A SIMPLE PROOF OF LEGENDRE'S CONJECTURE

DANIEL EDUARDO RUIZ

ABSTRACT. We analyze intervals of numbers whose prime factors are all $> y$. In this article we stablish and prove a lower bound of prime numbers contained in the interval $[x^a, (x+1)^a)$ where $a, x \in \mathbb{R}^+$ and $x > 2$, $a \geq 2$, then if $a = 2$, the interval $[x^2, (x+1)^2)$ contains at least 1 prime.

1. Introduction

In this article we give a very simple proof of Legendre's Conjecture, and we give a generalization for the sub intervals containing at least a prime in $[x^a, (x+1)^a)$ where $a, x \in \mathbb{R}^+$ and $x > 2$, $a \geq 2$ given by:

$$\pi((x+1)^a) - \pi(x^a) \geq \left\lfloor \frac{(x+1)^a - x^a}{2\left\lfloor ((x+1)^a - 1)^{\frac{1}{2}} \right\rfloor} \right\rfloor$$

The applied to $a = 2$ the interval $[x^2, (x+1)^2)$ contains at least 1 prime, and Legendre's conjecture states the interval between any consecutive perfect squares contains a prime.

2. Non n-smooth numbers

Theorem 2.1. *For $x, z > y \geq 2$ the interval $[x, x + 2p_{\pi(y)} - 2]$ contains at least a non y-smooth number.*

Proof. In Section 5,Theorem 4 of [3][p.65] is proved that $\Psi(x+z, y) \leq \Psi(x, y) + \Psi(z, y)$ for $x, z > y \geq 2$. Then we can get the alternative in equation for the dual problem or the complementary problem see [2][p.275]. Define $\Phi(x, y)$ to be the number of integers up to x whose prime factors are all $> y$:

$$\Psi(x+z, y) \leq \Psi(x, y) + \Psi(z, y) \tag{2.1}$$

A simple identity:

$$\Psi(a, y) + \Phi(a, y) = a \implies \Psi(a, y) = a - \Phi(a, y) \tag{2.2}$$

$$x + z - \Phi(x+z, y) \leq x - \Phi(x, y) + z - \Phi(z, y) \tag{2.3}$$

$$-\Phi(x+z, y) \leq -\Phi(x, y) - \Phi(z, y) \tag{2.4}$$

$$\Phi(x+z, y) \geq \Phi(x, y) + \Phi(z, y) \tag{2.5}$$

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$$\Phi(x+z,y)-\Phi(x,y)\geq\Phi(z,y) \tag{2.6}$$

To find an interval with at least a prime number the solution returning to (2.6) is: $\Phi(x+z,y)-\Phi(x,y)\geq 1$ is the same for $\Phi(z,y)\geq 1$, Bertrand Postulate states between n and $2n-2$ exists at least a prime see [1, p.435-436]. Then for $x,q<y\leq 2$, then:

$$z\geq 2p_{\pi(y)}-2 \implies \Phi(z,y)\geq 1 \implies \Phi(x+z,y)-\Phi(x,y)\geq 1 \tag{2.7}$$

Notice the number is $p_k<2p_{\pi(y)}-2$ by definition a prime is free of prime factors disctint to p. □

3. Prime numbers

Theorem 3.1. *The number of primes in the interval $[x^a,(x+1)^a)$ where $a,x\in\mathbb{R}^+$ and for $x>2$and $a\geq 2$:*

$$\pi((x+1)^a)-\pi(x^a)\geq\left\lfloor\frac{(x+1)^a-x^a}{2\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor}\right\rfloor \tag{3.1}$$

Proof. Assume, for the sake of contradiction, that b and $b\leq(n+1)^m-1$ is the product of two factors or more f_n

$$f_n>\left\lfloor((n+1)^m-1)^{\frac{1}{2}}\right\rfloor \tag{3.2}$$

Then an example of k:

$$b=f_1f_2\dots f_k \tag{3.3}$$

f_n are integers, the next integer to the bracket expression must be greater than the not rounded, since the rounding is less than one for the expression, and the next number is one unit greater to the expression

$$\left\lfloor((n+1)^m-1)^{\frac{1}{2}}\right\rfloor=((n+1)^m-1)^{\frac{1}{2}}-r \text{ and } 0\leq r<1 \tag{3.4}$$

