) = \left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right].$$

Furthermore,

$$Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_1)}{n}\right) \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_\varepsilon),$$

and $Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)$ commutes with $\mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_{\varepsilon/4})$. Therefore,

$$A_1(t) = \left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right] A \left[Ext\text{-}\exp\left(-\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(-\frac{itH_{0,\varkappa}(g_2)}{n}\right)\right]$$

depends on $g(\boldsymbol{x})$ only in the region $\boldsymbol{O}_\varepsilon$, and by the free propagation property (15.4),

$$A_1 \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_{(t/n)+\varepsilon})$$

We continue step by step, and after $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ steps by using hyperinfinite induction principle, see ref. [10], we conclude that

$$A_n(t) = \left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right]\right)^n A \times$$

$$\times\left(\left[Ext\text{-}\exp\left(-\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(-\frac{itH_{0,\varkappa}(g_2)}{n}\right)\right]\right)^n$$

depends on $g(\boldsymbol{x})$ only in the region $\boldsymbol{O}_{t+n\varepsilon}$ and

$$A_1(t) \in \mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_{t+n\varepsilon})$$

Since ε can be chosen arbitrarily, $A_n(t)$ depends on $g(\boldsymbol{x})$ only in the region #- $\overline{\boldsymbol{O}}_t$, the #- closure of $\boldsymbol{O}_t$, and

$$A_n(t) \in \bigcap_{\varepsilon>0}\mathfrak{C}^{\#}_{\varkappa}(\boldsymbol{O}_{t+\varepsilon}).$$

Thus $A_n(t)$ commutes with any local observable B localized in #-open region of space $\boldsymbol{O}'$ such that $\boldsymbol{O}'$ and $\boldsymbol{O}_t$ are disjoint. As this is true for each $n \in {}^*\mathbb{N}\backslash\mathbb{N}$, it is true for

$$\sigma_t(A) = \text{strong }\#\text{-}\lim_{n\to^*\infty} A_n(t).$$

Hence $\sigma_t(A)$ is local and it depends on $g(\boldsymbol{x})$ only in the region #- $\overline{\boldsymbol{O}}_t$, where we choose $g(\boldsymbol{x}) = \lambda$. Thus we conclude that the spatial cut-off has been removed and the resulting theory is local.

16. Semi-boundedness of the total Hamiltonian

16. Reduction to a Problem with Discrete Momentum We use the non-Archimedean Fock space representation for our field $\varphi_{\varkappa}^{\#}(\boldsymbol{x}), \boldsymbol{x} \in {}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}$. The Fock non-Archimedean Hubert space $\mathcal{F}^{\#}$ is a direct sum

$$\mathcal{F}^{\#} = Ext\text{-}\bigoplus_{n=0}^{{}^{*}\infty} \mathcal{F}_{n}^{\#},$$

where $\mathcal{F}_{n}^{\#}$ is the space of n non-interacting particles, i.e. $\mathcal{F}_{n}^{\#}$ is the space of symmetric square #-integrable functions, i.e.$L_{2}^{\#}\left({}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}\right)$ functions of n variables. Let $\boldsymbol{k} = (k_1, k_2, k_2) \in {}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}$

$$\mu(\boldsymbol{k}) = (\boldsymbol{k}^2 + \mu_0^2)^{1/2} = (k_1^2 + k_2^2 + k_3^2 + m_0^2)^{1/2},$$

$$\varphi_{\varkappa}^{\#-}(\boldsymbol{x}) = Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}} Ext\text{-}\exp(i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\, a(\boldsymbol{k})\theta(|\boldsymbol{k}|, \varkappa)[\mu(\boldsymbol{k})]^{-1/2} d^{\#3}k, \tag{16.1}$$

$$\varphi_{\varkappa}^{\#+}(\boldsymbol{x}) = Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}} Ext\text{-}\exp(i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\, a^{*}(-\boldsymbol{k})\theta(|\boldsymbol{k}|, \varkappa)[\mu(\boldsymbol{k})]^{-1/2} d^{\#3}k, \tag{16.2}$$

$$\theta(|\boldsymbol{k}|, \varkappa) = 1 \text{ if } |\boldsymbol{k}| \leq \varkappa \text{ and } (|\boldsymbol{k}|, \varkappa) = 0 \text{ if } |\boldsymbol{k}| > \varkappa,$$

and $\varphi_{\varkappa}^{\#}(\boldsymbol{x}) = \varphi_{\varkappa}^{\#-}(\boldsymbol{x}) + \varphi_{\varkappa}^{\#+}(\boldsymbol{x})$, where $a(\boldsymbol{k})$ and $a^{*}(\boldsymbol{k})$ are the annihilation and creation operators,

$$[a(\boldsymbol{k}), \boldsymbol{a}^{*}(\boldsymbol{k}')] = \delta^{\#}(\boldsymbol{k} - \boldsymbol{k}'). \tag{16.3}$$

By definition,

$$: \varphi_{\varkappa}^{\#p}(\boldsymbol{x}) := \sum_j \binom{p}{j} \varphi_{\varkappa}^{\#+}(\boldsymbol{x})^j \varphi_{\varkappa}^{\#-}(\boldsymbol{x})^{p-j}. \tag{16.4}$$

Remark 16.1 Remind that Wick product differs from the ordinary product in that all the annihilators are placed to the right and the creators are placed to the left. $: \varphi_{\varkappa}^{\#p}(\boldsymbol{x}):$ is not an operator, but it is a densely defined bilinear form. We take Fourier transforms to compute

$$Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}} : \varphi_{\varkappa}^{\#p}(\boldsymbol{x}): h(\boldsymbol{x}) d^{\#3}x = \sum_j \binom{p}{j} Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3p}} a^{*}(-\boldsymbol{k}_1) \cdots a^{*}(-\boldsymbol{k}_j) a(\boldsymbol{k}_j) \cdots a(\boldsymbol{k}_p) \times \tag{16.5}$$

$$\times\, Ext\text{-}\hat{h}(\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_p) \prod_{i=1}^{p} \theta(\|\boldsymbol{k}_i\|, \varkappa)[\mu(\boldsymbol{k}_i)]^{-1/2} d^{\#3}k_i,$$

where $Ext\text{-}\hat{h}$ is the Fourier transform of h. We assume h is in $L_2^{\#}$ and so $Ext\text{-}\hat{h}$ is in $L_2^{\#}$ also. Since $\mu(\boldsymbol{k}) \sim |\boldsymbol{k}|$ for large $|\boldsymbol{k}|$, one can show that

$$Ext\text{-}\hat{h}(\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_p) \prod_{i=1}^{p} \theta(\|\boldsymbol{k}_i\|, \varkappa)[\mu(\boldsymbol{k}_i)]^{-1/2} \in L_2^{\#}. \tag{16.6}$$

It is well known that (16.6) implies that each integral on the right side of (16.5) is an operator defined on the domain $D(N^{p/2})$ of $N^{p/2}$. This domain is the set of $\psi = \psi_0, \psi_1, \ldots, \psi_j \in \mathcal{F}_j^{\#}$ with

$$Ext\text{-}\sum_n n^{p/2} \|Ext\text{-}\prod_{i=1}^{n} \theta(\|\boldsymbol{k}_i\|, \varkappa)\, \psi_n\|_{\#2}^2 < {}^{*}\infty. \tag{16.7}$$

Thus (16.5) is an operator defined on $D(N^{p/2})$. Similarly $H_{0.\varkappa} + Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}} P(: \varphi_{\varkappa}^{\#}(\boldsymbol{x})): d^{\#3}x$ is an operator defined on the #-dense domain, $D(H_{0.\varkappa}) \cap D(N^{d/2})$, where d is the degree of the polynomial P. We approximate now (16.5) by a hyperfinite sum. Choose numbers $\delta \approx 0$ and $\varkappa \in {}^{*}\widetilde{\mathbb{R}}_{c}^{\#} \backslash {}^{*}\widetilde{\mathbb{R}}_{c,\mathrm{fin}}^{\#}$. We define now an hyperfinite approximation in configuration space. Under this approximation, the momentum space variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}$ is replaced by a discrete variable $\boldsymbol{k} \in \Gamma_{\delta}^{3}$

$$\Gamma_{\delta}^{3} = \{\boldsymbol{k} = (k_1, k_2, k_3) | k_i = \delta n_i, n_i \in {}^{*}\mathbb{Z}; i = 1,2,3\} \tag{16.8}$$

Thus we define $\mathcal{F}_{V}^{\#}$, the Fock space for hyperfinite volume $V^3 = \delta^{-3}$ as

$$\mathcal{F}_V^{\#} = \mathfrak{C}\left(l_2^{\#}(\Gamma_V^3)\right) = {}^*\mathbb{C}^{\#} \oplus l_2^{\#}(\Gamma_V^3) \oplus \{l_2^{\#}(\Gamma_V^3) \otimes_s l_2^{\#}(\Gamma_V^3)\} \cdots \quad (16.9)$$

We choose now one to one correspondence ${}^*\mathbb{Z} \leftrightarrow {}^*\mathbb{Z}\delta \times {}^*\mathbb{Z}\delta \times {}^*\mathbb{Z}\delta = \Gamma_\delta^3$ given by vector-function $\wp(m)$

$$\wp(m) = \{k_1(m), k_2(m), k_3(m)\} = \boldsymbol{k}(m) \quad (16.10)$$

and such that

$$\wp(-m) = -\wp(m). \quad (16.11)$$

And we define now

$$\Gamma_{\varkappa,\delta}^3 = \{\boldsymbol{k} \in \Gamma_\delta^3 | |\boldsymbol{k}| \leq \varkappa\}. \quad (16.12)$$

We set now

$$a_\delta(\boldsymbol{k}(m)) = (\delta)^{-3/2}\left[Ext\text{-}\int_0^\delta d^{\#}l_1\, Ext\text{-}\int_0^\delta d^{\#}l_2\, Ext\text{-}\int_0^\delta d^{\#}l_3\, a(\boldsymbol{k}(m) + \boldsymbol{l})\right], \quad (16.13)$$

$$a_\delta^*(\boldsymbol{k}(m)) = (\delta)^{-3/2}\left[Ext\text{-}\int_0^\delta d^{\#}l_1\, Ext\text{-}\int_0^\delta d^{\#}l_2\, Ext\text{-}\int_0^\delta d^{\#}l_3\, a^*(\boldsymbol{k}(m) + \boldsymbol{l})\right]. \quad (16.14)$$

Then one obtains

$$\left[a_\delta^*(\boldsymbol{k}(m_1)), a_\delta(\boldsymbol{k}(m_2))\right] = \boldsymbol{\delta}_{m_1 m_2} = \begin{cases} 1 \text{ if } m_1 = m_2 \\ 0 \text{ if } m_1 \neq m_2 \end{cases} \quad (16.15)$$

Let

$$H_{0,\varkappa,\delta} = Ext\text{-}\sum_{\boldsymbol{k} \in \Gamma_{\varkappa,\delta}^3} \mu(\boldsymbol{k})\, a_\delta^*(\boldsymbol{k}) a_\delta(\boldsymbol{k}). \quad (16.16)$$

One can check that each ψ in $D(H_{0.\varkappa})$ is in $D(H_{0,\varkappa,\delta})$ also and that

$$\#\text{-}\lim_{\delta \to_{\#} 0} H_{0,\varkappa,\delta}\psi = H_{0.\varkappa}\,\psi. \quad (16.17)$$

Next we approximate (16.5) by

$$:\varphi_{\varkappa,\delta}^{\#p}(\boldsymbol{x}): = \delta^{3p/2} \sum_j \binom{p}{j} Ext\text{-}\sum_{\boldsymbol{k} \in \Gamma_{\varkappa,\delta}^3} a_\delta^*(-\boldsymbol{k}_1) \cdots a_\delta^*(-\boldsymbol{k}_j) a_\delta(\boldsymbol{k}_j) \cdots a_\delta(\boldsymbol{k}_p) \times \quad (16.18)$$

$$\times\, Ext\text{-}\hat{h}([\boldsymbol{k}_1] + \cdots + [\boldsymbol{k}_p]) \prod_i [\mu([\boldsymbol{k}_i])]^{-1/2},$$

where

$$\hat{h}_\delta(\boldsymbol{k}) = Ext\text{-}\int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta} \int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta} \int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta} \left(Ext\text{-}\exp(i\langle \boldsymbol{k}, \boldsymbol{x}\rangle)\right) h(\boldsymbol{x})\, d^{\#3}x$$

and $[\boldsymbol{k}] = ([k_1], [k_2], [k_3])$, where

$$[k_1] = \sup\{l_1 | (l_1, l_2, l_3) \in \Gamma_\delta^3, l_1 \leq k_1\}, [k_2] = \sup\{l_2 | (l_1, l_2, l_3) \in \Gamma_\delta^3, l_2 \leq k_2\},$$

$$[k_3] = \sup\{l_3 | (l_1, l_2, l_3) \in \Gamma_\delta^3, l_3 \leq k_3\}$$

is the integral part of $\boldsymbol{k}$ relative to the lattice Γ_δ^3. Since $h \in L_1^{\#}$, $\hat{h}_\delta$ is #-continuous and

$$Ext\text{-}\hat{h}([\boldsymbol{k}_1] + \cdots + [\boldsymbol{k}_p]) \prod_i [\mu([\boldsymbol{k}_i])]^{-1/2} \to_{\#} Ext\text{-}\hat{h}(\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_p) \prod_{i=1}^p \theta(\|\boldsymbol{k}_i\|, \varkappa)[\mu(\boldsymbol{k}_i)]^{-1/2}$$

uniformly. Let $D_0^{\#}$ be the set of states $\psi = \{\psi_0, \psi_1, \ldots\}$ with $\psi_n(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = 0$ for $n < {}^*\infty$ or $Ext\text{-}\sum_i |k_i| < {}^*\infty$ large. If $\phi, \psi \in D_0^{\#}$ then

$$\#\text{-}\lim_{\delta\to_{\#}0}\langle\phi,:\varphi_{\varkappa,\delta}^{\#p}(\boldsymbol{x}):\psi\rangle_{\#}=\langle\phi,Ext\text{-}\int_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#3}}:\varphi_{\varkappa}^{\#p}(\boldsymbol{x}):d^{\#3}x\psi\rangle. \quad (16.18)$$

Thus the bilinear form of

$$H_{\varkappa,\delta}=H_{0,\varkappa,\delta}+\sum_{p}b_{p}:\varphi_{\varkappa,\delta}^{\#p}(h): \quad (16.19)$$

$\#$-converges to $H_{\varkappa}$ on $D_{0}^{\#}\times D_{0}^{\#}$ where $b_{0},\ldots,b_{r}$ are the coefficients of $y_{0},y_{1},\ldots,y_{r}$ in the polynomial $P(y)$. Hence if the $H_{\varkappa,\delta}$ are semibounded with a lower bound independent of δ then $H_{\varkappa}$ is semibounded also. Let $\mathcal{F}_{\delta}^{\#}$ be the subspace of $\mathcal{F}^{\#}$ consisting of functions which are piece wise constant between lattice points. In other words,

$$\psi=\{\psi_{0},\psi_{1},\ldots,\psi_{n},\ldots\}\in\mathcal{F}_{\delta}^{\#}\text{ if}$$

$$\psi_{n}(\boldsymbol{k}_{1},\ldots,\boldsymbol{k}_{n})=\psi_{n}([\boldsymbol{k}_{1}],\ldots,[\boldsymbol{k}_{n}]).$$

Let $\mathcal{F}_{\varkappa,\delta}^{\#}$ be the subspace of $\mathcal{F}_{\delta}^{\#}$ defined by the restriction

$$\psi_{n}(\boldsymbol{k}_{1},\ldots,\boldsymbol{k}_{n})=0\text{ if }[\boldsymbol{k}_{i}]\notin\Gamma_{\varkappa,\delta}^{3}$$

for some $i,1\leq i\leq n.$

The operators $a_{\delta}^{*}(\boldsymbol{k})$ and $a_{\delta}(\boldsymbol{k}),\boldsymbol{k}\in\Gamma_{\varkappa,\delta}^{3}$, leave $\mathcal{F}_{\varkappa,\delta}^{\#}$ invariant and act irreducibly on $\mathcal{F}_{\varkappa,\delta}^{\#}$. We set now $\delta=2^{-\nu}$, $\varkappa=2^{\nu}$ and observe that $\mathcal{F}_{2^{\nu},2^{-\nu}}^{\#}$ increases monotonically with v and that

$$D_{0}^{\#\prime}=D_{0}^{\#}\cap\bigcup\nolimits_{v}\mathcal{F}_{2^{\nu},2^{-\nu}}^{\#}$$

is $\#$-dense in $\mathcal{F}^{\#}$ and $H_{\varkappa}\subset\#\text{-}\overline{(H_{\varkappa}\upharpoonright D_{0}^{\#\prime})}$. Thus it is sufficient to prove the semiboundedness of

$$H_{\varkappa,\delta}\upharpoonright\left(D\left(H_{0,\varkappa}\right)\cap D\left(N_{\varkappa}^{d/2}\right)\cap\mathcal{F}_{\varkappa,\delta}^{\#}\right)$$

with a lower bound independent of δ.

17. Diagonalizing the potential. In this subsection we give a new representation of $\mathcal{F}_{\varkappa,\delta}^{\#}$ in which the interaction term of $:\varphi_{\varkappa}^{\#p}(h):$ is a multiplication operator. Let

$$q\left(\boldsymbol{k}(|m|)\right)=\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_{\delta}\left(\boldsymbol{k}(m)\right)+a_{\delta}^{*}\left(\boldsymbol{k}(m)\right)+a_{\delta}\left(-\boldsymbol{k}(m)\right)+a_{\delta}^{*}\left(-\boldsymbol{k}(m)\right)\right],$$

$$q\left(\boldsymbol{k}(-|m|)\right)=i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[-a_{\delta}\left(\boldsymbol{k}(|m|)\right)+a_{\delta}^{*}\left(\boldsymbol{k}(|m|)\right)+a_{\delta}\left(-\boldsymbol{k}(|m|)\right)-a_{\delta}^{*}\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p\left(\boldsymbol{k}(|m|)\right)=i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_{\delta}\left(\boldsymbol{k}(m)\right)-a_{\delta}^{*}\left(\boldsymbol{k}(m)\right)+a_{\delta}\left(-\boldsymbol{k}(m)\right)-a_{\delta}^{*}\left(-\boldsymbol{k}(m)\right)\right],$$

$$p\left(\boldsymbol{k}(-|m|)\right)=\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{\boldsymbol{1/2}}\left[a_{\delta}\left(\boldsymbol{k}(|m|)\right)+a_{\delta}^{*}\left(\boldsymbol{k}(|m|)\right)-a_{\delta}\left(-\boldsymbol{k}(|m|)\right)-a_{\delta}^{*}\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p_{m}=p\left(\boldsymbol{k}(m)\right),\ q_{m}=q\left(\boldsymbol{k}(m)\right),$$

$$q\left(\boldsymbol{k}(m)\right)=\begin{cases}q\left(\boldsymbol{k}(|m|)\right)\text{ if }m>0,\\ q\left(-\boldsymbol{k}(|m|)\right)\text{ if }m<0,\end{cases}$$

for $0\neq\boldsymbol{k}\in\Gamma_{\delta}^{3}$ and let

$$q_{0}=(\mu_{0}/2)^{1/2}\left[a_{\delta}\left(\boldsymbol{0}\right)+\boldsymbol{a}_{\boldsymbol{\delta}}^{*}\left(\boldsymbol{0}\right)\right],$$

$$p_{0}=i(\mu_{0}/2)^{1/2}\left[a_{\delta}\left(\boldsymbol{0}\right)-\boldsymbol{a}_{\boldsymbol{\delta}}^{*}\left(\boldsymbol{0}\right)\right].$$

Using the equations mentioned above one can compute that

$$H_{0,\varkappa,\delta} = Ext\text{-}\sum_{m\in{}^*\mathbb{Z},|\boldsymbol{k}(m)\leq\varkappa|} 2^{-1}\left[p_m^2 + \mu^2(\boldsymbol{k}(m))q_m^2 - \mu(\boldsymbol{k}(m))\right]. \quad (17.1)$$

