Note On The Transformation From STT or Local Coordinate System To The NTT Coordinate System

Abdelmajid Ben Hadj Salem, Expert Geodesist

- Final Version - 17/10/2024

Abstract

In this note, we give the details of how to calculate the parameters of the transformation from STT or local coordinate system to the NTT coordinate system the Tunisian geodetic terrestrial system.

Résumé

Dans cette note, nous donnons les détails de la façon de calculer les paramètres de la transformation du système géodésique de coordonnées STT ou des systèmes de coordonnées locales au système de coordonnées NTT, le système géodésique terrestre tunisien.

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Contents

1. Introduction

The purpose of this note is to provide detailed procedures and explanations of:

1- The transformation of the coordinates of points of the STT geodetic coordinate system to the NTT geodetic system.

2- The transformation of the coordinates of points of a local coordinate system to the NTT geodetic system.

We remark that only the Carthage34 and Carthage86 systems use STT coordinates. The NTT geodetic system uses UTM coordinates.

2. Transforming coordinates of points from the STT Geodetic System to the NTT Geodetic System

2.1. Notations

Let be a point $M(x,y)$ with plane coordinates (x,y) in the STT system. It is represented in the figure (Fig.1), $O'x$ to the North and $O'y$ to the West, O' is the origin of the $O'xy$ coordinate system. The coordinates are taken from the formulas of the plane representation Lambert-North Tunisia or Lambert-South Tunisia.

We have created an orthonormal coordinate system (O,X,Y) such that the OX,OY are oriented respectively to the East and North and the coordinates of $M(X,Y)$ in this coor-

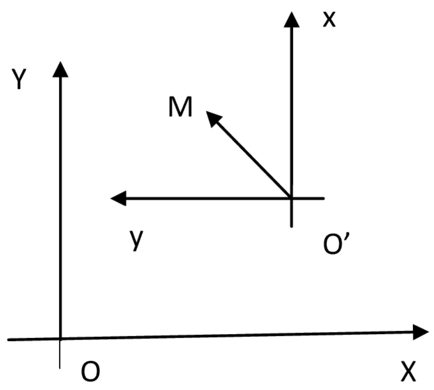

Figure 1 The coordinates STT (x, y)

dinate system are given by the following formulas:

$$\begin{cases} X = 500\,000.00\,m - y \\ Y = 300\,000.00\,m + x \end{cases} \tag{1}$$

$500,000.00, m$ and $300,000.00, m$ denote the constants X and Y. These constants are identical for the representations mentioned above Lambert-North Tunisia or Lambert-South Tunisia and they allow us to have positive coordinates (X, Y).

In the terrestrial geodetic system of NTT coordinates, we denote the plane coordinates of the point M by (X', Y') obtained from the plane representation UTM Tunisia (zone 32) with the longitude origin 3° East of Greenwich.

2.2. The Choose of how to transform the coordinates of points from the STT Geodetic System to the NTT Geodetic System

Having adopted the NTT terrestrial geodetic system, the secondary network was calculated based on the primordial geodetic points. The calculation was made by sheet at a scale of 1 : 50 000 of the Lambert cartographic division. In each sheet, there is generally a large number of known points (more than 20) with coordinates in NTT and Carthage34 or Carthage86.

Let the sheet $n°I$, N the number of common points, and we note their coordinates by $(X_k, Y_k)_{k=1,N}$ and $(X'_k, Y'_k)_{k=1,N}$ in Carthage34 (or Carthage86) and in NTT.

To maintain the conformity of the parcels obtained by the coordinates resulting from the transformations, the OTC has chosen the third-degree polynomial transformations, i.e.:

$$Z = Z(z) = \sum_{n=1}^{n=3} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \tag{2}$$

$$z = X + iY \ \ (STT) \Longrightarrow Z = X' + iY' \ \ (NTT) \tag{3}$$

$$a_n = \alpha_n + i\beta_n \quad \text{complex constants and } \ \alpha_n, \beta_n \ \text{ reals} \tag{4}$$