$$f_n\geq\left\lfloor((n+1)^m-1)^{\frac{1}{2}}\right\rfloor+1=((n+1)^m-1)^{\frac{1}{2}}-r+1 \text{ and } 0<1-r\leq 1 \tag{3.5}$$

$$f_n\geq\left\lfloor((n+1)^m-1)^{\frac{1}{2}}\right\rfloor+1=((n+1)^m-1)^{\frac{1}{2}}-r+1>((n+1)^m-1)^{\frac{1}{2}} \tag{3.6}$$

$$f_n>((n+1)^m-1)^{\frac{1}{2}} \tag{3.7}$$

$$f_1f_2=b>(n+1)^m-1 \tag{3.8}$$

The contradiction is a is not greater than. And for more than two:

$$f_1f_2\dots f_k=b>((n+1)^m-1)^{\frac{k}{2}} \tag{3.9}$$

Then it is only possible for $k=1$

$$f_1=b>\left\lfloor((n+1)^m-1)^{\frac{1}{2}}\right\rfloor \tag{3.10}$$

Let $s \in [x^a,(x+1)^a)$ as a non $\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$-smooth number, then returning to equation (2.6), and considering the worst case, the maximum value for the next prime:

(3.11)

$$\Phi(x^a+2\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor-2,\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor)-\Phi(x^a,\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor)\geq 1$$

Then we could extend for all the sub intervals in the main interval:

$$[x^a+2k\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor,x^a+2(k+1)\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor-2] \tag{3.12}$$

To simplify the bounds:

$$[x^a+2k\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor,x^a+2(k+1)\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor) \tag{3.13}$$

Then by dividing the total length and the length of the sub intervals:

(3.14)

$$\Phi((x+1)^a,\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor)-\Phi(x^a,\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor)\geq N=\left\lfloor\frac{(x+1)^a-x^a}{2\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor}\right\rfloor$$

Notice that the interval contains N non $\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$-smooth numbers s_i and for equations : (3.9) and (3.10) doesn't have more than 1 factor of the form:

$$f>\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor \tag{3.15}$$

Since s_i is not divisible by any prime $p\leq\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$ as consequence its factors are $f>\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$. Then s_i only has one factor $s_i=f>\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$ and $s_i=f$ are not divisible by any $p\leq\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor$. Hence all s_i are prime numbers a total of N, and then taking the complete possibles sub intervals:

$$\pi((x+1)^a)-\pi(x^a)\geq\left\lfloor\frac{(x+1)^a-x^a}{2\left\lfloor((x+1)^a-1)^{\frac{1}{2}}\right\rfloor}\right\rfloor \tag{3.16}$$

Notice continuous values for x,a are valid, because all the proofs are for continuous variable, and values for $x>2$ and $a\geq 2$. Because if $a<2$ the number and for $x<2$ is trivial beacuse of the inequation (2.6) and $x,z>y\geq 2$ the result in Section 5 Theorem 4 of [3]. □

Theorem 3.2. *The interval* $[x^2,(x+1)^2)$ $x\in\mathbb{R},x>2$*, contains at least 1 prime.*

Proof. $[n^2,(n+1)^2]$ is a especial case of 3.1[theorem] for $a=2$, then:

$$\pi((x+1)^2)-\pi(x^2)\geq\left\lfloor\frac{(x+1)^2-x^2}{2\left\lfloor((x+1)^2-1)^{\frac{1}{2}}\right\rfloor}\right\rfloor \tag{3.17}$$

The square root must be $x<r<x+1$ and the rounded is the lower value.

$$\pi((x+1)^2)-\pi(x^2)\geq\left\lfloor\frac{(x+1)^2-x^2}{2x}\right\rfloor \tag{3.18}$$

$$\pi((x+1)^2)-\pi(x^2)\geq\left\lfloor\frac{2x+1}{2x}\right\rfloor \tag{3.19}$$

$$\pi((x+1)^2) - \pi(x^2) \geq 1 \tag{3.20}$$

□

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Cerro de Coatepec S/N, Ciudad Universitaria, Universitaria, 50110 Toluca de Lerdo, Méx.

Email address: druizc005@alumno.uaemex.mx