We replace now p_m and q_m by unitarily equivalent operators. Let

$$\mathcal{H}^{\#}_{\varkappa,\delta} = Ext\text{-}\otimes_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}}\mathcal{H}^{\#}_{\boldsymbol{k}},$$

where $\mathcal{H}^{\#}_{\boldsymbol{k}}$ is $L^{\#}_2\left({}^*\widetilde{\mathbb{R}}^{\#}_c\right)$ with respect to the Gaussian #-measure

$$\phi^2_{\boldsymbol{k}}(q)d^{\#}q = (\mu(\boldsymbol{k})/\pi_{\#})^{1/2}\left(Ext\text{-}\exp(-\mu(\boldsymbol{k})q^2)\right)d^{\#}q. \quad (17.2)$$

There is a unitary equivalence between $\mathcal{H}^{\#}_{\varkappa,\delta}$ and $\mathcal{F}^{\#}_{\varkappa,\delta}$ which sends q_m into multiplication by q in the factor $\mathcal{H}^{\#}_{\boldsymbol{k}(m)}$ and p_m into the operator

$$\phi^{-1}_{\boldsymbol{k}}(q)i\left(\frac{d^{\#}}{d^{\#}q}\right)\phi_{\boldsymbol{k}}(q)$$

again acting in the factor $\mathcal{H}^{\#}_{\boldsymbol{k}}$. The proof of this statement is essentially generalized von Neumann's uniqueness theorem for irreducible representations of the commutation relations. We identify $\mathcal{H}^{\#}_{\varkappa,\delta}$ and $\mathcal{F}^{\#}_{\varkappa,\delta}$ and we identify q_m, etc. with its image, multiplication by q, etc. Let

$$H_{\mu(\boldsymbol{k})} = 2^{-1}\phi^{-1}_{\boldsymbol{k}}(q)\left[-\left(\frac{d^{\#}}{d^{\#}q}\right)^2 + \mu(\boldsymbol{k})q^2\right]\phi_{\boldsymbol{k}}(q) = \quad (17.3)$$

$$= -2^{-1}\left(\frac{d^{\#}}{d^{\#}q}\right)^2 + \mu(\boldsymbol{k})q\left(\frac{d^{\#}}{d^{\#}q}\right)$$

acting on $\mathcal{H}^{\#}_{\boldsymbol{k}}$. Now $-H_{\mu(\boldsymbol{k})}$ is the #-infinitesimal generator of a known Markoff process and furthermore the operator $Ext\text{-}\exp\left(-H_{\mu(\boldsymbol{k})}\right)$ is an integral operator and the kernel can be computed explicitly. In particular

$$\left(Ext\text{-}\exp\left(-H_{\mu(\boldsymbol{k})}\right)\psi\right)(q) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#}_c} p^t(q,q')\psi(q')\phi^2_{\boldsymbol{k}}(q')d^{\#}q' \quad (17.4)$$

for $\psi\in\mathcal{H}^{\#}_{\boldsymbol{k}}$, where

$$p^t(q,q') = [1 - Ext\text{-}\exp(-\mu t)]\left\{Ext\text{-}\exp\left[-\frac{\mu\left(q'-(Ext\text{-}\exp(-\mu t))q\right)^2}{1-Ext\text{-}\exp(-2\mu t)}\right] + \mu q'^2\right\}. \quad (17.5)$$

Let q now denote a variable in a Euclidean space $E_{\varkappa}$ and let q have coordinates $q_m = q\left(\boldsymbol{k}(m)\right)$. Then

$$\phi^2_{\varkappa}(q)d^{\#3}q = Ext\text{-}\prod_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}}\phi^2_{\varkappa}\left(q(\boldsymbol{k})\right)d^{\#}q(\boldsymbol{k}) \quad (17.6)$$

is the product of the #-measures (17.2) and

$$\mathcal{H}_{\varkappa,\delta} = L^{\#}_2(\phi^2_{\varkappa}(q)d^{\#}q).$$

In addition to the function space $L^{\#}_2$, we will have to consider $L^{\#}_r(\phi^2_{\varkappa}(q)d^{\#}q)$. Since $Ext\text{-}\int\phi^2_{\varkappa}(q)d^{\#}q = 1$, we have $L^{\#}_{r_2}\subset L^{\#}_{r_1}$ if $r_1\leq r_2$. .

Lemma 17.1. $Ext\text{-}\exp\left(-H_{0,\varkappa,\delta}\right)$ is a contraction operator on $L^{\#}_r$, $1\leq r\leq{}^*\infty$. If $T\leq t, 1<p$ and $r<{}^*\infty$ it is a contraction from $L^{\#}_p$ to $L^{\#}_r$, for some T not depending on δ. If p is bounded away from one and r is bounded then T does not depend on p or r.

Now we show that the interaction term $:\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}):$ is a polynomial in the q' s. Let

$$\varphi^{\#}_{\varkappa,\delta}(\boldsymbol{x}) = \delta^{3/2}Ext\text{-}\sum_{\boldsymbol{k}(m)\in\Gamma^3_{\varkappa,\delta}} Ext\text{-}\exp(i\langle\boldsymbol{k}(m),\boldsymbol{x}\rangle)[\mu([\boldsymbol{k}(m)])]^{-1/2}\left(\boldsymbol{a}_{\boldsymbol{\delta}}\left(\boldsymbol{k}(m)\right) + a^*_{\delta}\left(-\boldsymbol{k}(m)\right)\right) \quad (17.7)$$

Since

$$[\mu([\boldsymbol{k}(m)\,])]^{-1/2}\left(\boldsymbol{a_\delta}\left(\boldsymbol{k}(m)\right)+a_{\delta}^{*}\left(-\boldsymbol{k}(m)\right)\right)=\begin{cases}q\left(\boldsymbol{k}(|m|)\right)+iq\left(-\boldsymbol{k}(|m|)\right)\text{ if } m>0\\ \sqrt{2}q_0 \text{ if } m=0\\ q\left(\boldsymbol{k}(|m|)\right)-iq\left(-\boldsymbol{k}(|m|)\right)\text{ if } m<0\end{cases}$$

$\varphi^{\#}_{\varkappa,\delta}(\boldsymbol{x})$ and $\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x})$ are polynomials in the q's. We use the canonical formula

$$\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x})=\sum_{j=0}^{[p/2]}\frac{p!(2^{-j})}{(p-2j)!j!}c^{j}_{\varkappa}:\varphi^{\#p-2j}_{\varkappa,\delta}(\boldsymbol{x}): \tag{17.8}$$

to conclude by induction on p that $:\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}):$ is also a polynomial in the q's. In (18.2.8) the coefficient

$$\frac{p!(2^{-j})}{(p-2j)!j!}$$

is just the number of ways of selecting j unordered pairs from p objects and $c_\varkappa$ is defined by the formula

$$c_\varkappa=\delta^3\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}}[\mu([\boldsymbol{k}\,])]^{-1/2}$$

we have the bound

$$c_\varkappa\le K_1\varkappa^2 \tag{17.9}$$

where K_1 is independent of $\varkappa$ and δ. Thus

$$:\varphi^{\#p}_{\varkappa,\delta}(h):\;=Ext\text{-}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}:\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}):h(\boldsymbol{x})\,d^{\#3}x$$

is a polynomial in the q' s, as desired.
Let

$$P(y)=b_0+b_1y+\cdots+b_dy^d$$

be the polynomial in y and let

$$V_{\varkappa,\delta}=\sum_{0\le p\le d}b_p:\varphi^{\#p}_{\varkappa,\delta}(h): \tag{17.10}$$

denote our approximate interaction term, as in (18.1.19).
Lemma 17.2. For some constant K_2 , independent of δ and $\varkappa$, we have

$$-K_2\varkappa^d\le V_{\varkappa,\delta}. \tag{17.11}$$

Proof We use (17.8) to remove the Wick ordering in (17.10) and obtain

$$V_{\varkappa,\delta}=\sum_p a_p(c_\varkappa)\left\{Ext\text{-}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}\int_{-\pi_\#/\delta}^{\pi_\#/\delta}:\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}):h(\boldsymbol{x})\,d^{\#3}x\right\}$$

where $a_p(c_\varkappa)$ is a polynomial in $c_\varkappa$ of degree at most $[(d-p)/2]$. The coefficients of a_p depend only on the coefficients of P, and so we have an estimate

$$\left|a_p(c_\varkappa)\right|\le K'\times c_\varkappa^{[(d-p)/2]}.$$

Since $a_d=b_d>0$ and since d is even by hypothesis, it follows that

$$0<\sum_p a_p(c_\varkappa)\,y^p \text{ for } K''(1+c_\varkappa)<|y|^2$$

and

$$-c_{\varkappa}^{(d/2]}K''' \leq \sum_p a_p(c_\varkappa)\, y^p$$

for all y. We bound $c_\varkappa$ by (17.9) and the proof is complete.
Lemma 17.3 Function $V_{\varkappa,\delta} \in$ for all $r < {}^*\infty$ and if $\lambda \leq \varkappa$, then

$$\left\|V_{\varkappa,\delta} - V_{\lambda,\delta}\right\|_{\#2j}^{2j} \leq (dj)! \times K_3^j \times (\varkappa^{2d} - \lambda^{2d})^j, \tag{17.12}$$

where K_3 is a constant which is independent of δ, λ and $\varkappa$.
Proof We use the particle representation, $\mathcal{F}_{\varkappa,\delta}^{\#}$, in place of the representation $\mathcal{H}_{\varkappa\delta} = L_2^{\#}(\phi_\varkappa^2(q)d^\# q)$. Now $1 \in \mathcal{H}_{\varkappa\delta}$ corresponds to the vacuum state $\Omega_0 = \{1,0,0,\ldots\} \in \mathcal{F}^{\#}$

so

$$\left\|V_{\varkappa,\delta} - V_{\lambda,\delta}\right\|_{\#2j}^{2j} = Ext\text{-}\int \left(V_{\varkappa,\delta} - V_{\lambda,\delta}\right)^{2j} \phi_\varkappa^2(q) d^\# q = \tag{17.13}$$

$$= \langle \left(V_{\varkappa,\delta} - V_{\lambda,\delta}\right)^j \Omega_0, \left(V_{\varkappa,\delta} - V_{\lambda,\delta}\right)^j \Omega_0 \rangle_\# = \left\|\left(V_{\varkappa,\delta} - V_{\lambda,\delta}\right)^j \Omega_0\right\|_\#^2.$$

We set $\lambda = 0$ above and get

$$\left\|V_{\varkappa,\delta}\right\|_{\#2j}^{2j} = \left\|V_{\varkappa,\delta}^{2j}\Omega_0\right\|_\#^2$$

and so $V_{\varkappa,\delta} \in L_r^{\#}$ for all $r < {}^*\infty$. We return to (17.13) and note that $V_{\varkappa,\delta} - V_{\lambda,\delta}$ is a sum of $d2^d$ terms of the form

$$A = b_p \delta^{3p/2} Ext\text{-}\sum_{\lambda \leq |\boldsymbol{k}_i| \leq \varkappa} \left[Ext\text{-}\hat{h}\left(\sum_{i=1}^p \boldsymbol{k}_i\right) a^\#(\pm \boldsymbol{k}_i) \prod_{i=1}^p \left[\mu(\boldsymbol{k}_i)\right]^{-1/2} \right] \tag{17.14}$$

where in the summation over $\boldsymbol{k}_i$ we have $\boldsymbol{k}_i \in \Gamma_{\varkappa,\delta}^3$ for $1 \leq p \leq d, p \leq d$, and $\boldsymbol{k}_i \notin \Gamma_{\lambda,\delta}^3$ for at least one i.
Summing again over the same range of $\boldsymbol{k}_i$ we get

$$\delta^{3p} Ext\text{-}\sum_{\lambda \leq |\boldsymbol{k}_i| \leq \varkappa} \left[Ext\text{-}\hat{h}\left(\sum_{i=1}^p \boldsymbol{k}_i\right) \prod_{i=1}^p \left[\mu(\boldsymbol{k}_i)\right]^{-1/2} \right] \leq K_4 \times (\varkappa^{2d} - \lambda^{2d}) \tag{17.15}$$

and K_4 is independent of $\lambda, \varkappa$ and δ.
Let ψ be a state with at most l particles. It follows from (17.15) and the form of A that

$$\|A\Omega_0\|_\#^2 \leq \left((l+p)!/l!\right) \times_4 \times (\varkappa^{2d} - \lambda^{2d})\, \|\psi\|_\#^2$$

and furthermore $A\psi$ is a state with at most $l + p$ particles. Thus if we have operators $A_1, \ldots, A_j$ of the form (17.14),

$$\left\|A_1 \cdots A_j \Omega_0\right\|_\#^2 \leq (dj)!\, K_4^j \lambda^{2dj}.$$

Hence

$$\left\|\left(V_{\varkappa,\delta} - V_{\lambda,\delta}\right)^j \Omega_0\right\|_\#^2 \leq (dj)! \times K_3^j \times (\varkappa^{2d} - \lambda^{2d})^j,$$

and the proof is complete.

18. Path space and corresponding #-measure

Let q now denote a variable in a Euclidean space $E_\varkappa^3 = {}^*\widetilde{\mathbb{R}}_c^{\#3}$ amd let $C^\#$ be the space of #-continuous paths

$q=q(s)\in E_{\varkappa}^{3}, 0\leq s<{}^{*}\infty$. There is a #-measure on $C^{\#}$ intrinsically associated with the semigroup $Ext\text{-}\exp(-tH_{0,\varkappa,\delta})$. To define this #-measure we set $q_{\boldsymbol{k}}=q(\boldsymbol{k})$ and

$$p_{\boldsymbol{k}}^{t}(q_{\boldsymbol{k}},q'_{\boldsymbol{k}})=\phi_{\boldsymbol{k}}^{2}(q'_{\boldsymbol{k}})d^{\#}q'_{\boldsymbol{k}}=\mathbf{Pr}\{q_{\boldsymbol{k}}(t)=q'_{\boldsymbol{k}}|q_{\boldsymbol{k}}(0)=q_{\boldsymbol{k}}\} \quad (18.1)$$

the probability that $q_{\boldsymbol{k}}(t)=q'_{\boldsymbol{k}}$ if it is known that $q_{\boldsymbol{k}}(0)=q_{\boldsymbol{k}}\cdot p_{\boldsymbol{k}}^{t}$ is defined by (17.5) we have added a subscript $\boldsymbol{k}$ to indicate the dependence on $\mu=\mu(\boldsymbol{k})$. Let

$$p_{\varkappa}^{t}(q\,,q')=Ext\text{-}\prod_{\boldsymbol{k}\in\Gamma_{\varkappa,\delta}^{3}}p_{\boldsymbol{k}}^{t}(q_{\boldsymbol{k}},q'_{\boldsymbol{k}}). \quad (18.2)$$

The $\sigma^{\#}$- field [19] of #-measurable subsets of $C^{\#}$ is generated by the sets

$$q(s_i)\in B_i, 1\leq i\leq j, \quad (18.3)$$

where B_i is a #-Borel subset of $E_{\varkappa}^{3}$. The #-measure of (18.3) is

$$Ext\text{-}\int_{B_j\times\cdots\times B_j}Ext\text{-}\prod_{i=2}^{j}p_{\varkappa}^{s_i-s_{i-1}}(q(s_i)\,,q(s_{i-1})\,)\,\phi_{\varkappa}^{2}(q(s_i))d^{\#}q(s_i)\phi_{\varkappa}^{2}(q(0))d^{\#}q(0) \quad (18.4)$$

if $s_1=0<s_2<\cdots<s_j$. The definition (18.4) is forced by the definition (18.1) together with the Markov character of the process, the stipulation that each coordinate $q_{\boldsymbol{k}}$ of q defines an independent process and the specification of $\phi_{\boldsymbol{k}}^{2}(q)d^{\#}q$ as the probability distribution of the initial point $q(0)$ of the path q.
If $V_1,\dots,V_j\in L_j^{\#}(E_{\varkappa}^{3},\phi_{\boldsymbol{k}}^{2}(q)d^{\#}Q)$ then we compute

$$Ext\text{-}\int Ext\text{-}\prod_i V_i(q(s_i))\,d^{\#}Q=Ext\text{-}\int V_1(q(0))\phi_{\varkappa}^{2}(q(0))d^{\#}q(0)\times \quad (18.5)$$

$$\times$$
$$[Ext\text{-}\exp(-(s_1-s_0)H_{0,\varkappa,\delta})V_2Ext\text{-}\exp(-(s_2-s_1)H_{0,\varkappa,\delta})(\dots(V_{j-1}Ext\text{-}\exp(-(s_i-s_{i-1})H_{0,\varkappa,\delta})V_j)\dots)](q(0))$$

and

$$\left|Ext\text{-}\int Ext\text{-}\prod_i V_i(q(s_i))\,d^{\#}Q\right|\leq Ext\text{-}\prod_i\|V_i\|_{\#j} \quad (18.6)$$

using (18.4) and the fact that $Ext\text{-}\exp(-H_{0,\varkappa,\delta}))$ is a contraction on $L_r^{\#}$. Furthermore (18.5) and (18.6) remain valid when some of the times s_i coincide.
Lemma 18.1 Let V be a polynomial function on $E_{\varkappa}^{3}$. Then $Ext\text{-}\int_0^t V(q(s))d^{\#}s\in L_p^{\#}(C^{\#},d^{\#}Q)$ for all $p<{}^{*}\infty$. and

$$\left\|Ext\text{-}\int_0^t V(q(s))d^{\#}s\right\|_{\#j}\leq t\|V_i\|_{\#j}$$

for $j\in{}^{*}\mathbb{N}$ an even positive integer.
Lemma 18.2 Let $r\in[1,2)$. There is a finite T independent of δ such that if $t\geq T$ and if ϕ and ψ in $L_2^{\#}\left(L_2^{\#}(\phi_{\varkappa}^{2}(q)d^{\#}q)\right)$ then $\phi(q(0))\psi(q(t))\in L_r^{\#}(C^{\#},d^{\#}Q)$ and

$$\|\phi(q(0))\psi(q(t))\|_{\#r}\leq\|\phi\|_{\#2}\times\|\psi\|_{\#2}.$$

The T can be chosen independently of r provided r is bounded away from 2.