2.3. Calculation of the parameters of the chosen transformation

From the equations (2,3,4) and let M_1 be the first common point, we obtain the two following equations :

$$X' = \alpha_0 + \alpha_1 X - \beta_1 Y + \alpha_2 X^2 - 2\beta_2 XY - \alpha_2 Y^2 + \alpha_3 X^3 - 3\beta_3 X^2 Y - 3\alpha_3 XY^2 + \beta_3 Y^3$$
$$Y' = \beta_0 + \beta_1 X + \alpha_1 Y + \beta_2 X^2 + 2\alpha_2 XY - \beta_2 Y^2 + \beta_3 X^3 + 3\alpha_3 X^2 Y - 3\beta_3 XY^2 - \alpha_3 Y^3$$

or

$$X' = \alpha_0 + \alpha_1 X - \beta_1 Y + \alpha_2(X^2 - Y^2) - 2\beta_2 XY + \alpha_3(X^3 - 3XY^2) + \beta_3(Y^3 - 3X^2Y) \tag{5}$$

$$Y' = \beta_0 + \alpha_1 Y + \beta_1 X + 2\alpha_2 XY + \beta_2(X^2 - Y^2) + \alpha_3(3X^2Y - Y^3) + \beta_3(X^3 - 3XY^2) \tag{6}$$

The unknowns are $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$, so 8 parameters to determine. The equations (5,6) are written in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & X & -Y & X^2 - Y^2 & -2XY & X^3 - 3XY^2 & Y^3 - 3X^2Y \\ 0 & 1 & Y & X & 2XY & X^2 - Y^2 & 3X^2Y - Y^3 & X^3 - 3XY^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} X' \\ Y' \end{pmatrix} \tag{7}$$

The number of unknowns is greater than the number of equations, so we proceed by the least-squares method using the N common points. For example, for the first point, we have the relationship:

$$\begin{pmatrix} 1 & 0 & X_1 & -Y_1 & X_1^2 - Y_1^2 & -2X_1Y_1 & X_1^3 - 3X_1Y_1^2 & Y_1^3 - 3X_1^2Y_1 \\ 0 & 1 & Y_1 & X_1 & 2X_1Y_1 & X_1^2 - Y_1^2 & 3X_1^2Y_1 - Y_1^3 & X_1^3 - 3X_1Y_1^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} X_1' \\ Y_1' \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{8}$$

and (v_1, v_2) the residues for point 1.
Writing the previous equation for N points, we get the least-squares equation:

$$A.U = L + V \tag{9}$$

The matrix of coefficients is $A = {}_{2N}A_8$, the vector of unknowns $U = {}_8U_1$, the vector of observables $L = {}_{2N}L_1$ and the vector of residues $V = {}_{2N}V_1$, we successively obtain the matrix A:

$$A=\begin{pmatrix}
1 & 0 & X_1 & -Y_1 & X_1^2-Y_1^2 & -2X_1Y_1 & X_1^3-3X_1Y_1^2 & Y_1^3-3X_1^2Y_1\\
0 & 1 & Y_1 & X_1 & 2X_1Y_1 & X_1^2-Y_1^2 & X_1^3-3X_1Y_1^2 & Y_1^3-3X_1^2Y_1\\
1 & 0 & X_2 & -Y_2 & X_2^2-Y_2^2 & -2X_2Y_2 & X_2^3-3X_2Y_2^2 & Y_2^3-3X_2^2Y_2\\
0 & 1 & Y_2 & X_2 & 2X_2Y_2 & X_2^2-Y_2^2 & 3X_2^2Y_2-Y_2^3 & X_2^3-3X_2Y_2^2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 0 & X_i & -Y_i & X_i^2-Y_i^2 & -2X_iY_i & X_i^3-3X_iY_i^2 & Y_i^3-3X_i^2Y_i\\
0 & 1 & Y_{i+1} & X_{i+1} & 2X_{i+1}Y_{i+1} & X_{i+1}^2-Y_{i+1}^2 & 3X_{i+1}^2Y_{i+1}-Y_{i+1}^3 & X_{i+1}^3-3X_{i+1}Y_{i+1}^2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 0 & X_{2N-1} & -Y_{2N-1} & X_{2N-1}^2-Y_{2N-1}^2 & -2X_{2N-1}Y_{2N-1} & X_{2N-1}^3-3X_{2N-1}Y_{2N-1}^2 & Y_{2N-1}^3-3X_{2N-1}^2Y_{2N-1}\\
0 & 1 & Y_{2N} & X_{2N} & 2X_{2N}Y_{2N} & X_{2N}^2-Y_{2N}^2 & 3X_{2N}^2Y_{2N}-Y_{2N}^3 & X_{2N}^3-3X_{2N}Y_{2N}^2
\end{pmatrix}$$