19. The generalized Feynman Kac formula. The generalized Feynman Kac formula states that

$$\langle\phi,Ext\text{-}\exp(-tH_{\varkappa,\delta})\psi\rangle_{\#}=Ext\text{-}\int\phi(q(0))\left\{Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t V_{\varkappa,\delta}(q(s))\,d^{\#}s\right]\right)\right\}\psi(q(s))d^{\#}Q. \quad (19.1)$$

The RHS of (19.1) is bounded by

$$\left\|\phi\big(q(0)\big)\psi\big(q(s)\big)\right\|_{\#p'} \times \left\|Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t V_{\varkappa,\delta}\big(q(s)\big)\,d^{\#}s\right]\right)\right\|_{\#p} \leq$$

$$\leq \|\phi\|_{\#2} \times \|\psi\|_{\#2} \times \left\|Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t V_{\varkappa,\delta}\big(q(s)\big)\,d^{\#}s\right]\right)\right\|_{\#p}$$

for $p > 2$ and for t large, by Lemma 18..2. Thus

$$\left\|Ext\text{-}\exp\big(-tH_{\varkappa,\delta}\big)\right\|_{\#} \leq \left\|Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t V_{\varkappa,\delta}\big(q(s)\big)\,d^{\#}s\right]\right)\right\|_{\#p}$$

and therefore

$$t^{-1}\left\{Ext\text{-}\ln\left[\left\|Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t V_{\varkappa,\delta}\big(q(s)\big)\,d^{\#}s\right]\right)\right\|_{\#p}\right]\right\} \leq H_{\varkappa,\delta}. \quad (19.2)$$

Let

$$I_{\lambda} = Ext\text{-}\int_0^t V_{\varkappa,\delta}\big(q(s)\big)\,d^{\#}s.$$

Then by Lemma 17.2 we have

$$-tK_2\lambda^d \leq I_{\lambda}.$$

Let K_5 , . . . denote positive constants depending only on t and the polynomial P and let $\mathbf{Pr}\{q|\cdot\}$ denote the #-measure defined by $d^{\#}Q$.
Then by Lemma 18.1 we get

$$\mathbf{Pr}\{q|I_{\varkappa} \leq -tK_2\lambda^d - 1\} \leq \mathbf{Pr}\{q|[I_{\varkappa} - I_{\lambda}] \geq 1\} \leq Ext\text{-}\int [I_{\varkappa} - I_{\lambda}]^{2j}\,d^{\#}Q \leq t^{2j}\left\|I_{\varkappa,\delta} - I_{\lambda,\delta}\right\|_{\#2j}^{2j}. \quad (19.3)$$

From (19.3) by Lemma 17.3,see (17.12) we get

$$\mathbf{Pr}\{q|I_{\varkappa} \leq -tK_2\lambda^d - 1\} \leq [(dj)!] \times t^{2j} \times K_3^j \times (\varkappa^{2d} - \lambda^{2d})^j. \quad (19.3)$$

Lemma 19.1 Let f be ${}^*\mathbb{R}_c^{\#}$- valued function on a probability #-measure space $(M, \Sigma^{\#}, \mu^{\#})$, see ref.[19], and let $m_f(x) = \mu^{\#}\{q|f(q) \geq x\}$. Let $F\colon {}^*\mathbb{R}_c^{\#} \to {}^*\mathbb{R}_c^{\#}$ be a bounded positive, monotone non-decreasing $C^{\#1}$ function. Then

$$Ext\text{-}\int_M F\big(f(q)\big)d^{\#}\mu^{\#} = Ext\text{-}\int_{-{}^*\infty}^{{}^*\infty} F(x)d^{\#}\,m_f(x) = -F(-{}^*\infty) + Ext\text{-}\int_{-{}^*\infty}^{{}^*\infty} m_f(x)F^{\#'}(x)\,d^{\#}x. \quad (19.4)$$

In particular, by the generalized monotone convergence theorem:

$$Ext\text{-}\int_M \big[Ext\text{-}\exp\big(f(q)\big)\big]d^{\#}\mu^{\#} = Ext\text{-}\int_{-{}^*\infty}^{{}^*\infty} [Ext\text{-}\exp(x)]m_f(x)d^{\#}x. \quad (19.5)$$

From (18.4.3) we get

$$\mathbf{Pr}\{q|-I_{\varkappa} \geq K_2\lambda^d + 1\} \leq [(dj)!] \times t^{2j} \times K_3^j \times (\varkappa^{2d} - \lambda^{2d})^j. \quad (19.6)$$

From (19.6) and (19.5) finally we get

$$Ext\text{-}\int \big[Ext\text{-}\exp\big(-pI_{\varkappa}(q)\big)\big]d^{\#}Q \leq [(dj)!] \times t^{2j} \times K_3^j Ext\text{-}\int_0^{\varkappa} [Ext\text{-}\exp(x)](\varkappa^{2d} - x^{2d})^j d^{\#}x < {}^*\infty\,. \quad (19.7)$$

Thus $Ext\text{-}\int \big[Ext\text{-}\exp\big(-pI_{\varkappa}(q)\big)\big]d^{\#}Q$ is bounded independently of δ and combining this with (7.2) we have $H_{\varkappa,\delta}$ bounded below by a constant which is independent of δ; according to section 18.1this proves Theorem 19.1.

Theorem 19.1 Let h be a nonnegative function in $L_1^\# \cap L_2^\#$. Suppose that the polynomial P in ((18.2.10)) has even degree and that the leading coefficient is positive. Then Hamiltonian $H_\varkappa$ is bounded from below.

20. Alternate derivation without the use of functional integration

We will give an alternate derivation of the results mentioned in sections above without the use of functional integration, central in section 19. We consider a Hamiltonian of the form

$$H_\varkappa = H_{0.\varkappa} + V_\varkappa, \tag{20.1}$$

where $H_{0.\varkappa}$ is the free Hamiltonian of a particle of mass μ_0 expressed in terms of the neutral scalar field $\varphi_\varkappa^\#(\boldsymbol{x})$ and its momentum conjugate $\pi_\varkappa^\#(\boldsymbol{x})$:

$$H_{0.\varkappa} = Ext\text{-}\int_0^1 d^\# x_1 \left(Ext\text{-}\int_0^1 d^\# x_2 \left(Ext\text{-}\int_0^1 d^\# x_3 : [\nabla\varphi_\varkappa^{\#2}(\boldsymbol{x}) + \mu_0^2 \varphi_\varkappa^{\#2}(\boldsymbol{x}) + \pi_\varkappa^{\#2}(\boldsymbol{x})]: \right)\right) \tag{20.2}$$

As is evident from (17.2) we are working in a periodic box $B = [0,1]^3$. $V_\varkappa$ is a polynomial function of the $\varphi_\varkappa^\#(\boldsymbol{x})$. We denote by ${}^N H_{0.\varkappa}$ and ${}^N V_\varkappa, N \in {}^*\mathbb{N}\backslash\mathbb{N}$ the parts of $H_{0.\varkappa}$ and $V_\varkappa$ depending only on the creation and annihilation operators of the N lowest-energy modes of the free Hamiltonian and such that $|\boldsymbol{k}| \leq \varkappa$. We always imagine we are working with ${}^N H_{0.\varkappa}$ and ${}^N V_\varkappa$, but derive inequalities independent of N.

Theorem 20.1 Assume for each finite $\alpha > 0$ that there is an M_α such that

$$\langle 0| Ext\text{-}\exp\left(-\alpha({}^N V_\varkappa)\right)|0\rangle \leq M_\alpha,$$

where $|0\rangle$ denotes the vacuum of the free field. Then there is a B such that

$${}^N H_{0.\varkappa} + {}^N V_\varkappa \geq B, \text{ for all } N.$$

Actually as will be seen it is not necessary to satisfy the condition above for all α, but only for some sufficiently large α that one can calculate. We refer to section 18.3 for the result that the conditions of the theorem are satisfied for a large class of self-interactions.

We apply the notation

$$\varphi_{\varkappa,\delta}^\#(\boldsymbol{x}) = Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma_{\varkappa,\delta}^3} Ext\text{-}\exp(i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\left(\boldsymbol{a}_\delta(\boldsymbol{k}) + a_\delta^*(-\boldsymbol{k})\right) \tag{20.3}$$

and define for $\boldsymbol{k} \in \Gamma_{\varkappa,\delta}^3$

$$q_0 = (\mu_0/2)^{1/2}\left[a_\delta(\boldsymbol{0}) + \boldsymbol{a}_\delta^*(\boldsymbol{0})\right],\ p_0 = i(\mu_0/2)^{1/2}\left[a_\delta(\boldsymbol{0}) - \boldsymbol{a}_\delta^*(\boldsymbol{0})\right], \tag{20.4}$$

$$q\left(\boldsymbol{k}(|m|)\right) = \left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_\delta\left(\boldsymbol{k}(m)\right) + a_\delta^*\left(\boldsymbol{k}(m)\right) + a_\delta\left(-\boldsymbol{k}(m)\right) + a_\delta^*\left(-\boldsymbol{k}(m)\right)\right],$$

$$q\left(\boldsymbol{k}(-|m|)\right) = i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[-a_\delta\left(\boldsymbol{k}(|m|)\right) + a_\delta^*\left(\boldsymbol{k}(|m|)\right) + a_\delta\left(-\boldsymbol{k}(|m|)\right) - a_\delta^*\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p\left(\boldsymbol{k}(|m|)\right) = i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_\delta\left(\boldsymbol{k}(m)\right) - a_\delta^*\left(\boldsymbol{k}(m)\right) + a_\delta\left(-\boldsymbol{k}(m)\right) - a_\delta^*\left(-\boldsymbol{k}(m)\right)\right],$$

$$p\left(\boldsymbol{k}(-|m|)\right) = \left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{\boldsymbol{1/2}}\left[a_\delta\left(\boldsymbol{k}(|m|)\right) + a_\delta^*\left(\boldsymbol{k}(|m|)\right) - a_\delta\left(-\boldsymbol{k}(|m|)\right) - a_\delta^*\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p_m = p\left(\boldsymbol{k}(m)\right),\ q_m = q\left(\boldsymbol{k}(m)\right).\ \boldsymbol{q}\left(\boldsymbol{k}(m)\right) = \begin{cases}\boldsymbol{q}\left(\boldsymbol{k}(|m|)\right) \textbf{ if } m > 0,\\ \boldsymbol{q}\left(-\boldsymbol{k}(|m|)\right) \textbf{ if } m < 0.\end{cases}$$

In terms of these variables,

$$H_{0,\varkappa,\delta} = Ext\text{-}\sum_{m\in{}^*\mathbb{Z},|k(m)\leq\varkappa|} 2^{-1}\left[p_m^2 + \mu^2(k(m))q_m^2 - \mu(k(m))\right] = Ext\text{-}\sum_{m\in{}^*\mathbb{Z},|k(m)\leq\varkappa|} H_m. \quad (20.5)$$

We represent these operators on the $L_2^{\#}$ space of ${}^*\mathbb{R}_c^{\#N}$ with #-measure μ the product of the #-measures μ_m

$$d^{\#}\mu_m = (\omega_m/\pi_{\#})^{1/2}\left(Ext\text{-}\exp(-\omega_m q_m^2)\right)d^{\#}q_m \quad (20.6)$$

with q_m a multiplicative operator and

$$p_m = i(\partial^{\#}/\partial^{\#}q_m) - \omega_m q_m. \quad (20.7)$$

Where

$$\omega_m = (\boldsymbol{k}^2(m) + \mu_0^2)^{1/2} = (k_1^2(m) + k_2^2(m) + k_3^2(m) + \mu_0^2)^{1/2}.$$

A complete set of eigenfunctions for H_m is given by

$$\phi_{m,n}(q_m) = (2^n n!^{\#})^{-1/2} A_n\left(q_m(\omega_m)^{1/2}\right), n \in {}^*\mathbb{N}, \quad (20.8)$$

$$n!^{\#} = Ext\text{-}\prod_{0<p\leq n} p,\ 2^n = Ext\text{-}\prod_{0<p\leq n} 2,$$

$$A_n(z) = (-1)^n\left(Ext\text{-}\exp(z^2)\right)\frac{d^{\#n}}{d^{\#}z^n}\left(Ext\text{-}\exp(-z^2)\right).$$

The chief inequality we will exploit is the following numerical inequality for $x, y \in {}^*\mathbb{R}_c^{\#}$, $y \geq 0$:

$$xy \leq Ext\text{-}\exp(x) + Ext\text{-}\ln(y). \quad (20.9)$$

The expectation value of the interaction $V_\varkappa$ in a state with ${}^*\mathbb{C}_c^{\#}$-function F is given by

$$\langle F|V_\varkappa|F\rangle = Ext\text{-}\int(|F|^2 V_\varkappa)d^{\#}\mu\,. \quad (20.10)$$

We apply (18.3.10) with $x = rV_\varkappa$ and $y = r^{-1}F^2$ to derive the inequality

$$-\langle F|V_\varkappa|F\rangle \leq Ext\text{-}\int\left(Ext\text{-}\exp(-rV_\varkappa)\right)d^{\#}\mu + \frac{1}{r}\left[Ext\text{-}\int|F|^2\left(Ext\text{-}\ln(|F|^2)\right)d^{\#}\mu\right] - \frac{1}{r}\left(Ext\text{-}\ln(r)\right). \quad (20.11)$$

Here r is a numerical factor to be fixed later. Note that

$$Ext\text{-}\int\left(Ext\text{-}\exp(-rV_\varkappa)\right)d^{\#}\mu = \langle 0|Ext\text{-}\exp(-r\,V_\varkappa)|0\rangle. \quad (20.12)$$

We intend now to bound the second term on the right side of (20.11) by the expectation value of $H_{0.\varkappa}$ in the state F. We consider the following equation:

$$\left[Ext\text{-}\int|F|^2\left(Ext\text{-}\ln(|F|^2)\right)d^{\#}\mu\right] = \quad (20.13)$$

$$= \frac{2}{\lambda}\left(Ext\text{-}\int F^* H_{0.\varkappa}F\,d^{\#}\mu\right) + \frac{1}{\lambda}\frac{d^{\#}}{d^{\#}t}\left(Ext\text{-}\int\left[\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)^*\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)\right]^{1+\lambda t}d^{\#}\mu\right)\Big|_{t=0},$$

which easily follows for functions F nice enough so that all the integrals exist and the differentiation may be moved inside the integral, a dense subspace in $L_2^{\#}$. We do not discuss domain questions. We rewrite (20.11) using (20.13):

$$-\langle F|V_\varkappa|F\rangle \leq Ext\text{-}\int\left(Ext\text{-}\exp(-rV_\varkappa)\right)d^{\#}\mu + \frac{2}{\lambda r}\langle F\left|H_{0.\varkappa}\right|F\rangle - \frac{1}{r}\left(Ext\text{-}\ln(r)\right) + \quad (20.14)$$

$$+\frac{1}{\lambda r}\frac{d^{\#}}{d^{\#}t}\left(Ext\text{-}\int\left[\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)^*\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)\right]^{1+\lambda t}d^{\#}\mu\right)\Big|_{t=0}.$$

The theorem we are after is established provided $\lambda r \geq 2$ and we can bound the last term in (20.14). The remainder of the paper is devoted to a study of

$$Ext\text{-}\int\left[\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)^{*}\left(Ext\text{-}\exp(-tH_{0.\varkappa})\right)\right]^{1+\lambda t}d^{\#}\mu = Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^{\#}\mu. \quad (20.15)$$

We consider, corresponding to any g in $L_2^{\#}(\mu)$, its expression as a sum of products of the functions in (20.8):

$$g(q) = Ext\text{-}\sum_{i_{1\cdot,\cdot\cdot,}i_N} C_{i_{1\cdot,\cdot\cdot,}i_N}\left\{Ext\text{-}\prod_s(2^{i_s}i_s!^{\#})^{-1}\left(Ext\text{-}\exp(i_s)A_{i_s}\left(q_s(\omega_s)^{1/2}\right)\right)\right\}. \quad (20.16)$$

The q_s are merely the q_k in some order. The coefficient's $C_{i_{1\cdot,\cdot\cdot,}i_N}$ are now considered as functions on the discrete space whose points are the indices of the C's. To the point $(i_{1\cdot,\cdot\cdot,}i_N)$ is associated the point mass $Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)$. With this measure, the transformation T that carries a set of C's into the corresponding function g as in (20.16) is norm preserving as a map from $l_2^{\#}$ to $L_2^{\#}$. We will later show that T is norm decreasing as a map from $l_1^{\#}$ to $L_4^{\#}$. Assuming this for a moment, we complete the proof of the theorem. We apply the generalized Riesz-Thorin convexity theorem to the transformation T obtaining

$$Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^{\#}\mu \leq$$

$$\leq\left[Ext\text{-}\left(\sum_{i_{1\cdot,\cdot\cdot,}i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\times\left|\left(Ext\text{-}\exp\left(-\omega_{i_{1\cdot,\cdot\cdot,}i_N}t\right)\right)\times C_{i_{1\cdot,\cdot\cdot,}i_N}\right|^{\frac{2(1+\lambda t)}{1+3\lambda t}}\right)\right]^{\frac{1+3\lambda t}{2(1+\lambda t)}} \quad (20.17)$$

with

$$\omega_{i_{1\cdot,\cdot\cdot,}i_N} = Ext\text{-}\sum_s i_s\omega_s. \quad (20.18)$$

In the right-hand side of (20.17) we apply the generalized Holder inequality to obtain an expression involving the weighted sum of the squares of the absolute values of the C's which is equal to one:

$$Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^{\#}\mu \leq$$

$$\leq\left[Ext\text{-}\left(\sum_{i_{1\cdot,\cdot\cdot,}i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\left(Ext\text{-}\exp\left(-\omega_{i_{1\cdot,\cdot\cdot,}i_N}\times\frac{2(1+\lambda t)}{2\lambda}\right)\right)\right)\right]^{2\lambda t}. \quad (20.19)$$

It follows that

$$\frac{d^{\#}}{d^{\#}t}\left(Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^{\#}\mu\right)\Big|_{t=0} \leq$$

$$2\lambda\times\left\{Ext\text{-}\ln\left[Ext\text{-}\left(\sum_{i_{1\cdot,\cdot\cdot,}i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\left(Ext\text{-}\exp\left(-\omega_{i_{1\cdot,\cdot\cdot,}i_N}\times\frac{2(1+\lambda t)}{2\lambda}\right)\right)\right)\right]\right\}. \quad (20.20)$$

If $\mu_0/\lambda > 2$, this gives an inequality with finite right hand side in the #-limit $N \to {}^*\infty$. It is clear that the theorem is now reduced to establishing that T is #-norm decreasing from $l_1^{\#}$ to $L_4^{\#}$.

Lemma 20.1. Let S be the space of sequences $\{C_\gamma\}$, $\gamma = 0, 1, \ldots, N$ with #-measure at γ, $Ext\text{-}\exp(2\gamma)$; and Y the space of functions on ${}^*\mathbb{R}_c^{\#}$ with #-measure

$$(1/\pi_{\#})^{1/2}\left(Ext\text{-}\exp(-x^2)\right)d^{\#}x, \quad (20.21)$$

and T the operator from S to Y given by

$$T\{C_\gamma\} = Ext\text{-}\sum_\gamma C_\gamma\frac{\left(Ext\text{-}\exp(\gamma)\right)A_\gamma(x)}{[2^\gamma\gamma!^{\#}]} \quad (20.22)$$

with $A_\gamma(x)$ the γ-th Hermite polynomial $A_\gamma(x) = (-1)^n\left(Ext\text{-}\exp(x^2)\right)\frac{d^{\#n}}{d^\# x^n}\left(Ext\text{-}\exp(-x^2)\right)$; then, T is #-norm decreasing from $l_1^\#$ to $L_4^\#$.
It is easy to see that this lemma would follow from establishing the inequality

$$\left|\left(\frac{1}{\pi_\#}\right)^{\frac{1}{2}}\left(Ext\text{-}\exp(-a-b-c-d)\right)\times\right.$$

$$\left.\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[2^{a+b+c+d}(a!^\#)(b!^\#)(c!^\#)(d!^\#)]^{-1/2}A_a(x)A_b(x)A_c(x)A_d(x)\left(Ext\text{-}\exp(-x^2)\right)d^\# x\right)\right| \leq 1_{\widetilde{{}^*\mathbb{R}_c^\#}} \quad (20.23)$$

for all integers $a, b, c \in {}^*\mathbb{N}$ and $d \geq 0$; actually, it is sufficient to let $a = b = c = d$. We use the generating function

$$Ext\text{-}\exp(-t^2 + 2tZ) = Ext\text{-}\sum_{N\in{}^*\mathbb{N}}\frac{t^N}{N!^\#}A_N(Z) \quad (20.24)$$

to obtain

$$\left(\frac{1}{\pi_\#}\right)^{\frac{1}{2}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}\left(Ext\text{-}\exp(-x^2)\right)A_a(x)A_b(x)A_c(x)A_d(x)d^\# x\right) = \quad (20.25)$$

$$= \frac{(a!^\#)(b!^\#)(c!^\#)(d!^\#)}{\frac{1_{\widetilde{{}^*\mathbb{R}_c^\#}}}{2_{\widetilde{{}^*\mathbb{R}_c^\#}}}(a+b+c+d)!^\#}\times 2_{\widetilde{{}^*\mathbb{R}_c^\#}}{}^{\frac{1}{2}(a+b+c+d)}\times\ (rs+rt+ru+st+su+tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}}$$

where pick-a-power means to find the coefficient of the monomial $r^a s^b t^c u^q$ in the expansion of the expression. Note that $a + b + c + d$ is even or the integral vanishes. We make the crude estimate

$$(rs+rt+ru+st+su+tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}} \leq \quad (20.26)$$

$$\leq 2_{\widetilde{{}^*\mathbb{R}_c^\#}}{}^{-\frac{1}{2}(a+b+c+d)}\times(r+s+t+u)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}}$$

Now we get

$$(r+s+t+u)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}} = \frac{(a+b+c+d)!^\#}{(a!^\#)(b!^\#)(c!^\#)(d!^\#)}. \quad (20.27)$$

Denoting the left-hand side of (20.23) by $\mathfrak{I}$ and using (20.26) we obtain

$$\mathfrak{I} \leq \left(Ext\text{-}\exp(-a-b-c-d)\right)\times\frac{[(a+b+c+d)!^\#]\times 2^{-\frac{1}{2}(a+b+c+d)}}{[(a!^\#)(b!^\#)(c!^\#)(d!^\#)]^{1/2}\left[\frac{1}{2}(a+b+c+d)\right]!^\#}. \quad (20.28)$$

It is easily verify that

$$\mathfrak{I} \leq 1_{\widetilde{{}^*\mathbb{R}_c^\#}}. \quad (20.29)$$

The inequality (20.29) finalized the proof of theorem.

21. Strong #-convergence of operators
In this section we study the sum $A + B$ of two #-selfadjoint operators on a non-Archimedean Banach spaces over field $\widetilde{\mathbb{C}}_c^\#$, and we find sufficient conditions for $C = A + B$ to be #-selfadjoint. Our technique is to approximate B by a hyper infinite sequence of bounded in $\widetilde{\mathbb{R}}_c^\#$ #-selfadjoint operators $\{B_n\}, n \in {}^*\mathbb{N}$ and so to approximate C by #-selfadjoint operators $C_n = A + B_n$.We answer these three questions separately:1.When do the operators C_n have a #-lim C? 2.When is C a #-selfadjoint operator? 3.When is $C = A + B$? In Theorem 21.8

we give a set of estimates on the relative size of A and B which ensure a positive answer to all three questions. Hence these estimates show that $A+B=C$ is #-selfadjoint. In another paper [17], we used Theorem 21.8 to prove the existence of a self-interacting, causal quantum field in 4-dimensional space-time. Formally this field theory is Lorentz covariant and has non-trivial scattering; this application was the motivation for the present results. In order to investigate the meaning of $\#\text{-}\lim_{n\to {}^*\infty} C_n$, we give a new definition for the strong #-convergence of a hyper infinite sequence of operators. Consequences of this definition are worked out in this section below and we give also estimates on operators C_n which are sufficient to ensure that the $\#\text{-}\lim_{n\to {}^*\infty} C_n = C$ exists and that operator C is maximal symmetric or #-selfadjoint. This result is given in Theorem 21.5 and Corollary 21.6. We investigate also whether $\#\text{-}\lim_{n\to {}^*\infty} C_n = C$ is equal to $A+B$. We combine this work in Theorem 21.8, our second main theorem, where B is a singular, but nearly positive #-selfadjoint perturbation of a positive #-selfadjoint operator A. To illustrate this theorem, let $A \geq I$ and let B be essentially #-selfadjoint on domain $D^{\#} = \cap_{n\in {}^*\mathbb{N}} D(A^n)$. Assume now that, for some $\beta > 0$ and some α,

$$A^{-(1-\beta)}\mathrm{B}A^{-(1-\beta)} \text{ and } A^{\beta}BA^{\alpha} \tag{21.1}$$

are #-densely defined, bounded in $\widetilde{\mathbb{R}}_c^{\#}$ operators. Also, for some positive $a, \varepsilon \in \widetilde{\mathbb{R}}_{c+}^{\#}$satisfying: $2a+\varepsilon<1$, suppose that there is a constant $b \in \widetilde{\mathbb{R}}_c^{\#}$ such that, as bilinear forms on $D^{\#}\times D^{\#}$,

$$0 \leq aA + B + b, \tag{21.2}$$

$$0 \leq \varepsilon A^2 + [A^{1/2},[A^{1/2},B]] + b. \tag{21.3}$$

Then $A+B$ is # -selfadjoint. We see from this example that neither the operator B nor the bilinear form B need be bounded relative to A. While it may not appear evident, the conditions (21.1)-(21.3) are closely related to a more easily understandable estimate on $D^{\#}\times D^{\#}$,

$$A^2 + B^2 c(A+B)^2 + c. \tag{21.4}$$

In fact, estimates (21.1)-(21.3) are chosen because they allow us not only to prove (1.4), but also the similar inequality where B is replaced by B_n. Let us now see that if $A+B$ is #-selfadjoint, then (1.4) must hold for every vector in $D(A+B) = D(A)\cap D(B)$.

Proposition 21.1 Let A and B be #-closed operators. Then $A+B$ is #-closed if and only if there is a constant $c \in \widetilde{\mathbb{R}}_c^{\#}$ such that for all $\psi \in D(A+B)$

$$\|A\psi\|_{\#} + \|B\psi\|_{\#} \leq \|(A+B)\psi\|_{\#} + c\|\psi\|_{\#} \tag{21.5}$$

and (21.5) is equivalent to (21.4) on $D(A+B)\times D(A+B)$.

Proof: Certainly (21.5) implies that $A+B$ is #-closed. Conversely, assume that $A+B$ is #-closed and introduce the #-norms on $D(A+B) = D(A)\cap D(B)$,

$$\|\psi\|_{\#1} \triangleq \|\psi\|_{\#} + \|A\psi\|_{\#} + \|B\psi\|_{\#}, \tag{21.6}$$

$$\|\psi\|_{\#2} \triangleq \|\psi\|_{\#} + \|(A+B)\psi\|_{\#} \tag{21.7}$$

Then $D(A+B), \|\psi\|_{\#2}$ is a non-Archimedean Banach space because $A+B$ is #-closed. The identity map from $D(A+B), \|\psi\|_{\#2}$ to $D(A+B), \|\psi\|_{\#1}$ has a #-closed graph because A, B, and $A+B$ are #-closed. By the #-closed graph theorem, the identity map is #-continuous; hence

$$\|\psi\|_{\#1} \leq c\,\|\psi\|_{\#2}. \tag{21.8}$$

Proposition 21.2 Let $A \geq I, B$ be #-selfadjoint operators with $D^{\#} \subset D(B)$ and suppose that (21.2) and (21.3) hold. Then (21.4) is valid on domain $D^{\#}\times D^{\#}$.

Proof The operators A^2, B^2, AB, BA, and $A^{1/2}BA^{1/2}$ define bilinear forms on $D^{\#}\times D^{\#}$. Using (21.2) and (21.3), we have the inequality:

$$A^2+B^2=(A+B)^2-2A^{1/2}BA^{1/2}-[A^{1/2},[A^{1/2},B]]\leq(A+B)^2+(2a+\varepsilon)A^2+2Ab+b,$$

which establishes (21.4).

Let $\mathscr{G}$(C) be the graph of the operator C. For any hyper infinite sequence $\{C_n\}, n\in {}^*\mathbb{N}$ of #-densely defined operators we define

$$\mathcal{L}_{{}^*\infty}(C)=\{\varphi,\chi|\varphi=\#\text{-}\lim_{n\to{}^*\infty}\varphi_n\in D(C_n),\chi=\#\text{-}\lim_{n\to{}^*\infty}C_n\varphi_n\}. \quad (21.9)$$

In general, $\mathcal{L}_{{}^*\infty}(C)$ will not be the graph of an operator. If the hyper infinite sequence $\{C_n\}, n\in {}^*\mathbb{N}$ #-converges strongly on a #-dense domain D to an operator C^*, namely, $C^*\psi=\#\text{-}\lim_{n\to{}^*\infty}C_n^*\psi, \psi\in D$,then $\mathcal{L}_{{}^*\infty}$ is the graph of some operator C^*. In particular, if each C_n is self #-adjoint, and if the C_n #-converge on a #-dense set D to an operator C is defined on D,then $\mathcal{L}_{{}^*\infty}=\mathcal{L}_{{}^*\infty}(C_{{}^*\infty})$ and $C_{{}^*\infty}$ is a symmetric extension of C.

Definition 21.1 G-#-CONVERGENCE. The hyper infinite sequence of operators $\{C_n\}, n\in {}^*\mathbb{N}$ #-converge strongly to $C_{{}^*\infty}$in the sense of graphs, written

$$C_n\to_{\#G}C_{{}^*\infty} \quad (21.10)$$

if $\mathcal{L}_{{}^*\infty}$ is the graph of a #-densely defined operator $C_{{}^*\infty}$.

Remark 21.1 Note that for a hyper infinite sequence of uniformly bounded in $\widetilde{\mathbb{R}}_c^\#$ operators $C_n^*, n\in {}^*\mathbb{N}$ such that $C_n\to_{\#G}C_{{}^*\infty}$, $C_{{}^*\infty}$ is the usual strong #-limit of the operators $\{C_n\}, n\in {}^*\mathbb{N}$ and is everywhere defined.

Definition 21.2.R -#-CONVERGENCE. Let the resolvents $R_n(z)=(C_n-z)^{-1}, n\in {}^*\mathbb{N}$ exist for some $z\in\widetilde{\mathbb{R}}_c^\#$, and be uniformly hyper bounded in n. The operators C_n #-converge strongly to $C_{{}^*\infty}$in the sense of resolvents, written az

$$C_n\to_{\#R}C_{{}^*\infty} \quad (21.11)$$

if the resolvents $R_n(z)$ #-converge strongly to an operator R(z), which has a #-densely defined inverse.

Remark 21.2 Note that in that case, the operator $C_{{}^*\infty}=R^{-1}(z)+z$ exists for all $z\in\widetilde{\mathbb{C}}_c^\#$, for which the strong #-limit of the $R_n(z)$ exists, and $R^{-1}(z)+z$ is independent of z.

Remark 21.3 Note that G -#-convergence is weaker than R -#-convergence, in the case $C_n=C_n^*$ at least, because, as we shall show, in this case $C_n\to_{\#R}C_{{}^*\infty}$ implies $C_n\to_{\#G}C_{{}^*\infty}$. It seems likely that G -#-convergence is strictly weaker than R -#-convergence; this could be established by giving an example for which $C_n^*=C_n\to_{\#G}C_{{}^*\infty}$ with $C_{{}^*\infty}$ not maximal symmetric. The importance of G -#-convergence is that it is technically easier to verify-and gives less information about the #-limit-than R -#-convergence, while automatically selecting the correct domain in the case that R -#-convergence also holds. The most familiar examples of G -#-convergence occur where there is $C_n$strong #-convergence on a #-dense domain.

A less trivial example occurs where there is $D(C_n)$ is independent of $n\in {}^*\mathbb{N}$, but apparently $D(C)\cap D(C_n)=\{0\}$. We have the following connection between G and R -#-convergence for a hyper infinite sequence of #-selfadjoint operators.

Proposition 21.3 Let $C_n\in {}^*\mathbb{N}$ be #-selfadjoint.

(a) The domain $D_{{}^*\infty}=\{\varphi|\{\varphi,\chi\}\in\mathcal{L}_{{}^*\infty}$ for some $\chi\}$ is #-dense in H and only if $C_n\to_{\#G}C_{{}^*\infty}$, and in this case $C_{{}^*\infty}$ is necessarily symmetric.

(b) If $R_n(z)=(C_n-z)^{-1}, n\in {}^*\mathbb{N}$ #-converges to a bounded in $\widetilde{\mathbb{R}}_c^\#$ operator $R(z)$ for an unbounded set of z's with $\|zR(z)\|_\#$ bounded uniformly in $z\in\widetilde{\mathbb{C}}_c^\#$, and $n\in {}^*\mathbb{N}$ and if $C_n\to_{\#G}C_{{}^*\infty}$, ,then each $R(z)$ is invertible.

(c) If $R_n(z)$ #-converges to an invertible $R(z)$, then $C_n\to_{\#R}C_{{}^*\infty}$.

(d) If $C_n\to_{\#R}C_{{}^*\infty}$.,then $C_n\to_{\#G}C_{{}^*\infty}$,, $\mathcal{L}_{{}^*\infty}=\mathcal{L}(C)$, and C is maximal symmetric.

(e) Conversely, if $C_n\to_{\#G}C_{{}^*\infty}$, where C is maximal symmetric, then $C_n\to_{\#R}C_{{}^*\infty}$.

Proof. (a) Suppose that $D_{{}^*\infty}$ is #-dense and let $\{0,\psi\}\in\mathcal{L}_{{}^*\infty}$. Then, for some θ_n in each $D(C_n)$, $\theta_n\to_\# 0$, $C_n\theta_n\to_\#\psi$.

Let $\varphi \in \mathcal{L}_{\,^*\infty}$, φ=#-$\lim_{n\to\,^*\infty} \varphi_n$, χ =#-$\lim_{n\to\,^*\infty} C_n\varphi_n$.
Then, $0 = \langle 0,\psi\rangle_\# =$ #-$\lim_{n\to\,^*\infty} \langle\theta_n, C_n\varphi_n\rangle_\# =$ #-$\lim_{n\to\,^*\infty} \langle C_n\theta_n, \varphi_n\rangle_\# = \langle\psi,\varphi\rangle_\#$, so $\psi \in D^{\perp}_{\,^*\infty}$ and $\psi = 0$. Thus $\mathcal{L}_{\,^*\infty}$ is the graph of an operator C with domain $D_{\,^*\infty}$, and since $D_{\,^*\infty}$ #-dense by hypothesis, $C_n \to_{\#G} C_{\,^*\infty}$.
Conversely, if $C_n \to_{\#G} C_{\,^*\infty}$, we have $D_{\,^*\infty} = D(C_{\,^*\infty},)$ and, since C, is assumed to be #-densely defined, $D_{\,^*\infty}$ is #-dense. The expectation values $\langle\varphi, C_{\,^*\infty}\varphi\rangle_\# =$ #-$\lim_{n\to\,^*\infty} \langle\varphi_n, C_n\varphi_n\rangle_\#$ are real and so $C_{\,^*\infty}$, is symmetric.
(b) Let z_1 and z_2 be any two z's for which #-$\lim_{n\to\,^*\infty} R_n(z)$ exists. Then $R(z)$ satisfies the resolvent equation $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z)$ from which it follows that $R(z)$ has a range and a null space which are independent of z. If the null space is zero, $R(z)$ is invertible, and it is sufficient to show that, for large $|z|$, $\|zR(z)\psi + \psi\|_\# < \|\psi\|_\#$. Since $D_{\,^*\infty}$ is #-dense, we choose $\varphi \in D_{\,^*\infty}$ so that $\|\psi - \varphi\|_\# < \varepsilon \approx 0$.
Then,$\|zR(z)\psi + \psi\|_\# \leq \|zR(z)(\psi - \varphi)\|_\# + \|zR(z)\varphi + \varphi\|_\# + \|\varphi - \psi\|_\#$. For the z under consideration, $\|zR_n(z)\|_\#$ and $\|zR(z)\|_\#$ are uniformly bounded in $\widetilde{\mathbb{R}}^\#_c$ by a constant $M \in \widetilde{\mathbb{R}}^\#_{c+}$. Thus, $\|zR(z)\psi + \psi\|_\# \leq (M+1)\varepsilon + \|zR(z)\varphi + \varphi\|_\#$. Since $\{\varphi,\chi\} \in \mathcal{L}_{\,^*\infty}$,there exists $\varphi_n \in D(C_n)$ such that $\|\varphi_n - \varphi\|_\# \to_\# 0$ and $\|C_n\varphi_n - \chi\|_\# \to_\# 0$.Thus ‖C_{n}φ_{n}‖_{} is uniformly bounded in n∈¦^{*}ℕ¦.Thus

$$\|zR(z)\varphi - \varphi\|_\# \leq \|z(R(z) - R_n(z))\varphi\|_\# + \|zR_n(z)(\varphi_n - \varphi)\|_\# + \|zR_n(z)\varphi_n + \varphi_n\|_\# +$$

$$+\|\varphi_n - \varphi\|_\# \leq |z|\|(R(z) - R_n(z))\varphi\|_\# + (M+1)\,\|\varphi - \varphi_n\|_\# + \|(C_n - z)^{-1}C_n\varphi_n\|_\#.$$

We can choose $|z|$ sufficiently infinite large so that $\|(C - z)^{-1}\|_\# \leq M\,|z|$ is infinite small, then the last term above is small, uniformly in $n \in {}^*\mathbb{N}$. With this fixed value of z, we choose n large enough so that the first two terms are infinite small, and we conclude that $\|zR(z)\varphi + \varphi\|_\#$ is arbitrarily infinite small for infinite large |z| and that the null space of $R(z)$ is zero.
(c) In order to show that $C_n \to_{\#R} C_{\,^*\infty}$, we need only show that $C = R(z)^{-1} + z$ is #-densely defined. We show that $\{\text{range}R(z)\}^\perp \subset \text{null}R(z)$ which implies (c). To prove the inclusion we may suppose that $y = \text{Im}(z) \neq 0$,because $R_n(z) + R(z)$ for z in an #-open subset of the complex plane $\widetilde{\mathbb{C}}^\#_c$. If $\psi^\perp\{\text{range}R(z)\}$ and if $\notin$ null$R(z)$, then $\|\psi\|_\# < \|iyR(z)\psi + \psi\|_\# =$ #-$\lim_{n\to\,^*\infty} \|iyR_n(z)\psi + \psi\|_\# \leq \|\psi\|_\#$, which is a contradiction.
(d) Let $C_n \to_{\#R} C_{\,^*\infty}$, and let $\{\varphi,\theta\} \in \mathcal{L}(C)$.Then, for some $\chi \in H^\#$ and some z, $\varphi = R(z)\chi =$ #-$\mathit{lim}_{n\to\,^*\infty} \varphi_n =$ #-$\lim_{n\to\,^*\infty} R_n(z)\chi$ and $C_n\varphi_n$ =(C_n-z) φ_n+zφ_nχ+$\varphi z_n \to_\#$χ+$z\varphi = C\varphi$.Thus, $L(C) \subset \mathscr{L}(C_{\,^*\infty})$ so $D(C) \subset D_{\,^*\infty}$and since $D(C)$ is #-dense by assumption, $D_{\,^*\infty}$ is #-dense also. By (a), $C_n \to_{\#G} C_{\,^*\infty}$ and $C_{\,^*\infty}$. is a symmetric extension of C. However, $C_{\,^*\infty}$ - is #-closed and it has a resolvent $R(z)$; therefore, C has defect indices $(0,n)$ and is maximal symmetric. Thus, $C_{\,^*\infty}. = C$. (e) Suppose that $\mathcal{L}_{\,^*\infty}, = \mathscr{L}(C_{\,^*\infty})$ is the graph of a maximal symmetric operator. For non-real z we have $\|R_n(z)\|_\#$bounded uniformly in $n \in {}^*\mathbb{N}$. Let $\chi \in H^\#$,and write χ in the form $\chi = (C_{\,^*\infty} \pm i)\psi$ for some $\psi \in D(C_{\,^*\infty})$. Then, $\psi =$ #-$\lim_{n\to\,^*\infty} \psi_n$,$C_{\,^*\infty}$ψ= #-$\lim_{n\to\,^*\infty} C_n\psi_n$ and
$\chi =$ #-$\lim_{n\to\,^*\infty} (C_{\,^*\infty}.\pm\mathrm{i})\psi_n =$ #-$\lim_{n\to\,^*\infty} \psi_n$. Therefore
#-$\lim_{n\to\,^*\infty} R_n(\mp\mathrm{i})$χ= #-$\lim_{n\to\,^*\infty} R_n(\mp\mathrm{i})\chi_n = \psi = \left(C_{\,^*\infty} \pm i\right)^{-1}$and so $C_n \to_{\#R} C_{\,^*\infty}$
Remark 21.4 In case the #-limit of the $C_n, n \in {}^*\mathbb{N}$ is actually #-selfadjoint, there are further connections between G and R #-convergence.
Theorem 21.4 Let C_n be #-selfadjoint. The follouiing conditions are equivalent; (a) $C_n \to_{\#G} C$, and $C = C^*$. (b) $C_n \to_{\#R} C$, and $C = C^*$. (c) The hyper infinite sequences $R_n(z)$ and $[R_n(z)]^*$, $n \in {}^*\mathbb{N}$ #-converge strongly and #-$\lim_{n\to\,^*\infty} R_n(z)$ is invertible for some z. (d) Statement (c) holds for all non-real $z \in \widetilde{\mathbb{C}}^\#_c$.
Proof. The theorem follows from Proposition 21.3 and sect.24.
Now we give estimates which are sufficient to assure that it G #-convergent sequence of operators is R -#-convergent, and that the #-limit is maximal symmetric or #-selfadjoint. In order to measure the rate of

#-convergence, we introduce a #-selfadjoint operator $N \geq I$ and the associated non-Archimedean Hilbert spaces H_λ with the scalar product

$$\langle \psi, \psi \rangle_{\#\lambda} = \langle N^{\lambda/2}\psi, N^{\lambda/2}\psi \rangle_{\#}. \tag{21.12}$$

By standard identifications we have for $\lambda \geq 0$: $H_\lambda \subset H_0 \subset H_{-1}$ and $H_0 = H$. If $D: H_\alpha \to H_\beta$ is a #-densely defined, bounded in $\widetilde{\mathbb{R}}_c^{\#}$ operator from H_α to H_β, we let $\|D\|_{\#\alpha,\beta}$ denote its #-norm. Setting $\|D\|_{\#} = \|D\|_{\#0,0}$ we obtain

$$\|D\|_{\#\alpha,\beta} = \|N^{\beta/2} D N^{-\alpha/2}\|_{\#}. \tag{21.13}$$

Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators, and consider the following three conditions.