$$U=\begin{pmatrix}\alpha_0\\ \beta_0\\ \alpha_1\\ \beta_1\\ \alpha_2\\ \beta_2\\ \alpha_3\\ \beta_3\end{pmatrix};\quad L=\begin{pmatrix}X'_1\\ Y'_1\\ X'_2\\ Y'_2\\ \vdots\\ X'_i\\ Y'_{i+1}\\ \vdots\\ X'_{2N-1}\\ Y'_{2N}\end{pmatrix},\quad V=\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_i\\ v_{i+1}\\ \vdots\\ v_{2N-1}\\ v_{2N}\end{pmatrix} \tag{10}$$

The least-squares solution is obtained by minimizing the sum of the squares of the residuals, i.e. $\sum_i v_i^2 = V^t.V = ||V||^2$. Now the squared norm of the residual vector is the scalar function as a function of the U vector of the unknowns, i.e.:

$$F(U) = V^t.V = ||V||^2 = (A.U - L)^t.(A.U - L) =$$
$$(U^tA^t - L^t).(A.U - L) = U^t.(A^t.A).U - 2L^t.A.U + L^t.L \tag{11}$$

We denote $N = A^t.A = {}_8N_8$, this matrix is square called the normal matrix. It is invertible because the matrix N is defined positive, i.e. Let W be a vector 8×1, then $W^T.N.W = W^t.A^t.A.W = (A.W)^t.(A.W) = ||W||_A^2 \geq 0$ by defining a norm by $||W||_A$ the usual norm of the vector $A.W$, but a norm checks whether $||H||_A = 0 \Longrightarrow H = 0$. It follows that if R verifies $N.R = G$, then this equation has a single solution equal to $R = N^{-1}.G$.

It is shown that $min\, F(U) \Longrightarrow dF(U) = 0$. Recall that if $y = X.X = X^t.X = ||X||^2$, then $dy = 2X^t.dX = 2dX^t.X$, as a result:

$$dF = d(||A.U||^2 - 2L^t.A.U + ||L||^2)$$

The vector L and the matrix A are independent of the vector U, so we obtain:

$$dF(U) = 2.(A.dU)^t.(A.U) - 2L^t.A.dU = 2dU^t(A^tA).U - 2dU^tA^t.L = 2dU^t.(N.U - A^t.L) \tag{12}$$

then:

$$dF(U) = 0 \Longrightarrow N.U - A^tL = 0 \Longrightarrow U = N^{-1}A^t.L = (A^t.A)^{-1}.A^t.L \tag{13}$$

We find the solution of the least squares method:

$$\boxed{\overline{U} = \begin{pmatrix} \bar{\alpha_0} \\ \bar{\beta_0} \\ \bar{\alpha_1} \\ \bar{\beta_1} \\ \bar{\alpha_2} \\ \bar{\beta_2} \\ \bar{\alpha_3} \\ \bar{\beta_3} \end{pmatrix} = (A^t.A)^{-1}.A^t.L} \tag{14}$$

Thus the 8 parameters are calculated. We compute the vector of the residuals $V = A.\overline{U} - L$, we keep the points used having $|v_j| \leq 0.10\,m$ and we iterate the process. Let be a point P with STT coordinates (x_P, y_P). We calculate its coordinates (X_P, Y_P) by the formulas (1). Its coordinates (X'_P, Y'_P) in the NTT system are given by:

$$\begin{pmatrix} X'_P \\ Y'_P \end{pmatrix} = \begin{pmatrix} 1 & 0 & X_P & -Y_P & X_P^2 - Y_P^2 & -2X_PY_P & X_P^3 - 3X_PY_P^2 & Y_P^3 - 3X_P^2Y_P \\ 0 & 1 & Y_P & X_P & 2X_PY_P & X_P^2 - Y_P^2 & 3X_P^2Y_P - Y_P^3 & X_P^3 - 3X_PY_P^2 \end{pmatrix} . \begin{pmatrix} \bar{\alpha_0} \\ \bar{\beta_0} \\ \bar{\alpha_1} \\ \bar{\beta_1} \\ \bar{\alpha_2} \\ \bar{\beta_2} \\ \bar{\alpha_3} \\ \bar{\beta_3} \end{pmatrix}$$

3. Transforming coordinates of points from a local system to the NTT geodetic system

For this case, considering that the area subject to the coordinate transformation is generally very small with an area of the order of $100 - 400\,km^2$, the transformation used is the two-dimensional Helmert transformation. This transformation is written in vector form:

$$\boxed{\boldsymbol{X'} = T + s.R(\theta).\boldsymbol{X}} \tag{15}$$

where:

- $\boldsymbol{X}$ is the vector of components $(X, Y)^t$ and (X, Y) denotes the coordinates in the local system.

- $\boldsymbol{X'}$ is the component vector $(X', Y')^t$, (X', Y') denote the planimetric coordinates in the NTT system.

- T is the translation vector of components $(T_x, T_y)^t$ between the local and NTT systems.

- s is the scaling factor between the 2 systems.

- $R(\theta)$ is the rotation matrix 2×2 to switch from the local system to the NTT system.

Developing (15), we obtain:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} T_x \\ T_y \end{pmatrix} + s \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} \tag{16}$$

Taking as auxiliary unknowns:

$$v = s.sin\theta \tag{17}$$

$$u = s.cos\theta \tag{18}$$

the system (16) becomes :

$$\begin{aligned} X' &= T_x + X.u - Y.v \\ Y' &= T_y + X.v + Y.u \end{aligned} \tag{19}$$

The unknowns T_x, T_y, u and v will be determined by the least-squares method using the common points in both systems. Having u and v, it follows :

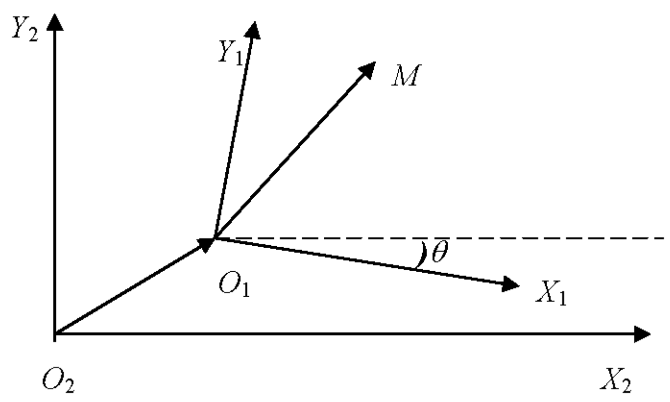

Figure 2 Helmert Model

$$\begin{cases} s = \sqrt{u^2 + v^2} \\ tg\theta = \dfrac{v}{u} \end{cases} \tag{20}$$

3.1. Least-Squares Resolution

We solve the system (19) by the least-squares method. We assume the data of N common points between the local system and NTT:

- $(X'_i, Y'_i)_{i=1,N}$ in the NTT system.
- $(X_i, Y_i)_{i=1,N}$ in the local system.

We denote:

$$\overline{X} = \frac{\sum_1^N X_i}{N}, \quad \overline{Y} = \frac{\sum_1^N Y_i}{N}, \quad \overline{X'} = \frac{\sum_1^N X'_i}{N}, \quad \overline{Y'} = \frac{\sum_1^N Y'_i}{N} \tag{21}$$

the respective coordinates of the centers of gravity of the common points, in the local and NTT systems.

We denote also:

$$\begin{aligned} x_i &= X_i - \overline{X} \\ y_i &= Y_i - \overline{Y} \\ x'_i &= X'_i - \overline{X'} \\ y'_i &= Y'_i - \overline{Y'} \end{aligned} \tag{22}$$

In this case, the system (19) becomes:

$$\begin{aligned} x'_i &= T_x + x_i.u - y_i.v \\ y'_i &= T_y + x_i.v + y_i.u \end{aligned} \tag{23}$$

Let (T_x^0, T_y^0, u_0, v_0) be an approach solution of the system. Then we denote:

$$\begin{aligned} T_x &= T_x^0 + dt_x \\ T_y &= T_y^0 + dt_y \\ u &= u_0 + du \\ v &= v_0 + dv \end{aligned} \tag{24}$$