(i) Suppose that $C_n - C_m$ is a #-densely defined, bounded in $\widetilde{\mathbb{R}}_c^{\#}$ operator from H_λ to $H_{-\lambda}$, for some λ,and that as $n, m \to {}^*\infty$

$$\|D\|_{\#\lambda,-\lambda} \to_{\#} 0_{{}^*\widetilde{\mathbb{R}}_c^{\#}}. \tag{21.14}$$

(ii) Suppose that, for some p and for an unbounded set of $z = x + iy \in \widetilde{\mathbb{C}}_c^{\#}$ in the sector $|x| \leq \text{const} \times |y|$,

$$\|R_n(z)\|_{\#\mu,\lambda} \leq M(z), \tag{21.15}$$

where the bound $M(z)$ is uniform in $n \in {}^*\mathbb{N}$.

(iii) Suppose that, for the above z's,

$$\|R_n(\bar{z})\|_{\#\mu,\lambda} \leq M(z). \tag{21.16}$$

Theorem 21.5 Let $C_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-selfadjoint operators with a common domain, such that $C_n \to_{\#G} C$ ${}_{{}^*\infty}$. If the conditions (i) and (ii) mentioned above hold, then $C_n \to_{\#R} C$ ${}_{{}^*\infty}$ and C is maximal symmetric.
Corollary 21.6 If in addition to the hypothesis of Theorem 21.5, condition (iii) also holds, then C is #-selfajoint.
Remark 21.5 (1) If $\mu = 0$ in (ii), then the resolvents #-converge uniformly. (2) If the C_n are uniformly semibounded in $\widetilde{\mathbb{R}}_c^{\#}$ from below, then we may choose the z in condition (ii) to be infinite large negative numbers. In that case the conclusion of the Theorem 21.5 is that $C_n \to_{\#R} C = C^*$.

We consider now a singular perturbation B of a #-selfadjoint operator A. We give estimates on B which ensure that the sum $A + B$ is #-selfadjoint.
Abbreviation 21.1 We abbreviate $A^{-\#}$ instead #-$\bar{A}$.
Definition 21.3 A #-core of an operator C is a domain D contained in $D(C)$ such that $C = (C \upharpoonright D)^{-\#}$.
Lemma 21.7 Let ,$A_n, n \in {}^*\mathbb{N}$,$B, B_n, n \in {}^*\mathbb{N}$ and $C_n = A + B_n, n \in {}^*\mathbb{N}$ be #-selfadjoint operators with a common #-core D. Assume the hypotheses of Theorem 21.5 and Corollary 21.6 for $C_n, n \in {}^*\mathbb{N}$ and suppose also that, for $\theta \in D$,

$$\|(A - A_n)\theta\|_{\#} + \|(B - B_n)\theta\|_{\#} \to_{\#} 0 \text{ as } n \to {}^*\infty, \tag{21.17}$$

$$\|A\theta\|_{\#}^2 + \|B\theta\|_{\#}^2 \leq \text{const.} \times \|\theta\|_{\#}^2 + \text{const.} \times \|C_n\theta\|_{\#}^2, \tag{21.18}$$

with constants independent of n. Then $A + B$ is #-selfadjoint and $C_n \to_{\#R} (A + B)$.
Proof. Let $\psi_n = R_n(z)\chi$. We have the inequality

$$\|A_n\psi_n\|_{\#} \leq const. \times \|\psi_n\|_{\#} + \text{const.} \times \|C_n\psi_n\|_{\#} \leq \text{const.} \times \|(C_n - z)\psi_n\|_{\#} = \text{const.} \times \|\chi\|_{\#}$$

Thus, $|\langle A\theta,\psi\rangle_{\#}| \leq \|A\theta\|_{\#}\|\psi-\psi_n\|_{\#} + \|(A-A_n)\theta\|_{\#}\times\|\psi_n\|_{\#} + const.\|\theta\|_{\#}\times\|\psi\|_{\#}$ and therefore $|\langle A\theta,\psi\rangle_{\#}| \leq const.\|\theta\|_{\#}\times\|\psi\|_{\#}$ for $\theta\in D$ and $\psi = R(z)\chi$. It follows that $\psi\in D(A\upharpoonright^{*} D) = D(A)$ and similarity we obtain that $\psi\in D(B)$. Since $D(C)$ is the range of $R(z)$,we have shown that $D(C) = D(A+B)$. Since $\langle\theta,(A+B)\psi\rangle_{\#} = \langle(A+B)\theta,\psi\rangle_{\#} = \#\text{-}\lim_{n\to{}^{*}\infty}\langle C_n\theta,\psi_n\rangle_{\#} = \#\text{-}\lim_{n\to{}^{*}\infty}\langle\theta,C_n\psi_n\rangle_{\#} = \langle\theta,C\psi\rangle_{\#}$ we have $C\subset A+B$.Thus $A+B$ is a symmetric extension of the #-selfadjoint operator C, and so $A+B=C$.

Remark 21.6 As hypothesis for our next theorem, our second main result, we assume that $N\leq A$ and that N and A commute. Let

$$D^{*\infty}(A) = \bigcap_{n\in{}^{*}\mathbb{N}} D(A^{n}) \tag{21.19}$$

the elements of $D^{*\infty}(A)$ are called $C^{*\infty}$ vectors for A. Assume that $D^{*\infty}(A)$ is a #-core for the #-selfadjoint operator B. Also assume that, as bilinear forms on $D^{*\infty}\times D^{*\infty}$ and for some α and ε in the indicated ranges,

$$0\leq\alpha N+B+\text{const.},0\leq\alpha<1_{{}^{*}\widetilde{\mathbb{R}}_c^{\#}}/2_{{}^{*}\widetilde{\mathbb{R}}_c^{\#}}, \tag{21.20}$$

$$0\leq\varepsilon A^2+\text{const}\times B+[A^{1/2},[A^{1/2},B]]+\text{const.},2\alpha+\varepsilon<1_{{}^{*}\widetilde{\mathbb{R}}_c^{\#}}. \tag{21.21}$$

Let B be a bounded in $\widetilde{\mathbb{R}}_c^{\#}$ operator from H_v to H_{-v} and from H_α to H_β for some α,β and $,\beta>0$, where (H_α is defined following Theorem 21.4. If $v\geq 2_{{}^{*}\widetilde{\mathbb{R}}_c^{\#}}$, assume that for all $\varepsilon>0$

$$0\leq\varepsilon N^{\mu+2}+[N^{(\mu+1)/2},[N^{(\mu+1)/2},B]]+\text{const.} \tag{21.22}$$

as bilinear forms on $D^{*\infty}\times D^{*\infty}$, for some $\mu > v-2_{{}^{*}\widetilde{\mathbb{R}}_c^{\#}}$.

Theorem 21.6. Under the above hypothesis, $A+B$ is #-selfadjoint.

Proof Let $N = Ext\text{-}\int_1^{*\infty}\lambda d^{\#}E_\lambda$, and let $B_n = (E_nBE_n)^{\#-}, A_n$=A. E_n leaves D invariant and so the domain of B_n contains D. For $\theta\in D$

$$|\langle\theta,B_n\theta\rangle_{\#}| = |\langle E_n\theta,BE_n\theta\rangle_{\#}|\leq\|B\|_{\#\lambda,-\lambda}\times\left\|N^{\nu/2}E_n\theta\right\|_{\#}^{2}\leq n^{\nu}\|B\|_{\#\lambda,-\lambda}\times\|\theta\|_{\#}^{2} \tag{21.23}$$

and so B_n is bounded in $\widetilde{\mathbb{R}}_c^{\#}$ operator and essentially #-selfadjoint on D. By Corollary 21.8, $C_n = A+B_n$ is essentially #-selfadjoint on D and $D(C_n) = D(A)$.

Let $D_0\subset D$ be those vectors with #-compact support relative to the spectral #-measure of the operator A. In other words, if $\psi\in D_0$, there exist constants α and β such that, for all $\mu\geq 0$, $\|A^{\mu}\psi\|_{\#}^{2}\leq\alpha\beta^{\mu}$.It is clear that the vectors $(C_n+b)D_0$ are #-dense in $H^{\#}$ and are #-analytic vectors for $N^{\mu/2}$ for any μ. Thus by generalized Nelson's theorem [19], $N^{\mu/2}$ is essentially #-selfadjoint on $(C_n+b)D_0$, and hence $(C_n+b)D_0$ is #-dense in H_μ} every μ. In terms of $H^{\#}$

$$N^{\mu/2}(C_n+b)D\supset N^{\mu/2}(C_n+b)D_0 \tag{21.24}$$

is #-dense in $H^{\#}$ for every μ. For $\lambda>v$ and for $m\leq n$

$$\|B_n-B_m\|_{\#\lambda,-\lambda} = \|N^{-\lambda/2}(B_n-B_m)N^{-\lambda/2}\|\leq 2\left\|N^{-\frac{\lambda-v}{2}}(I-E_m)\right\|_{\#}\left\|N^{\nu/2}BN^{\nu/2}\right\|_{\#}$$

$$\leq 2m^{-(\lambda-\nu)/2}\|B\|_{\#\lambda,-\lambda}. \tag{21.25}$$

The inequality (21.21) is preserved under the substitution $B\to B_n$. To see this, we multiply (21.21) by E_n on the left and right and notice that

$$\varepsilon E_nA^2E_n+\text{const.}\times E_n\leq\varepsilon A^2+\text{const.} \tag{21.26}$$

Similarly we see that (21.20) is preserved under the substitution $B\to B_n$. Thus the bounds (21.18) follow from Proposition 21.2 applied to the operators A and B_n. To prove (21.17), let $\theta\in D$. Then,

$$\|(B-B_n)\theta\|_\# \leq (\|I-E_n)B\theta\|_\# + \|B(I-E_n)\theta\|_\# \leq$$

$$n^{-\beta/2}\|B\|_{\alpha\beta} \times \|\|N^{\alpha/2}\theta\|_\# + n^{-1}\|B\|_{\alpha\beta} \times \|N^{1+(\alpha/2)}\theta\|_\#.$$

We set $\mu = \nu + 2$ and use Lemma 21.7. Because of the uniform lower bound (21.18), we can find infinite large negative numbers $-c$ bounded away from the spectrum of C_n. If $\nu < 2_{^*\widetilde{\mathbb{R}}_c^\#}$, then we set $\mu = 0_{^*\widetilde{\mathbb{R}}_c^\#}, \lambda = 2_{^*\widetilde{\mathbb{R}}_c^\#}$, and, because of (21.18), we have

$$\|R_n(-c)\|_{\mu\lambda} \times \|NR_n(-c)\|_{\mu\lambda} \times \|AR_n(-c)\|_{\mu\lambda} \leq \text{const.} \tag{21.27}$$

Now we assume $\nu \geq 2$ and we use (21.22) to bound

$$\|R_n(-c)\|_{\mu\lambda} = \|N^{(\mu+2)/2}R_n(-c)N^{-\mu/2}\|_{\mu\lambda}. \tag{21.28}$$

Since a bounded in $\widetilde{\mathbb{R}}_c^\#$ operator is determined by its action on any #-dense domain and since $N^{\mu/2}(C_n + c)$ maps onto a #-dense subset of $H^\#$, it is sufficient to show that

$$N^{\mu+2} \leq \text{const.} \times (C_n + c)\, N^\mu (C_n + c) \tag{21.29}$$

as a bilinear form on $D \times D$. We expand the right side as

$$(C_n + c)N^\mu(C_n + c)\ \varepsilon^2 N^{\mu+2} +$$

$$+P + \varepsilon(A - \varepsilon N + B_n + c)N^{\mu+1}\{(A - \varepsilon N + B_n + c)\} + \varepsilon N^{\mu+1}(A - \varepsilon N + B_n + c) \tag{21.30}$$

with $0 \leq P$ and so it is sufficient to show that, for some $\varepsilon > 0, \varepsilon \approx 0$

$$0 \leq \varepsilon N^{\mu+2} + 2N^{(\mu+1)/2}(A - \varepsilon N + B_n + c/2)\, N^{(\mu+1)/2} + [N^{(\mu+1)/2}, [\, N^{(\mu+1)/2}, B_n]] \tag{21.31}$$

on the domain $D \times D$. In these inequalities, c may be chosen independently of ε and hence c is an arbitrarily large constant. By (21.22), the sum of the first, third and fourth terms are positive and, by (21.20), the second term is positive. We have verified the hypotheses of Lemma 21.7, and the theorem follows; $A + B$ is # -selfadjoint.

Definition 21.4 Let $M, N, \ldots$ be a #-closed linear manifolds of a non-Archimedean Banach space Z. We denote by S_M the unit sphere of M (the set of all $u \in M$ with $\|u\|_\# = 1$). For any two #-closed linear sub manifolds M, N of Z, we set: (1) $\delta(M,N) = \sup_{u \in S_M}\{\text{dist}(u, N)$, (2) $\hat{\delta}(M,N) = \max[\delta(M,N), \delta(N,M)]$.

Remark 21.7 Note that (1) has no meaning if $M = 0$; in this case we define $\delta(0,N) = 0$ for any N. On the other hand $\delta(M,0) = 1$ if $M \neq 0$, as is seen from the definition, $\delta(M,N)$ can also be characterized as the smallest number δ such that (3) $\text{dist}(u,N) \leq \delta\|u\|_\#$ for all $u \in M$.

Definition 21.5 $\hat{\delta}\,(M,N)$ will be called the gap between the manifoldsM, N.

Lemma 21.8 (1) $\delta(M,N) = 0$ if and only if $M \subset N$. (2) $\hat{\delta}$ (M,N) $= 0$ if and only if $M = N$.
(3) $\hat{\delta}\,(M,N) = \hat{\delta}$ (N,M). (4) $0 \leq \delta(M,N) \leq 1, 0 \leq \hat{\delta}\,(M,N) \leq 1$.

Proof. Directly from the definitions.

Definition 21.6 We set: (1) $(M,N) = \sup_{u \in S_M}\text{dist}(u, S_N)$, (2) $\hat{d}(M,N) = \max[d(M,N), d(N,M)]$. Note that (1) does not make sense if either M or N is 0. In such cases we set (3) $d(0,N) = 0$ for any N; $\hat{d}(M,0) = 2$ for $M \neq 0$.

Lemma 21.9 d and $\hat{d}$ satisfy the triangle inequalities : (1) $d(L,N) \leq d(L,M) + d(M,N)$,
(2) $\hat{d}\,(L,N) \leq \hat{d}\,(L,M) + \hat{d}\,(M,N)$.

Proof. The second inequality follows from the first, which in turn follows directly from the definition.

Definition 21.7 We say that hyper infinite sequence $T_n, n \in {}^*\mathbb{N}$ #-converges to operator T $(T_n \rightarrow_{g\#} T)$ in the generalized sense if $\hat{\delta}\,(T_n,T) \rightarrow_\# 0$.

Theorem 21.7 Let $T: H \rightarrow H$ be a #-selfadjoint operator. Then there is a $\delta > 0, \delta \approx 0$ such that any #-closed symmetric operator S with $\hat{\delta}(S,T) < \delta$ is #-selfadjoint, where $\hat{\delta}(S,T)$ denotes the gap between S and T.

Corollary 21.8 Let T, T_n. be #-closed symmetric operators and let $\{T_n\}, n \in {}^*\mathbb{N}$ #-converge to T in the generalized sense, see Definition 21.7. If T is #-selfadjoint, then T_n is #-selfadjoint for sufficiently large $n \in {}^*\mathbb{N}\backslash\mathbb{N}$.
Corollary 21.9 With the hypothesis of Theorem 21.6, $A + \gamma B \to_{\#R} A$ as $y \to_{\#} 0$.
Proof. This is a special case of Corollary 10(a)
Corollary 21.10 Let B, B_j be singular perturbations of A, each satisfying the hypothesis of Theorem 21.6, with constants independent of j.
(a) If $B_j - B_l$ is a #-densely defined bounded in ${}^*\widetilde{\mathbb{R}}_c^\#$ operator from H_λ to $H_{-\lambda}$ for a sufficiently large λ, if $\|B_j - B_l\|_{\#\lambda,-\lambda} \to_{\#} 0$ and $B_j \to_{\#G} B$, then

$$A + B_j \to_{\#R} A + B. \tag{21.32}$$

(b) If (a) is true for $\lambda \leq 2$ and if there is a #-core D of C with $\subset \bigcap_j D(B_j)$, then the resolvents of $A + B_j$ #-converge in #-norm to the resolvent of $A + B$.
22. Construction of ${}^*\widetilde{\mathbb{R}}_c^\#$
Definition 22.1 Let $a_n, n \in \mathbb{N}$ be ${}^*\mathbb{R}_c^\#$- valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_c^\#$ such that:
(i) there is $M \in \mathbb{N}$ such that $\{a_n\}_{n=M}^{^*\infty}$ is monotonically decreasing ${}^*\mathbb{R}_{\text{cfin}}^\#$ - valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_{\text{cfin}}^\# \backslash\{0_{{}^*\mathbb{R}_c^\#}\}$, We denote these sequences by $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$, etc.
(ii) there is $N \in \mathbb{N}$ such that for all $n > N$, $a_n \neq 0_{{}^*\mathbb{R}_c^\#}$,
(iii) for all $n \in \mathbb{N}$, $a_n \not\approx 0_{{}^*\mathbb{R}_c^\#}$ and for any ε such that $\varepsilon > 0, \varepsilon \not\approx 0_{{}^*\mathbb{R}_c^\#}$ there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n < \varepsilon$ and we denote a set of the all these countable sequences by $\Delta_\omega^{+\downarrow 0}$.
Definition 22.2 (i) We define a set $\Delta_\omega^{-\downarrow 0}$ by $a_n \in \Delta_\omega^{-\downarrow 0} \Leftrightarrow -a_n \in \Delta_\omega^{+\downarrow 0}$.Note that$\Delta_\omega^{-\downarrow 0} = -\Delta_\omega^{+\downarrow 0}$.
(ii) We define a set $\Delta_\omega^{(-1)\downarrow 0}$ by $a_n \in \Delta_\omega^{(-1)\downarrow 0} \Leftrightarrow a_n^{-1} \in \Delta_\omega^{+\downarrow 0}$.Note that $\Delta_\omega^{(-1)\downarrow 0} = (\Delta_\omega^{+\downarrow 0})^{-1}$.
Definition 22.3 Let $a_n, n \in \mathbb{N}$ be ${}^*\mathbb{R}_c^\#$- valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_c^\#$ such that:
(i) there is $M \in \mathbb{N}$ such that $\{a_n\}_{n=M}^{^*\infty}$ is monotonically decreasing ${}^*\mathbb{R}_{c\approx}^\#$ - valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_{c+\approx}^\# \backslash\{0_{{}^*\mathbb{R}_c^\#}\}$,
(ii) there is $N \in \mathbb{N}$ such that for all $n > N$, $a_n \neq 0_{{}^*\mathbb{R}_c^\#}$,
(iii) we denote a set of the all these countable sequences by $\Delta_{\omega\approx}^{+\downarrow 0}$.
Definition 22.4 (i) We define a set $\Delta_{\omega\approx}^{-\downarrow 0}$ by $a_n \in \Delta_{\omega\approx}^{-\downarrow 0} \Leftrightarrow -a_n \in \Delta_{\omega\approx}^{+\downarrow 0}$.Note that: $\Delta_{\omega\approx}^{-\downarrow 0} = -\Delta_{\omega\approx}^{+\downarrow 0}$.
(ii) We define a set $\Delta_{\omega\approx}^{(-1)\downarrow 0}$ by $a_n \in \Delta_{\omega\approx}^{(-1)\downarrow 0} \Leftrightarrow a_n^{-1} \in \Delta_{\omega\approx}^{+\downarrow 0}$.Note that: $\Delta_{\omega\approx}^{(-1)\downarrow 0} = (\Delta_{\omega\approx}^{+\downarrow 0})^{-1}$.
Definition 22.5 Let $a_n, n \in \mathbb{N}$ be ${}^*\mathbb{R}_c^\#$- valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_c^\# \backslash\{0_{{}^*\mathbb{R}_c^\#}\}$ such that:
(i) there is $M \in \mathbb{N}$ such that $\{a_n\}_{n=M}^{^*\infty}$ is monotonically increasing ${}^*\mathbb{R}_{\text{cfin}}^\#$ - valued countable sequence $a: \mathbb{N} \to {}^*\mathbb{R}_{\text{cfin}}^\# \backslash\{0_{{}^*\mathbb{R}_c^\#}\}$, We denote these sequences by $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$, etc.
(ii) there is $N \in \mathbb{N}$ such that for all $n > N$, $a_n \neq 0_{{}^*\mathbb{R}_c^\#}$,
(iii) for all $n \in \mathbb{N}$, $a_n \not\approx 0_{{}^*\mathbb{R}_c^\#}$ and for any ε such that $\varepsilon > 0, \varepsilon \in {}^*\mathbb{R}_{\text{cfin}}^\#$ there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n > \varepsilon$ and we denote a set of the all these countable sequences by $\Delta_\omega^{+\downarrow\infty}$
Definition 22.6 (i) We define a set $\Delta_\omega^{-\downarrow\infty}$ by $a_n \in \Delta_\omega^{-\downarrow\infty} \Leftrightarrow -a_n \in \Delta_\omega^{+\downarrow\infty}$.Note that$\Delta_\omega^{-\downarrow\infty} = -\Delta_\omega^{+\downarrow\infty}$.
(ii) We define a set $\Delta_\omega^{(-1)\downarrow\infty}$ by $a_n \in \Delta_\omega^{(-1)\downarrow\infty} \Leftrightarrow a_n^{-1} \in \Delta_\omega^{+\downarrow 0}$.Note that $\Delta_\omega^{(-1)\downarrow\infty} = \Delta_\omega^{+\downarrow 0}$.
Definition 22.7 (i) We define the ordering relation $(\cdot\leq\cdot)$ on a set $\Delta_\omega^{+\downarrow 0} \times \Delta_\omega^{+\downarrow 0}$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ then $\{a_n\}_{n=0}^\infty \leq \{b_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n \leq b_n$.
(ii) We define the ordering relation $(\cdot\leq\cdot)$ on a set $\Delta_\omega^{-\downarrow 0} \times \Delta_\omega^{-\downarrow 0}$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_\omega^{-\downarrow 0}$, then $\{a_n\}_{n=0}^\infty \leq \{b_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n \leq b_n$
(iii) We define the ordering relation $(\cdot\leq\cdot)$ on a set $\Delta_{\omega\approx}^{+\downarrow 0} \times \Delta_{\omega\approx}^{+\downarrow 0}$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_{\omega\approx}^{+\downarrow 0}$ and $\{b_n\}_{n=0}^\infty \in \Delta_{\omega\approx}^{+\downarrow 0}$ then $\{a_n\}_{n=0}^\infty \leq \{b_n\}_{n=0}^\infty$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n \leq b_n$.
Definition 22.8 (i) We define the ordering relation $(\cdot\leq\cdot)$ on a set $\Delta_\omega^{+\downarrow 0} \times {}^*\mathbb{R}_{c+}^\#$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_\omega^{+\downarrow 0}$ and $x \in {}^*\mathbb{R}_{c+}^\#$, then $\{a_n\}_{n=0}^\infty \leq x$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n < x$.
(ii) We define the ordering relation $(\cdot\leq\cdot)$ on a set $\Delta_{\omega\approx}^{+\downarrow 0} \times {}^*\mathbb{R}_{c\approx+}^\#$ by: let $\{a_n\}_{n=0}^\infty \in \Delta_{\omega\approx}^{+\downarrow 0}$ and $\varepsilon \in {}^*\mathbb{R}_{c\approx+}^\#$, then