It follows the equations (23) are written as :

$$\begin{aligned} x'_i &= T_x^0 + dt_x + x_i.(u_0 + du) - y_i.(v_0 + dv) \\ y'_i &= T_y^0 + dt_y + x_i.(v_0 + dv) + y_i.(u_0 + du) \end{aligned} \tag{25}$$

We write these last 2 equations in the form of the least-squares equation:

$$A.U = L + W \tag{26}$$

with U the vector of the unknowns:

$$U = \begin{pmatrix} dt_x \\ dt_y \\ du \\ dv \end{pmatrix} \tag{27}$$

L is the observable vector:

$$L = \begin{pmatrix} x'_1 - T_x^0 - x_1 u_0 + y_1 v_0 \\ y'_1 - T_y^0 - x_1 v_0 - y_1 u_0 \\ \vdots \\ x'_i - T_x^0 - x_i u_0 + y_i v_0 \\ y'_i - T_y^0 - x_i v_0 - y_i u_0 \\ \vdots \\ x'_n - T_x^0 - x_n u_0 + y_n v_0 \\ y'_n - T_y^0 - x_n v_0 - y_n u_0 \end{pmatrix} \tag{28}$$

W the residues vector:

$$W = \begin{pmatrix} w_{x_1} \\ w_{y_1} \\ \vdots \\ w_{x_n} \\ w_{y_n} \end{pmatrix} \tag{29}$$

and A the matrix of the coefficients:

$$A = \begin{pmatrix} 1 & 0 & x_1 & -y_1 \\ 0 & 1 & y_1 & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_i & -y_i \\ 0 & 1 & y_i & x_i \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_N & -y_N \\ 0 & 1 & y_N & x_N \end{pmatrix} \tag{30}$$

3.2. The Solution by the least-squares

The solution of (26) by the least-squares method gives:

$$\overline{U} = (A^t A)^{-1} A^t L \tag{31}$$

We denote:

$$N = A^t A \tag{32}$$

which is called the normal matrix of the system (26). We then obtain:

$$N = \begin{pmatrix} n & 0 & \sum x_i & -\sum y_i \\ 0 & n & \sum y_i & \sum x_i \\ \sum x_i & \sum y_i & \sum(x_i^2 + y_i^2) & 0 \\ -\sum y_i & \sum x_i & 0 & \sum(x_i^2 + y_i^2) \end{pmatrix} \tag{33}$$

As we work with respect to the centers of gravity of the coordinates in the local and NTT systems, we then have by definition:

$$\sum x_i = \sum y_i = \sum x'_i = \sum y'_i = 0 \tag{34}$$

We denote also :

$$d_i^2 = x_i^2 + y_i^2 \tag{35}$$

Then the matrix N is written as :

$$N = \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & \sum d_i^2 & 0 \\ 0 & 0 & 0 & \sum d_i^2 \end{pmatrix} \tag{36}$$

The normal matrix is diagonal, its inverse is given by:

$$N^{-1} = \begin{pmatrix} \frac{1}{N} & 0 & 0 & 0 \\ 0 & \frac{1}{N} & 0 & 0 \\ 0 & 0 & \frac{1}{\sum d_i^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\sum d_i^2} \end{pmatrix} \tag{37}$$

But we know that:

$$\sigma_{\bar{U}}^2 = \sigma_0^2 . N^{-1} \tag{38}$$

where σ_0^2 is the unit variance factor given by:

$$\sigma_0^2 = \frac{W^t W}{N-4} = \frac{\sum_i^N w_i^2}{N-4} \tag{39}$$

From the equation (38), we obtain that :

$$\boxed{\sigma_{dt_x}^2 = \sigma_{dt_y}^2 = \frac{\sigma_0^2}{N}} \tag{40}$$

Therefore, we compute the unknowns $\bar{U}$ and we transform the other points of the local system to the official NTT system by:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} T_x \\ T_y \end{pmatrix} + s \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} . \begin{pmatrix} X \\ Y \end{pmatrix}$$

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Abdelmajid Ben Hadj Salem, Expert Geodesist

- Final Version - 17/10/2024

, E-mail: abenhadjsalem@gmail.com