$\{a_n\}_{n=0}^{\infty} \leq \varepsilon$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $a_n < \varepsilon$.
(iii) We define the ordering relation $(\cdot\leq\cdot)$ on a set ${}^*\mathbb{R}^{\#}_{c\approx+} \times \Delta^{+\downarrow 0}_{\omega\approx}$ by: let $\{a_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega\approx}$ and $\varepsilon \in {}^*\mathbb{R}^{\#}_{c\approx+}$, then $\varepsilon \leq \{a_n\}_{n=0}^{\infty}$ iff there is $N \in \mathbb{N}$ such that for all $n > N$: $\varepsilon < a_n$.

Proposition 221 Let $\{a_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega}$, then
(i) $\{a_n\}_{n=0}^{\infty} + \{b_n\}_{n=0}^{\infty} \triangleq \{a_n + b_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega}$;
(ii) $\{a_n\}_{n=0}^{\infty} - \{b_n\}_{n=0}^{\infty} \triangleq \{a_n - b_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega} \cup \Delta^{-\downarrow 0}_{\omega} \cup \left\{0_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty}$, where $\left\{0_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty}$ is a countable $0_{{}^*\mathbb{R}^{\#}_c}$- valued sequence;
(iii) $\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega}$;
Proof. Immediately from definitions.

Proposition 22.2 Let $\{a_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega}$ and $\{b_n\}_{n=0}^{\infty} \in \Delta^{(-1)\downarrow 0}_{\omega}$ then
$\{a_n\}_{n=0}^{\infty} \times \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \times b_n\}_{n=0}^{\infty} \in \Delta^{+\downarrow 0}_{\omega} \cup \Delta^{(-1)\downarrow 0}_{\omega} \cup \Delta^{+\downarrow\infty}_{\omega} \cup \left\{1_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty}$, where $\left\{1_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty}$ is a countable $1_{{}^*\mathbb{R}^{\#}_c}$-valued sequence;
Proof. Immediately from definitions.

Definition 22.9 (i) Let $\{a_n\}_{n=0}^{\infty} \in \mathcal{M}_{\infty} = \Delta^{+\downarrow 0}_{\omega} \cup \Delta^{-\downarrow 0}_{\omega} \cup \Delta^{+\downarrow\infty}_{\omega} \Delta^{-\downarrow\infty}_{\omega} \cup \left\{0_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty} \cup \left\{1_{{}^*\mathbb{R}^{\#}_c}\right\}_{n=0}^{\infty}$, and let $\{A_n\}_{n=0}^{{}^*\infty} \triangleq \widetilde{\{a_n\}}_{n=0}^{\infty}$ be a hyper infinite sequence

$$\{A_n\}_{n=0}^{{}^*\infty} \triangleq \widetilde{\{a_n\}}_{n=0}^{\infty} = \left(a_0, a_1, \ldots, a_n, \ldots, \{a_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \ldots\right), \tag{22.1}$$

i.e. for any infinite $m \in {}^*\mathbb{N}\backslash\mathbb{N}$, $A_m = \{a_n\}_{n=0}^{\infty}$.
(ii) we define a set $\widetilde{\mathcal{M}}_{\infty}$ by

$$\{A_n\}_{n=0}^{{}^*\infty} \in \widetilde{\mathcal{M}}_{\infty} \Leftrightarrow \left(\{A_n\}_{n=0}^{{}^*\infty} = \widetilde{\{a_n\}}_{n=0}^{\infty}\right) \wedge \left(\{a_n\}_{n=0}^{\infty} \in \mathcal{M}_{\infty}\right) \tag{22.2}$$

Definition 22.10 (i) Let $\Psi: {}^*\mathbb{N} \to \mathcal{M}_{\infty} \cup \left({}^*\mathbb{R}^{\#\,{}^*\mathbb{N}}_c\right)$ be a hyper infinite sequence and we denote these hyper infinite sequences by $\{\Psi_n\}_{n=0}^{{}^*\infty}, \{\Phi_n\}_{n=0}^{{}^*\infty}$,etc.,
(ii) we denote a set of the all these hyper infinite sequences by $\mathcal{R}_{{}^*\infty}$. Note that $\widetilde{\mathcal{M}}_{\infty} \subset \mathcal{R}_{{}^*\infty}$.
(iii) Let $\wp: {}^*\mathbb{N} \to \mathcal{M}_{\infty}$ be a hyperfinite sequence and we denote these hyperfinite sequences by $\{\mathcal{F}_n\}_{n=0}^{n=m}, m \in {}^*\mathbb{N}\backslash\mathbb{N}, \{\wp_n\}_{n=0}^{n=k}, k \in {}^*\mathbb{N}\backslash\mathbb{N}$, etc.,
(iii) Let $\{\mathcal{F}_n\}_{n=0}^{n=m}$ be $\{\mathcal{F}_n\}_{n=0}^{n=m}, m \in {}^*\mathbb{N}\backslash\mathbb{N}$, we define hyper infinite sequence $\widetilde{\{\mathcal{F}_n\}}_{n=0}^{n=m}$ by:
for $n \leq m, \widetilde{\{\mathcal{F}_n\}}_{n=0}^{n=m} = \{\mathcal{F}_n\}_{n=0}^{n=m}$, and for $n > m, \mathcal{F}_n = m$.
(iv) We denote a set of the all these hyper infinite sequences by $GF_{{}^*\infty}$.
(v) Let $\psi \in {}^*\mathbb{R}^{\#\,{}^*\mathbb{N}}_c$ be a hyper infinite sequence and we denote these hyper infinite sequences by $\{\psi_n\}_{n=0}^{{}^*\infty}, \{\phi_n\}_{n=0}^{{}^*\infty}, \{\theta_n\}_{n=0}^{{}^*\infty}$,etc.
(vi) Let $\psi \in {}^*\mathbb{R}^{\#\,{}^*\mathbb{N}}_c$ be a hyper infinite sequence .Assume that there is exists $N \in {}^*\mathbb{N}$ such that $\psi_n \neq 0_{{}^*\mathbb{R}^{\#}_c}$ for $n > N$. Define hyper infinite sequence $\{\phi_n\}_{n=0}^{{}^*\infty}$ of hyperreal numbers from ${}^*\mathbb{R}^{\#}_c$ as follows: for $n \leq N, \phi_n = 0_{{}^*\mathbb{R}^{\#}_c}$, and for $n > N$, $\phi_n = \frac{1_{{}^*\mathbb{R}^{\#}_c}}{\psi_n}$

$$\{\phi_n\}_{n=0}^{{}^*\infty} = \left(0_{{}^*\mathbb{R}^{\#}_c}, 0_{{}^*\mathbb{R}^{\#}_c}, \ldots, 0_{{}^*\mathbb{R}^{\#}_c}, \frac{1_{{}^*\mathbb{R}^{\#}_c}}{\psi_{N+1}}, \frac{1_{{}^*\mathbb{R}^{\#}_c}}{\psi_{N+2}}, \ldots\right). \tag{22.3}$$

This definition (22.3) makes sense since, for $n > N$, an is a nonzero hyperreal number, so $1_{{}^*\mathbb{R}^{\#}_c}/\Psi_n$ exists. Then $\psi_n \times \phi_n$ is equal to $\psi_n \times 0_{{}^*\mathbb{R}^{\#}_c} = 0_{{}^*\mathbb{R}^{\#}_c}$ for $n \leq N$, and equals $\psi_n \times \phi_n = \psi_n \times 1_{{}^*\mathbb{R}^{\#}_c}/\psi_n = 1_{{}^*\mathbb{R}^{\#}_c}$ for $n > N$. Thus hyper infinite sequence $\{\psi_n\}_{n=0}^{{}^*\infty}$ is invertible in the following sense

$$\left(\{\psi_n\}_{n=0}^{{}^*\infty}\right) \times \left(\{\phi_n\}_{n=0}^{{}^*\infty}\right) = \left(0_{{}^*\mathbb{R}^{\#}_c}, 0_{{}^*\mathbb{R}^{\#}_c}, \ldots, 0_{{}^*\mathbb{R}^{\#}_c}, 1_{{}^*\mathbb{R}^{\#}_c}, 1_{{}^*\mathbb{R}^{\#}_c}, \ldots\right). \tag{22.4}$$

If the equality (22.4) holds we say that hyper infinite sequence $\{\psi_n\}_{n=0}^{*\infty}$ multiplicative invertible and $\{\phi_n\}_{n=0}^{*\infty}$ is *multiplicative inverse* or *reciprocal* for hyper infinite sequence $\{\psi_n\}_{n=0}^{*\infty}$ and denote it by

$$\left(\{\psi_n\}_{n=0}^{*\infty}\right)^{-1_{*\mathbb{R}_c^\#}} \triangleq \{\phi_n\}_{n=0}^{*\infty}. \tag{22.5}$$

Note that

$$\left(\left(\{\psi_n\}_{n=0}^{*\infty}\right)^{-1_{*\mathbb{R}_c^\#}}\right)^{-1_{*\mathbb{R}_c^\#}} \equiv \{\psi_n\}_{n=0}^{*\infty}. \tag{22.6}$$

Definition 22.11 Let $\{\psi_n\}_{n=0}^{*\infty} \in {}^*\mathbb{R}_c^{\#\;{}^*\mathbb{N}}$ be a hyper infinite sequence such that $\{\psi_n\}_{n=0}^{*\infty} \not\equiv \{0_{*\mathbb{R}_c^\#}\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{*\infty}$ is not multiplicative invertible. This meant there is exists hyper infinite subsequence $\{\psi_{n_i}\}_{i=0}^{*\infty}$ such that $\psi_{n_i} \neq 0_{*\mathbb{R}_c^\#}, i \in {}^*\mathbb{N}$ and $\psi_n \equiv 0_{*\mathbb{R}_c^\#}$ if $\psi_n \notin \{\psi_{n_i}\}_{i=0}^{*\infty}$. Define now hyper infinite sequence $\{\phi_n\}_{n=0}^{*\infty}$ of hyperreal numbers from ${}^*\mathbb{R}_c^\#$ as follows: for $= n_i$, $\phi_n = \frac{1_{*\mathbb{R}_c^\#}}{\psi_{n_i}}$ and $\phi_n \equiv 0_{*\mathbb{R}_c^\#}$ if $\psi_n \notin \{\psi_{n_i}\}_{i=0}^{*\infty}$. Thus hyper infinite sequence $\{\psi_n\}_{n=0}^{*\infty}$ is invertible in the following sense

$$\{\psi_n\}_{n=0}^{*\infty} \times \{\phi_n\}_{n=0}^{*\infty} = \{\omega_n\}_{n=0}^{*\infty}, \tag{22.7}$$

where $\omega_n \equiv 1_{*\mathbb{R}_c^\#}$ if $n = n_i$ and $\omega_n \equiv 0_{*\mathbb{R}_c^\#}$ if $n \neq n_i, i \in {}^*\mathbb{N}$. If the equality (22.7) holds we say that hyper infinite sequence $\{\psi_n\}_{n=0}^{*\infty}$ multiplicative semi-invertible and $\{\phi_n\}_{n=0}^{*\infty}$ is *multiplicative semi-inverse* or semi-*reciprocal* for hyper infinite sequence $\{\psi_n\}_{n=0}^{*\infty}$ and denote it by

$$\left(\{\psi_n\}_{n=0}^{*\infty}\right)^{-1_{*\mathbb{R}_c^{\#\blacksquare}}} \triangleq \{\phi_n\}_{n=0}^{*\infty}. \tag{22.8}$$

Note that

$$\left(\left(\{\psi_n\}_{n=0}^{*\infty}\right)^{-1_{*\mathbb{R}_c^{\#\blacksquare}}}\right)^{-1_{*\mathbb{R}_c^{\#\blacksquare}}} \equiv \{\psi_n\}_{n=0}^{*\infty}. \tag{22.9}$$

Definition 22.12 (i) Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \in \Delta_\omega^{+\downarrow 0} \cup \Delta_\omega^{-\downarrow 0} \cup \Delta_\omega^{+\downarrow\infty} \Delta_\omega^{-\downarrow\infty} \cup \{0_{*\mathbb{R}_c^\#}\}_{n=0}^{\infty} \cup \{1_{*\mathbb{R}_c^\#}\}_{n=0}^{\infty}$ and $x, y \in {}^*\mathbb{R}_c^\#$, then we define

$$\{a_n\}_{n=0}^{\infty} \pm \{b_n\}_{n=0}^{\infty} \triangleq \{a_n \pm b_n\}_{n=0}^{\infty}, \tag{22.10}$$

$$x \pm y\{a_n\}_{n=0}^{\infty} \triangleq \{x \pm y a_n\}_{n=0}^{\infty}. \tag{22.11}$$

(ii) Let $\{\Psi_n\}_{n=0}^{*\infty}, \{\Phi_n\}_{n=0}^{*\infty} \in \mathcal{R}_{*\infty.}$ and $x, y \in {}^*\mathbb{R}_c^\#$, then we define

$$\{\Psi_n\}_{n=0}^{*\infty} \pm \{\Phi_n\}_{n=0}^{*\infty} \triangleq \{\Psi_n \pm \Phi_n\}_{n=0}^{*\infty}, \tag{22.12}$$

$$\{\Psi_n\}_{n=0}^{*\infty} \times \{\Phi_n\}_{n=0}^{*\infty} \triangleq \{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}, \tag{22.13}$$

$$x \pm y\{\Psi_n\}_{n=0}^{*\infty} \triangleq \{x \pm y\Psi_n\}_{n=0}^{*\infty}. \tag{22.14}$$

Definition 22.13 Let hyper infinite sequence $\{\Psi_n\}_{n=0}^{*\infty}$ be in $\mathcal{R}_{*\infty,}$, i.e. for all $n \in {}^*\mathbb{N}, \Psi_n \in \mathcal{M}_\infty \cup \left({}^*\mathbb{R}_c^{\#\;{}^*\mathbb{N}}\right)$. Say $\{\Psi_n\}_{n=0}^{*\infty}$ #-tends to $\widetilde{0_{*\mathbb{R}_c^\#}} \triangleq 0_{\widetilde{*\mathbb{R}_c^\#}}$, as $n \to {}^*\infty$.iff for any given $\varepsilon > 0_{*\mathbb{R}_c^\#}, \varepsilon \approx 0_{*\mathbb{R}_c^\#}$ there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $n > N, |\Psi_n| < \varepsilon$.

Definition 22.14 Let $\{\Psi_n\}_{n=0}^{*\infty}$ be a hyper infinite sequence such that for all $n \in {}^*\mathbb{N}, \Psi_n \in \mathcal{M}_\infty$.We call $\{\Psi_n\}_{n=0}^{*\infty}$a

Cauchy hyper infinite sequence if the difference between its terms #-tends to $0_{^*\mathbb{R}_c^\#}$. To be precise: given any $\varepsilon > 0_{^*\mathbb{R}_c^\#}, \varepsilon \approx 0_{^*\mathbb{R}_c^\#}$ there is a hypernatural number $N \in {}^*\mathbb{N}\backslash\mathbb{N}, N = N(\varepsilon)$ such that for any $m, n > N$, $|\Psi_n - \Psi_m| < \varepsilon$.

Theorem 22.1 Let $\{\Psi_n\}_{n=0}^{^*\infty}$ be in $\mathcal{R}_{^*\infty}$. If $\{\Psi_n\}_{n=0}^{^*\infty}$ is a #-convergent hyper infinite sequence (that is, $\Psi_n \to_\# \Phi$ as $n \to {}^*\infty$ for some$\Phi \in \mathcal{R}_{^*\infty}$), then $\{\Psi_n\}_{n=0}^{^*\infty}$ is a Cauchy hyper infinite sequence.

Proof. We know that$\Psi_n \to_\# \Phi$. Here is a ubiquitous trick: instead of using $\varepsilon \approx 0_{^*\mathbb{R}_c^\#}$ in the definition, start with an arbitrary infinitesimal $\varepsilon > 0, \varepsilon \approx 0_{^*\mathbb{R}_c^\#}$ and then choose $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ so that $|\Psi_n - \Phi| < \varepsilon/2$ when $n > N$. Then if $m, n > N$, we have $|\Psi_n - \Psi_m| = |(\Psi_n - \Phi) - (\Psi_m - \Phi)| \leq |\Psi_n - \Phi| + |\Psi_m - \Phi| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\{\Psi_n\}_{n=0}^{^*\infty}$ is a Cauchy hyper infinite sequence.

Theorem 22.2 If $\{\Psi_n\}_{n=0}^{^*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded in ${}^*\mathbb{R}_c^\#$; that is, there is some number $M \in {}^*\mathbb{R}_{c+}^\#$ such that $|\Psi_n| \leq M$ for all $n \in {}^*\mathbb{N}$.

Proof. Since $\{\Psi_n\}_{n=0}^{^*\infty}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in {}^*\mathbb{N}$ such that $|\Psi_n - \Psi_m| < 1$ whenever $m, n > N$. Thus, $|\Psi_{N+1} - \Psi_n| < 1$ for $> N$. We can rewrite this as $\Psi_{N+1} - 1 < \Psi_n < \Psi_{N+1} + 1$.This means that $|\Psi_n|$ is less than the maximum of $|\Psi_{N+1} - 1|$ and $|\Psi_{N+1} + 1|$. So, set $M \in {}^*\mathbb{R}_{c+}^\#$ larger than any number in the following hyperfinite list: $\{|\Psi_0|, |\Psi_1|, \ldots, |\Psi_N|, |\Psi_{N+1} - 1|, |\Psi_{N+1} + 1|\}$. Then for any term Ψ_n, if $n \leq N$, then $|\Psi_n|$ appears in the list and so $|\Psi_n| \leq M$; if $n > N$, then (as shown above) $||\Psi_n|$is less than at least one of the last two entries in the list, and so $|\Psi_n| \leq M$. Hence, M is a bound for the sequence.

Definition 22.15 Let S be a set of objects. A relation among pairs of elements of S is said to be an equivalence relation if the following three properties hold: (1) Reflexivity: for any $s \in S$, s is related to s. (2) Symmetry: for any s,t∈S, if s is related to t then t is related to s. (3) Transitivity: for any *s*,*t*,*r*∈S, if s is related to t and t is related to r, then s is related to *r*.

Remark 22.1 The following well known proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Proposition 22.3 Let S be a set, with an equivalence relation on pairs of elements. For s∈S, denote by [s] the set of all elements in S that are related to s. Then for any s,t∈S, either [s]=[t] or [s] and [t] are disjoint. The sets [s] for s∈S are called the equivalence classes, and they are the bins.

Corollary 22.1 If S is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of S.

Definition.22.16 Let $\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ and$\{\Psi_{2,n}\}_{n=0}^{^*\infty}\}$ be in $\mathcal{R}_{^*\infty}$. Say they are equivalent (i.e. related) if $|\Psi_{1,n} - \Psi_{2,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$.

Proposition 22.4 Definition 22.15 yields an equivalence relation on $\mathcal{R}_{^*\infty}$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive. (1) Reflexive: $\Psi_{1,n} - \Psi_{2,n} = 0_{^*\mathbb{R}_c^\#}, n \in {}^*\mathbb{N}$, and the hyper infinite sequence all of whose terms are $0_{^*\mathbb{R}_c^\#}$ clearly #-converges to $0_{^*\mathbb{R}_c^\#}$ -valued hyper infinite sequence So $\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{^*\infty}$. (2) Symmetric: Suppose that $\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{^*\infty}$, so $|\Psi_{1,n} - \Psi_{2,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$, as $n \to {}^*\infty$. But $\Psi_{2,n} - \Psi_{1,n} = -(\Psi_{1,n} - \Psi_{2,n})$, and since only the absolute value $|\Psi_{1,n} - \Psi_{2,n}| = |\Psi_{2,n} - \Psi_{1,n}|$ comes into play in Definition 22,15 it follows that $|\Psi_{2,n} - \Psi_{1,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$, as $n \to {}^*\infty$ as well. Hence, $\{\Psi_{2,n}\}_{n=0}^{^*\infty}$ is related to $\{\{\Psi_{1,n}\}_{n=0}^{^*\infty}$. (3) Transitive: Suppose $\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ is related to $\{\Psi_{2,n}\}_{n=0}^{^*\infty}$, and $\{\Psi_{2,n}\}_{n=0}^{^*\infty}$is related to $\{\Psi_{3,n}\}_{n=0}^{^*\infty}$. This means that $|\Psi_{1,n} - \Psi_{2,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$ and $|\Psi_{2,n} - \Psi_{3,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$, as $n \to {}^*\infty$. To be fully precise, let us fix infinite small $\in {}^*\mathbb{R}_{c+}^\#$; then there exists an $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that for all $n > N$, $|\Psi_{1,n} - \Psi_{2,n}| < \varepsilon/2$; also, there exists an $M \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that for all $n > M$, $|\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2$. Well, then, as long as $> \max(N, M)$, we have that

$$|\Psi_{2,n} - \Psi_{3,n}| = |(\Psi_{1,n} - \Psi_{2,n}) + (\Psi_{2,n} - \Psi_{3,n})| \leq |\Psi_{1,n} - \Psi_{2,n}| + |\Psi_{2,n} - \Psi_{3,n}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, choosing L equal to the max of N, M, we see that given infinite small $\varepsilon > 0$ we can always choose L so that for $n > L$,| $|\Psi_{1,n} - \Psi_{3,n}| < \varepsilon$. This means that $|\Psi_{1,n} - \Psi_{3,n}| \to_\# 0_{^*\mathbb{R}_c^\#}$, i.e. $\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ is related to $\{\Psi_{3,n}\}_{n=0}^{^*\infty}$.

So, we really have equivalence relation, and so by Corollary 22.1 the set $\mathcal{R}_{^*\infty}$. Is partitioned into disjoint subsets (equivalence classes).

Definition 22.17 The hyperreal numbers ${}^*\widetilde{\mathbb{R}}_c^\#$ are the equivalence classes $\left[\{\Psi_n\}_{n=0}^{^*\infty},\right]$ of Cauchy hyper infinite sequences of, as per Definition 22.16.That is, each such equivalence class is a hyperreal number in ${}^*\widetilde{\mathbb{R}}_c^\#$.

Definition 22.18 Let $\{a_n\}_{n=0}^{\infty} \in {}^*\widetilde{\mathbb{R}}_c^\#$. We define external countable sum $\widehat{Ext}$ - $\sum_{n=0}^{n=\infty} a_n$ by:

(i) $$\widehat{Ext}\text{ - }\sum_{n=0}^{n=\infty} a_n \triangleq (b_0, b_1, \dots, b_k, \dots, \{b_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \dots) = \widetilde{\{b_n\}}_{n=0}^{\infty}, \quad (22.15)$$

where $b_0 = a_0, \dots, b_k, \dots; b_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in \mathbb{N}$.

(ii) We set

$$\widehat{Ext}\text{ - }\sum_{n=0}^{n=\infty} a_n \equiv Ext\text{-}\sum_{n=0}^{n=\infty} a_n. \quad (22.16)$$

Remark 22.2 Note that in non-Archimedean field ${}^*\mathbb{R}_c^\#$ external countable sum $Ext\text{-}\sum_{n=0}^{n=\infty} a_n$is not preserved in general case under the substitution $a_n = b_n, n \in \mathbb{N}$, i.e. $a_n = b_n, n \in \mathbb{N} \Rightarrow_{s,w} Ext\text{-}\sum_{n=0}^{n=\infty} a_n = Ext\text{-}\sum_{n=0}^{n=\infty} b_n$. For example in non-Archimedean field ${}^*\widetilde{\mathbb{R}}_c^\#$ for countable sequence: $\alpha, \alpha r, \alpha r^2, \dots . \alpha r^n, \dots, n \in \mathbb{N}, r < 1$ we get

$$Ext\text{- }\sum_{n=0}^{n=\infty} \alpha r^n = \widehat{Ext}\text{ - }\sum_{n=0}^{n=\infty} \alpha r^n = \alpha \frac{1_{^*\widetilde{\mathbb{R}}_c^\#}}{1_{^*\widetilde{\mathbb{R}}_c^\#} - r}\left(1_{^*\widetilde{\mathbb{R}}_c^\#} - \widetilde{\{r^n\}}_{n=1}^{n=\infty}\right).$$

However in non-Archimedean field ${}^*\mathbb{R}_c^\#$ one obtains that

$$Ext\text{- }\sum_{n=0}^{n=\infty} \alpha r^n = \alpha \frac{1_{^*\mathbb{R}_c^\#}}{1_{^*\mathbb{R}_c^\#} - r},$$

see Example 22.1.

Example 22.1 Consider countable sequence: $\alpha, \alpha r, \alpha r^2, \dots . \alpha r^n, \dots, n \in \mathbb{N}, r < 1$.

$$\widehat{Ext}\text{ - }\sum_{n=0}^{n=\infty} \alpha r^n = \alpha \left\{\widetilde{\frac{1_{^*\mathbb{R}_c^\#} - r^n}{1_{^*\mathbb{R}_c^\#} - r}}\right\}_{n=0}^{n=\infty} = \alpha \frac{1_{^*\widetilde{\mathbb{R}}_c^\#}}{1_{^*\widetilde{\mathbb{R}}_c^\#} - r} - \alpha \left\{\widetilde{\frac{r^n}{1_{^*\mathbb{R}_c^\#} - r}}\right\}_{n=1}^{n=\infty} = \alpha \frac{1_{^*\widetilde{\mathbb{R}}_c^\#}}{1_{^*\widetilde{\mathbb{R}}_c^\#} - r} - \frac{\alpha}{1_{^*\widetilde{\mathbb{R}}_c^\#} - r} \widetilde{\{r^n\}}_{n=1}^{n=\infty}. \quad (22.17)$$

Notice that $\widetilde{\{r^n\}}_{n=1}^{n=\infty} \in \Delta_\omega^{+\downarrow 0}$.

Definition 22.19 Let $\{a_n\}_{n=0}^{n=m} \in GF_{^*\infty}, m \in {}^*\mathbb{N}\backslash\mathbb{N}$. We define external hyper finite sum $\widehat{Ext}$ - $\sum_{n=0}^{n=m} a_n$ by:

(i) $$\widehat{Ext}\text{ - }\sum_{n=0}^{n=m} a_n \triangleq (b_0, b_1, \dots, b_k, \dots, \{b_n\}_{n=0}^{n=m}, \{b_n\}_{n=0}^{n=m}, \dots) = \widetilde{\{b_n\}}_{n=0}^{n=m}, \quad (22.18)$$

where $b_0 = a_0, \dots, b_k, \dots, b_m; b_k = Ext\text{-}\sum_{n=0}^{n=k} a_n, k \in {}^*\mathbb{N}$.

(ii) We set

$$\widehat{Ext}\text{-}\sum_{n=0}^{n=m} a_n \equiv \widetilde{Ext}\text{-}\sum_{n=0}^{n=m} a_n, \quad (22.19)$$

where $\widetilde{Ext}\text{-}\sum_{n=0}^{n=m} a_n \triangleq \widetilde{c_m}, c_m = Ext\text{ - }\sum_{n=0}^{n=m} a_n$.

Example 22.2 Consider hyper finite sequence: $\alpha, \alpha r, \alpha r^2, \dots, \alpha r^m, m \in {}^*\mathbb{N}\backslash\mathbb{N}, r < 1$.

$$\widehat{Ext}\text{ - }\sum_{n=0}^{n=m} \alpha r^n = \alpha \left\{\widetilde{\frac{1_{^*\mathbb{R}_c^\#} - r^n}{1_{^*\mathbb{R}_c^\#} - r}}\right\}_{n=0}^{n=m} = \alpha \frac{1_{^*\mathbb{R}_c^\#}}{1_{^*\mathbb{R}_c^\#} - r} - \alpha \left\{\widetilde{\frac{r^n}{1_{^*\mathbb{R}_c^\#} - r}}\right\}_{n=1}^{n=m} = \alpha \frac{1_{^*\mathbb{R}_c^\#}}{1_{^*\mathbb{R}_c^\#} - r} - \frac{\alpha}{1_{^*\mathbb{R}_c^\#} - r} \widetilde{\{r^n\}}_{n=1}^{n=m}. \quad (22.20)$$

From (22.17) and (22.20) we obtain

$$\widehat{Ext}\text{ -}\sum_{n \in {}^*\mathbb{N}\backslash\mathbb{N}, n \le m} \alpha r^n = \frac{\alpha}{1_{^*\mathbb{R}_c^\#} - r}\left(\widetilde{\{r^n\}}_{n=1}^{n=\infty} - r^m\right) > 0_{^*\mathbb{R}_c^\#}. \quad (22.21)$$

Example 22.3 Consider hyper infinite sequence: $,\alpha r,\alpha r^2,\ldots.\alpha r^n,\ldots,n\in\mathbb{N},r<1.$
$n\in{}^*\mathbb{N}$.Type equation here.

$$\widehat{Ext}-\sum_{n\in{}^*\mathbb{N}}\alpha r^n\triangleq\#\text{- }\lim_{m\to{}^*\infty}\left(\widehat{Ext}-\sum_{n=0}^{n=m}\alpha r^n\right)=\alpha\frac{1_{{}^*\widetilde{\mathbb{R}}_c^\#}}{1_{{}^*\widetilde{\mathbb{R}}_c^\#}-r}-\left(\#\text{-}\lim_{m\to{}^*\infty}\alpha\left\{\widetilde{\frac{r^n}{1_{{}^*\mathbb{R}_c^\#}-r}}\right\}_{n=1}^{n=m}\right)=$$

$$=\alpha\frac{1_{{}^*\widetilde{\mathbb{R}}_c^\#}}{1_{{}^*\mathbb{R}_c^\#}-r}.\tag{22.22}$$

From (22.17) and (22.22) we obtain

$$\widehat{Ext}\text{ -}\sum_{n\in{}^*\mathbb{N}}\alpha r^n=\widetilde{\{r^n\}}_{n=1}^{n=\infty}>0_{{}^*\widetilde{\mathbb{R}}_c^\#}.\tag{22.23}$$

Definition 22.20 Let $s,t\in{}^*\widetilde{\mathbb{R}}_c^\#$, so there are Cauchy hyper infinite sequences $\{\Psi_n\}_{n=0}^{^*\infty}$ and $\{\Phi_n\}_{n=0}^{^*\infty}$ with $s=\{\Psi_n\}_{n=0}^{^*\infty}$ and $t=\{\Phi_n\}_{n=0}^{^*\infty}$.
(i) Define $s+t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n+\Phi_n\}_{n=0}^{^*\infty}$.
(ii) Define $s\times t$ to be the equivalence class of the hyper infinite sequence $\{\Psi_n\times\Phi_n\}_{n=0}^{^*\infty}$.
Proposition 22.5 The operations $+,\times$ in Definition 22.18 (i),(ii) are well-defined.
Proof. (i) Suppose that $\{\Psi_n\}_{n=0}^{^*\infty}=\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ and $\{\Phi_n\}_{n=0}^{^*\infty}=\{\Phi_{1,n}\}_{n=0}^{^*\infty}$ .Thus means that $\Psi_n-\Psi_{1,n}\to_\#0_{{}^*\mathbb{R}_c^\#}$, and $\Phi_n-\Phi_{1,n}\to_\#0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$. Then $(\Psi_n+\Phi_n)-(\Psi_{1,n}+\Phi_{1,n})=(\Psi_n-\Psi_{1,n})+(\Phi_n-\Phi_{1,n})$. Now, using the familiar $\frac{\varepsilon}{2}$ trick, you can construct a proof that this #-tends to $0_{{}^*\mathbb{R}_c^\#}$, and so $\left[\{\Psi_n+\Phi_n\}_{n=0}^{^*\infty}\right]=\left[\{\Psi_{1,n}+\Phi_{1,n}\}_{n=0}^{^*\infty}\right]$.
(ii) Again, suppose that $\{\Psi_n\}_{n=0}^{^*\infty}=\{\Psi_{1,n}\}_{n=0}^{^*\infty}$ and $\{\Phi_n\}_{n=0}^{^*\infty}=\{\Phi_{1,n}\}_{n=0}^{^*\infty}$. We wish to show that $\left[\{\Psi_n\times\Phi_n\}_{n=0}^{^*\infty}\right]=\left[\{\Psi_{1,n}\times\Phi_{1,n}\}_{n=0}^{^*\infty}\right]$,or, in other words, that $\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}\to_\#0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$. Well, we add and subtract one of the other cross terms, say $\Phi_n\times\Psi_{1,n}$:

$$\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}=\Psi_n\times\Phi_n+(\Phi_n\times\Psi_{1,n}-\Phi_n\times\Psi_{1,n})-\Psi_{1,n}\times\Phi_{1,n}=(\Psi_n\times\Phi_n-\Phi_n\times\Psi_{1,n})+$$

$$+(\Phi_n\times\Psi_n-\Phi_n\times\Psi_{1,n})+(\Phi_n\times\Psi_{1,n}-\Psi_{1,n}\times\Phi_{1,n})=\Phi_n\times(\Psi_n-\Psi_{1,n})+\Psi_{1,n}\times(\Phi_n-\Phi_{1,n}).$$

Hence, we have $|\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}|\leq|\Phi_n|\times|\Psi_n-\Psi_{1,n}|+|\Psi_{1,n}|\times|\Phi_n-\Phi_{1,n}|\leq R(|\Psi_n-\Psi_{1,n}|+|\Phi_n-\Phi_{1,n}|)$. Now, noting that both $\Psi_n-\Psi_{1,n}$ and $\Phi_n-\Phi_{1,n}$ #-tend to $0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$ and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2R$, we see that $\Psi_n\times\Phi_n-\Psi_{1,n}\times\Phi_{1,n}\to_\#0_{{}^*\mathbb{R}_c^\#}$ as $n\to{}^*\infty$.
Theorem 22.3 (i) Given any hyperreal number $s\in{}^*\widetilde{\mathbb{R}}_c^\#,s=\{\Psi_n\}_{n=0}^{^*\infty}$ such that $s\neq\left[\widetilde{0_{{}^*\mathbb{R}_c^\#}}\right]$ and there is N such that for all $n>N,\Psi_n$ is multiplicative invertible, then there is a hyperreal number $t\in{}^*\widetilde{\mathbb{R}}_c^\#$ such that $s\times t=\widetilde{1_{{}^*\mathbb{R}_c^\#}}\triangleq1_{{}^*\widetilde{\mathbb{R}}_c^\#}$. (ii) Given any hyperreal number $s\in{}^*\widetilde{\mathbb{R}}_c^\#,s=\{\Psi_n\}_{n=0}^{^*\infty}$ such that $s\neq\left[\widetilde{0_{{}^*\mathbb{R}_c^\#}}\right]$ and there is N such that for all $n>N,\Psi_n$ is multiplicative semi-invertible, then there is a hyperreal number $t\in{}^*\widetilde{\mathbb{R}}_c^\#$ such that $s\times t=[\omega_n]$, where $\omega_n\equiv1_{{}^*\mathbb{R}_c^\#}$ if $n=n_i$ and $\omega_n\equiv0_{{}^*\mathbb{R}_c^\#}$ if $n\neq n_i,i\in{}^*\mathbb{N}$.
Proof. (i) First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\widetilde{0_{{}^*\mathbb{R}_c^\#}}\triangleq0_{{}^*\widetilde{\mathbb{R}}_c^\#}$. In other words, $s=\{\Psi_n\}_{n=0}^{^*\infty}$ where $\{\Psi_n\}_{n=0}^{^*\infty}-0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ does not #-converge to $0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ as $n\to{}^*\infty$. From this, we are to deduce the existence of a hyperreal number $t=\{\Phi_n\}_{n=0}^{^*\infty}$ such that $s\times\ t=\left[\{\Psi_n\times\Phi_n\}_{n=0}^{^*\infty}\right]$ is the same equivalence class as $1_{{}^*\widetilde{\mathbb{R}}_c^\#}$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers from ${}^*\mathbb{R}_c^\#$ have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{\Psi_n\}_{n=0}^{^*\infty}$ does not #-tend to $0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ as $n\to{}^*\infty$, there's no reason any number of the terms in $\{\Psi_n\}_{n=0}^{^*\infty}$ can't equal $0_{{}^*\mathbb{R}_c^\#}$. However, it turns out that eventually, $\{\Psi_n\}_{n=0}^{^*\infty}\neq0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ by

Lemma 22.1.

Proof. (ii) Immediately from definitions and by Lemma 22.1.

Lemma 22.1 If $\{\Psi_n\}_{n=0}^{*\infty}$ is a Cauchy hyper infinite sequence which does not #-tend to $0_{^*\mathbb{R}_c^\#}$, then there is an $N \in {}^*\mathbb{N}$ such that: (i) for all $n > N$, $\Psi_n \neq 0_{^*\mathbb{R}_c^\#}$ or (ii) there is hyper infinite subsequence $\{\Psi_{n_i}\}_{n_i=0}^{*\infty} \subset \{\Psi_n\}_{n=0}^{*\infty}$ such that $\{\Psi_{n_i}\}_{n_i=M}^{*\infty} \neq 0_{^*\mathbb{R}_c^\#}$, $i \in {}^*\mathbb{N}, M \in {}^*\mathbb{N}\backslash\mathbb{N}$.

We will now use it to complete the proof of Theorem 22.3.

Let $N \in {}^*\mathbb{N}$ be such that $\Psi_n \neq 0_{^*\mathbb{R}_c^\#}$ for $n > N$. Define hyper infinite sequence $\{\Phi_n\}_{n=0}^{*\infty}$ of hyperreal numbers from ${}^*\mathbb{R}_c^\#$ as follows: for $n \leq N, \Phi_n = 0_{^*\mathbb{R}_c^\#}$, and for $n > N$, $\Phi_n = \frac{1_{^*\mathbb{R}_c^\#}}{\Psi_n}$. Thus we obtain hyper infinite sequence $\{\Phi_n\}_{n=0}^{*\infty} = \left(0_{^*\mathbb{R}_c^\#}, 0_{^*\mathbb{R}_c^\#}, \dots, 0_{^*\mathbb{R}_c^\#}, \frac{1_{^*\mathbb{R}_c^\#}}{\Psi_{N+1}}, \frac{1_{^*\mathbb{R}_c^\#}}{\Psi_{N+2}}, \dots\right)$. This makes sense since, for $n > N$, an is a nonzero hyperreal number, so $1_{^*\mathbb{R}_c^\#}/\Psi_n$ exists. Then $\Psi_n \times \Phi_n$ is equal to $\Psi_n \times 0_{^*\mathbb{R}_c^\#} = 0_{^*\mathbb{R}_c^\#}$ for $n \leq N$, and equals $\Psi_n \times \Phi_n = \Psi_n \times 1_{^*\mathbb{R}_c^\#}/\Psi_n = 1_{^*\mathbb{R}_c^\#}$ for $n > N$. Well, then, if we look at the hyper infinite sequence $\widetilde{1_{^*\mathbb{R}_c^\#}} \triangleq 1_{^*\widetilde{\mathbb{R}}_c^\#}$, we have $\widetilde{1_{^*\mathbb{R}_c^\#}} - \{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}$ is the hyper infinite sequence which is $\widetilde{1_{^*\mathbb{R}_c^\#}} - \widetilde{0_{^*\mathbb{R}_c^\#}} = 1_{^*\widetilde{\mathbb{R}}_c^\#}$ for $n \leq N$ and equals $1_{^*\widetilde{\mathbb{R}}_c^\#} - 1_{^*\widetilde{\mathbb{R}}_c^\#} = 0_{^*\widetilde{\mathbb{R}}_c^\#}$ for $n > N$. Since this hyper infinite sequence is eventually equal to $\widetilde{0_{^*\mathbb{R}_c^\#}}$, it #-converges to $0_{^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, and so $\left[\{\Psi_n \times \Phi_n\}_{n=0}^{*\infty}\right] = 1_{^*\widetilde{\mathbb{R}}_c^\#}$. This shows that $t = \left[\{\Phi_n\}_{n=0}^{*\infty}\right]$ is a multiplicative inverse to $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right]$.

Definition 22.21 Let $s \in {}^*\widetilde{\mathbb{R}}_c^\#$. Say that s is positive if $s \neq 0_{^*\widetilde{\mathbb{R}}_c^\#}$, and if $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right]$ for some Cauchy hyper infinite sequence such that for some $N, \Psi_n > 0_{^*\mathbb{R}_c^\#}$ for all $n > N$. Given two hyperreal numbers $s, t \in {}^*\widetilde{\mathbb{R}}_c^\#$, say that $s > t$ if $s - t$ is positive.

Theorem 22.3 Let $s, t \in {}^*\widetilde{\mathbb{R}}_c^\#$ be hyperreal numbers such that $s > t$, and let $r \in {}^*\widetilde{\mathbb{R}}_c^\#$. Then $s + r > t + r$.

Proof. Let $s = \left[\{\Psi_n\}_{n=0}^{*\infty}\right], t = \left[\{\Phi_n\}_{n=0}^{*\infty}\right]$ and $r = \left[\{\Theta_n\}_{n=0}^{*\infty}\right]$. Since $s > t$, i.e. $-t > 0_{^*\mathbb{R}_c^\#}$, we know that there is an $N \in {}^*\mathbb{N}$ such that, for $n > N$, $\Psi_n - \Phi_n > 0_{^*\mathbb{R}_c^\#}$ .So $\Psi_n > \Phi_n$ for $n > N$. Now, adding Θ_n to both sides of this inequality, we have $\Psi_n + \Theta_n > \Phi_n + \Theta_n$ for $n > N$, or $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) > 0_{^*\mathbb{R}_c^\#}$ for $n > N$. Note also that $(\Psi_n + \Theta_n) - (\Phi_n + \Theta_n) = \Psi_n - \Phi_n$ does not #-converge to $0_{^*\mathbb{R}_c^\#}$ as $n \to {}^*\infty$, by the assumption that $s - t > \widetilde{0_{^*\mathbb{R}_c^\#}}$. Thus, by Definition 22.19, this means that: $s + r = \left[\{\Psi_n + \Theta_n\}_{n=0}^{*\infty}\right] > \left[\{\Psi_n + \Phi_n\}_{n=0}^{*\infty}\right] = t + r$.

Remark 22.3 There is canonical imbedding

$$ {}^*\mathbb{R}_c^\# \hookrightarrow_i {}^*\widetilde{\mathbb{R}}_c^\# \tag{22.24}$$

defined by $i: a \mapsto [\tilde{a}]$.

23. Generalized convergence of hyper infinite sequences of #-closed operators

When we consider various perturbation problems related to #-closed operators, it is necessary to make precise what is meant by a "small" perturbation. This can be done in a most natural way by introducing a metric in the set $W(X, Y)$ of all #-closed operators from X to Y. If $T, S \in W(X, Y)$, their graphs $G(T), G(S)$ are #-closed linear manifolds in the product space $X \times Y$. Thus the "distance" between T and S can be measured by the "gap" between the closed linear manifolds. $G(T), G(S)$. In this way we are led to consider how to measure the gap of two #-closed linear manifolds of a non-Archimedean Banach space. In this paragraph we shall consider -closed linear manifolds $M, N, \dots$ of a non-Archimedean Banach space Z. We denote by S_M the unit sphere of M (the set of all $u \in M$ with $\|u\|_\# = 1$). For any two #-closed linear manifolds M, N of Z, we set

$$\delta(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, N), \tag{23.1}$$

$$\hat{\delta}(M, N) = \max[\delta(M, N), \delta(N, M)]. \tag{23.2}$$

Note that (23.1) has no meaning if $M\ =\emptyset$; in this case we define $\delta(0,N)=0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ for any N. On the other hand $\delta(M,0)=1_{{}^*\widetilde{\mathbb{R}}_c^\#}$ if $M\neq\emptyset$, as is seen from the definition. $\delta(M,N)$ can also be characterized as the smallest number δ such that

$$\operatorname{dist}(u,N)\leq\delta\|u\|_\# \tag{23.3}$$

for all $u\in M$.

Definition 23.1 The quantity $\hat{\delta}(M,N)$ is called the gap between M,N.

The following relations follow directly from the definition.

$$\delta(M,N)=0_{{}^*\widetilde{\mathbb{R}}_c^\#}\text{ if and only if }M\subset N. \tag{23.4}$$

$$\hat{\delta}(M,N)\ =0_{{}^*\widetilde{\mathbb{R}}_c^\#}\text{ if and only if }M=N. \tag{23.5}$$

$$\hat{\delta}\ (M,N)=\ \hat{\delta}\ (N,M). \tag{23.6}$$

$$0_{{}^*\widetilde{\mathbb{R}}_c^\#}\leq\delta(M,N)\leq 1_{{}^*\widetilde{\mathbb{R}}_c^\#},0_{{}^*\widetilde{\mathbb{R}}_c^\#}\leq\hat{\delta}(M,N)\ \leq 1_{{}^*\widetilde{\mathbb{R}}_c^\#}. \tag{23.7}$$

(23.5) and (23.6) suggest that $\delta(M,N)$ could be used to define a distance between M and N. But this is not possible, since the function $\delta(X,Y)$ does not in general satisfy the triangle inequality required of a distance function'.This inconvenience may be removed by modifying the definition (23.1)-(23.2). We set

$$d(M,N)=\sup_{u\in S_M}\operatorname{dist}(u,S_N), \tag{23.8}$$

$$\hat{d}(M,N)=\max[d(M,N),d(N,M)]. \tag{23.9}$$

(23.8) does not make sense if either M or N is 0. In such cases we set

$$d(0,N)=0_{{}^*\widetilde{\mathbb{R}}_c^\#}\text{ for any }N;d(M,0)=2_{{}^*\widetilde{\mathbb{R}}_c^\#}\text{ for }M\neq\emptyset. \tag{23.10}$$

Furthermore, d and $\hat{d}$ satisfy the triangle inequalities :

$$d(L,N)\leq d(L,M)+d(M,N),\hat{d}(L,N)\leq\hat{d}(L,M)+\hat{d}(M,N). \tag{23.11}$$

The second inequality of (23.11) follows from the first, which in turn follows easily from the definition. The proof will be left to the reader. The case when some of L,M,N are 0 should be considered separately; note (23.10). The set of all #-closed linear manifolds of Z becomes a metric space endowed with ${}^*\widetilde{\mathbb{R}}_c^\#$-valued metric if the distance between M,N is defined by $\hat{d}(L,N)$.

Definition 23.2 A hyper infinite sequence M_n,$n\in{}^*\mathbb{N}$ of #-closed linear manifolds #-converges to M if $\hat{d}\ (M_n,M)\to_\#\ 0_{{}^*\widetilde{\mathbb{R}}_c^\#}$ as $n\to{}^*\infty$. Then we write $M_n\to_\# M$ or #-$\lim_{n\to{}^*\infty}M_n=\ M$.

Remark 23.1 Although the gap $\hat{\delta}$ is not a proper distance function, it is more convenient than the proper distance function $\hat{d}$ for applications since its definition is slightly simpler. Furthermore, when we consider the topology of the set of all closed linear manifolds, the two functions give the same result. This is due to the following inequalities

$$\delta(M,N)\leq d(M,N)\leq 2\delta(M,N), \tag{23.12}$$

$$\hat{\delta}(M,N)\leq\hat{d}(M,N)\leq 2\hat{\delta}(M,N). \tag{23.13}$$

We set

$$\delta(T,S)=\delta(G(T),G(S))\,,\hat{\delta}(T,S)=\hat{\delta}(G(T),G(S))=\max[\delta(T,S),\delta(S,T)]. \tag{23.14}$$

The quantity $\hat{\delta}(T,S)$ will be called the gap between T and S. Similarly we can define the distance $\hat{d}(T,S)$ between T and S as equal to$\hat{d}(G(T),G(S))$. Under this distance function $W(X,Y)$ becomes a non-Archimedean metric space endowed with ${}^{*}\widetilde{\mathbb{R}}_{c}^{\#}$-valued metric. In this space the #-convergence of a hyper infinite sequence $T_n \in W(X,Y)$ to a $T \in W(X,Y)$ as $n \to {}^{*}\infty$ is defined by $\hat{d}(T_n,T) \to_{\#} 0_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}$. But since $\hat{\delta}(T,S) \leq \hat{d}(T,S) \leq 2\hat{\delta}(T,S)$ in virtue of (23.13), this is true if and only if $\hat{\delta}(T_n,T) \to_{\#} 0_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}$ as $n \to {}^{*}\infty$.

Definition 23.3 Let $T_n \in W(X,Y), n \in {}^{*}\mathbb{N}$. If $\hat{\delta}(T_n,T) \to_{\#} 0_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}$ as $n \to {}^{*}\infty$ we shall say that the operator T_n #-converges to T or $T_n \to_{\#} T$ in the generalized sense.

Theorem 23.1 Let $T: H^{\#} \to H^{\#}$ be a #-selfadjoint operator. Then there is a $\delta > 0$ such that any #-closed symmetric operator S with $\hat{\delta}(S,T) < \delta$ is #-selfadjoint, where $\hat{\delta}(S,T)$ denotes the gap between S and T.

Corollary 23.1 Let T, T_n. be #-closed symmetric operators and let hyper infinite $T_n, n \in {}^{*}\mathbb{N}$. #-converge to T in the generalized sense. If T is #-selfadjoint, then T_n is #-selfadjoint for sufficiently large $n \in {}^{*}\mathbb{N}$.

24. Strong convergence of the resolvent

Let $T_n, n \in {}^{*}\mathbb{N}, T_n \in W(X,X)$ be a hyper infinite sequence of #-closed operators in a non-Archimedean Banach space X. In this section we are briefly concerned with general considerations on strong #-convergence of the resolvents $R_n(\xi) = (T_n - \xi)^{-1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}}$. We remind the fundamental result on the #-convergence in #-norm of the resolvents: if $R_n(\xi)$ #-converges in #-norm to the resolvent $R(\xi) = (T - \xi)^{-1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}}$ of a #-closed operator T for some $\xi \in P(T)$, then the same is true for every $\xi \in P(T)$ (see Theorem 23.2 and Remark 23.2). There is no corresponding theorem for strong #-convergence of the resolvents. Nevertheless, we can prove several theorems on the set of points ξ where the $R_n(\xi)$are strongly #-convergent or bounded. Let us define the region of boundedness, denoted by Δ_b, for the hyper infinite sequence $R_n(\xi)$ as the set of all complex numbers $\xi \in {}^{*}\widetilde{\mathbb{C}}_{c}^{\#}$ such that $\xi \in P(T_n)$ for sufficiently large $n \in {}^{*}\mathbb{N}$ and the sequence $\| R_n(\xi)\|_{\#}$ is bounded [for n so infinite large that the $R_n(\xi)$ are defined]. Furthermore, let Δ_s be the set of all ξ such that s-#-$\lim_{n \to {}^{*}\infty} R_n(\xi) = R'(\xi)$ exists. Δ_s will be called the region of strong #-convergence for $R_n(\xi)$. Similarly we define the region Δ_b of #-convergence in #-norm for $R_n(\xi)$. Obviously we have $\Delta_u \subset \Delta_s \subset \Delta_b$.

Theorem 24.1 Δ_b is an #-open set in the complex plane ${}^{*}\widetilde{\mathbb{C}}_{c}^{\#}$. Operator $R_n(\xi)$ is bounded in ${}^{*}\widetilde{\mathbb{R}}_{c}^{\#}$ uniformly in n and ξ in any #-compact subset of Δ_b.

Proof. Let $\xi_0 \in \Delta_b$; for $|\xi - \xi_0| < 1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}/\| R_n(\xi_0)\|_{\#} = (\| R_n(\xi_0)\|_{\#})^{-1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}}$ we have the Neumann hyper infinite series

$$R_n(\xi) = Ext\text{-}\sum_{k=0}^{{}^{*}\infty}(\xi - \xi_0)\, [\| R_n(\xi_0)\|_{\#}]^{k}. \tag{24.1}$$

If $\| R_n(\xi_0)\|_{\#} \leq M_0$, then $\| R_n(\xi_0)\|_{\#} \leq M_0(1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}} - M_0 |\xi - \xi_0|^{-1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}})$ for $|\xi - \xi_0| < M_0^{-1_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}}$ The theorem follows immediately.

Remark 24.1 Theorem 24.1 implies that Δ_b consists of at most a hyper infinite number of #-connected #-open sets $\Delta_{b_1}, \Delta_{b_2}, \ldots$, (the components of Δ_b).

Theorem 24.2 Δ_s is relatively #-open and #-closed in Δ_b (so that Δ_s is the union of some of the components Δ_{b_k} of Δ_b). The strong #-convergence $R_n(\xi) \to_{\#} R(\xi)$ is uniform in each #-compact subset of Δ_s.

Remark 24.2 For convenience we call Δ_b also the region of boundedness for the sequence if there is no possibility of confusion. Similarly for Δ_s and Δ_b.

Remark 24.3 The strong #-convergence $R_n(\xi) \to_{\#} R(\xi)$ is uniform in ξ if $\|R_n(\xi) - R(\xi)\|_{\#} \to_{\#} 0_{{}^{*}\widetilde{\mathbb{R}}_{c}^{\#}}$ uniformly in ξ for each fixed $u \in X$.

Remark 24.4 The strong #-limit R′(ξ) of $R_n(\xi)$ for $\xi \in \Delta_s$ need not be the resolvent of an operator. In any case, however, $R'(\xi)$ satisfies the resolvent equation

$$R'(\xi_1) - R'(\xi_2) = (\xi_1 - \xi_2)R'(\xi_1)R'(\xi_2), \xi_1, \xi_2 \in \Delta_s, \tag{24.2}$$

as the strong #-limit of operators $R_n(\xi)$ which satisfy the same equation.

Definition 24.1 For this reason R′(ξ) is called a pseudo-resolvent. Note that $R'(\xi_1)$ and $R'(\xi_2)$ commute.

Remark 24.5 The Eq.(24.2) implies that the null space $N = N(R'(\xi))$ and the range $R = R(R'(\xi))$ of $R'(\xi)$ are independent of ξ. In fact, it follows from Eq.(24.2) that $R'(\xi_2)u = 0$ implies $R'(\xi_1)u = 0$ and that $u = R'(\xi_1)v$ implies $u = R'(\xi)w$ with $w = v - (\xi_1 - \xi_2)u$.
Remark 24.6 The pseudo-resolvent $R'(\xi)$ is a resolvent of an #-closed operator T if and only if $N = 0$.
Theorem 24.3 Let Δ_s be nonempty. There are the alternatives: either $R'(\xi)$ is invertible for no $\xi \in \Delta_s$ or $R'(\xi)$ is equal to the resolvent $R(\xi) = (T - \xi)^{-1_{\widetilde{{}^*\mathbb{R}_c^\#}}}$ of a unique operator $T \in W(X, X)$. In the latter case we have $\Delta_s = P(T) \cap \Delta_b$.
Proof. Only the last statement remains to be proved. We have $\Delta_s \subset P(T) \cap \Delta_b\}$ since $\Delta_s \subset P(T)$. To prove opposite inclusion, we note the identity

$$R_n(\xi) - R(\xi) = (1_{\widetilde{{}^*\mathbb{R}_c^\#}} + (\xi - \xi_0) R_n(\xi))(R_n(\xi_0) - R(\xi_0))(1_{\widetilde{{}^*\mathbb{R}_c^\#}} + (\xi - \xi_0)R(\xi)) \tag{24.3}$$

for $\xi, \xi_0 \in P(T) \cap \Delta_b$ this is a simple consequence of the resolvent equations for $R_n(\xi)$ and $R(\xi)$. If $\xi_0 \in \Delta_s$, we have s-#-lim $R_n(\xi_0) = R'(\xi_0) = R(\xi_0)$ so that Eq.(24.3) gives s-#-lim $R_n(\xi) = R(\xi)$ by the uniform boundedness of hyper infinite sequence $R_n(\xi), n \in {}^*\mathbb{N}$.This shows that $\xi \in \Delta_s$ and completes the proof.
Corollary 24.1 Let $T_n, n \in {}^*\mathbb{N}$ and T be sebe #-selfadjoint operators in a non-Archimedean Hilbert space, with the resolvents $R_n(\xi)$ and $R(\xi)$. If s-#-$\lim_{n\to{}^*\infty} R_n(\xi) = R(\xi)$ for some complex number $\xi \in {}^*\tilde{\mathbb{C}}_c^\#$, then the same is true for every nonreaal $\xi \in {}^*\tilde{\mathbb{C}}_c^\#$.
Proof. Since $\| R_n(\xi) \|_\# \leq 1_{\widetilde{{}^*\mathbb{R}_c^\#}}/|\mathrm{Im}\xi|$, all nonreal numbers ξ are included in Δ_b as well as in $P(T)$. Thus $P(T) \cap \Delta_b$ also includes all nonreal ξ, and the assertion follows from Theorem 24.3.
Definition 24.2 When the second alternative of Theorem 24.3 is realized, we shall say that $R_n(\xi)$ #-converges strongly to $R(\xi)$ on Δ_s, and that T_n #-converges strongly to T in the generalized sense.
A criterion for generalized strong convergence is given by:
Theorem 24.5 Let T_n,$T \in W(X, X)$ and let there be a #-core D of T such that each $u \in D$ belongs to $D(T_n)$ for sufficiently infinite large n and $T_n u \to_\# Tu$. If $P(T) \cap \Delta_b$ is not empty, T_n #-converges strongly to T in the generalized sense and $\Delta_s = P(T) \cap \Delta_b$.

25. Conclusion

A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the canonical C^*- algebra of bounded observables corresponding to this model satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant. For each Poincare transformation a, Λ and each bounded region O of Minkowski space we obtain a unitary operator U which correctly transforms the field bilinear forms $\varphi(x,t)$ for $(x,t) \in O$. The von Neumann algebra $\mathfrak{C}(O)$ of local observables is obtained as standard part of external nonstandard algebra $\mathcal{B}_\#(O)$.
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