Finite mathematics as the most general (fundamental) mathematics

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Abstract

The purpose of this paper is to explain at the simplest possible level why finite mathematics based on a finite ring of characteristic p is more general (fundamental) than standard mathematics. The belief of most mathematicians and physicists that standard mathematics is the most fundamental arose for historical reasons. However, simple *mathematical* arguments show that standard mathematics (involving the concept of infinities) is a degenerate case of finite mathematics in the formal limit $p \to \infty$: standard mathematics arises from finite mathematics in the degenerate case when operations modulo a number are discarded. Quantum theory based on a finite ring of characteristic p is more general than standard quantum theory because the latter is a degenerate case of the former in the formal limit $p \to \infty$.

List of Abbreviations

FM: finite mathematics

SM: standard mathematics

SR: special relativity

NM: nonrelativistic mechanics

QT: quantum theory

CT: classical theory

FQT: Quantum theory based on finite mathematics

SQT: Standard quantum theory

IR: irreducible representation

QFT: Quantum Field Theory

1 The main goal of this paper

SM deals with relations

$$a + b = c, \ a \cdot b = c, \ etc. \tag{1}$$

On the other hand, FM deals with relations

$$a + b = c \ (mod \, p), \ a \cdot b = c \ (mod \, p), \ etc. \tag{2}$$

where all the numbers $a, b, c, ...$ can take only values $0, 1, 2, ...p - 1$ and p is called characteristic of the ring. Therefore, in FM there are no infinities and all numbers do not exceed p in absolute value.

Before discussing these versions of mathematics, let's discuss the following. How should we treat mathematics: i) as a purely abstract science or ii) as a science that should describe nature? I am a physicist and have worked among physicists for most of my life. For them, only approach ii) is acceptable. However, when I discussed this issue with mathematicians and philosophers, I discovered that many of them view mathematics only from the point of view of i) and arguments related to the description of nature are not significant for them.

Perhaps the most famous mathematician who championed the approach i) was Hilbert. The goal of his approach is to find a complete and consistent set of axioms which will make it possible to conclude whether any mathematical statement is true or false. This problem is also formulated as the Entscheidungsproblem which asks for algorithms that consider statements and answers "Yes" or "No" according to whether the statements are universally valid, i.e., valid in every structure satisfying the axioms.

In the framework of i), the problem of foundation of mathematics is very difficult. This problem has been considered by many great mathematicians. The Gödel's incompleteness theorems state that mathematics involving standard arithmetic of natural numbers is incomplete and cannot demonstrate its own consistency. The problem widely discussed in the literature is whether the problems posed by the theorems can be circumvented by nonstandard approaches to natural numbers, e.g., by treating them in the framework of Peano arithmetic, Presburger arithmetic etc. However, as shown by Turing and others, in Hilbert's approach, the problem of foundation of mathematics remains.

The fact that Hilbert's approach does not raise the question of describing nature does not mean that this approach should be rejected out of hand. For example, Dirac's philosophy is: *"I learned to distrust all physical concepts as a basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics."* Dirac also said that for him the most important thing in any physical theory is the beauty of formulas in this theory. That is, he meant that sooner or later, in any beautiful mathematical theory, its physical meaning will be found. But even if it is not found, the beauty of the theory itself has aesthetic value. For example, in music we appreciate its beauty and do not demand that music should somehow describe nature.

Nevertheless, in this paper, we treat mathematics only as a tool for describing nature. In the framework of this approach, most mathematicians and physicists believe that, at the most fundamental level, nature is described by SM, and FM is needed only in some special model problems. In this regard, the question arises whether it is possible to give a definition when mathematics A is more general (fundamental) than mathematics B, and mathematics B is a degenerate case of mathematics A.

In [1] we have proposed the following

Definition: *Let theory A contain a finite nonzero parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when, the limit is already taken, one cannot return to A and B cannot reproduce all results of A. Then A is more general than B and B is a degenerate case of A.*

We have shown that, using this **Definition**, it is possible to prove *purely mathematically* some known facts which in the physical literature are explained from physical considerations, in particular:

1) NM is a degenerate case of SR in the formal limit $c \to \infty$ where it is usually said that c is the speed of light, but in fact, it is only a constant of the theory.;

2) CT is a degenerate case of QT in the formal limit $\hbar \to 0$ (where $\hbar$ is the Planck constant).

In applications to 1), **Definition** implies that SR is a more general (fundamental) theory than NM because any result of NM can be obtained in SR with some choice of c, and, on the other hand, NM cannot reproduce those results of SR where it is important that c is finite and not infinitely large.

Analogously, in applications to 2), **Definition** implies that QT is a more general (fundamental) theory than CT because any result of CT can be obtained in QT with some choice of $\hbar$, and on the other hand, CT cannot reproduce those results of QT where it is important that $\hbar$ is finite and not zero.

The main goal of this paper is to discuss the result of [1] that, as follows from **Definition**, contrary to the belief of many mathematicians and physicists described above:

Statement: *SM is a degenerate case of FM in the formal limit $p \to \infty$, where p is the characteristic of the ring in FM.*

This implies that FM is a more general (fundamental) theory than SM because any result of SM can be obtained in FM with some choice of p, and, on the other hand, SM cannot reproduce those results of FM where it is important that p is finite and not infinitely large.

As explained below, SM is a degenerate case of FM because SM is obtained from FM in the case when all operations modular a number are discarded. Also, as explained in [1], a consequence of this **Statement** is that, *for describing*

nature at the most fundamental level, the concepts of infinitesimals, infinitely large, limits, continuity etc. are not needed; they are needed only for describing nature approximately.

Kronecker's famous phrase is that God invented integers, and humans invented everything else. However, in view of this **Statement**, this phrase can be reformulated so that God came up with only finite sets of numbers, and everything else was invented by people.

One of the key problems of SQT (based on SM) is the problem of divergences: the theory gives divergent expressions for the S-matrix in perturbation theory. In renormalized theories, the divergences can be eliminated by renormalization where finite observable quantities are formally expressed as products and sums of singularities. From the mathematical point of view, such procedures are not legitimate but in some cases they result in impressive agreement with experiment. The most famous case is that the results for the electron and muon magnetic moments obtained at the end of 40th agree with experiment with the accuracy of eight decimal digits. In view of this and other successes of SQT, most physicists believe that agreement with the data is much more important than the rigorous mathematical substantiation.

At the same time, in non-renormalized theories, divergences cannot be eliminated by the renormalization procedure, and this is a great obstacle for constructing quantum gravity based on QFT. As the famous Nobel Prize laureate Steven Weinberg wrote in his book [2]: "*Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day*". The title of Weinberg's paper [3] is "Living with infinities".

However, as follows from **Statement**, in QT based on FM, the problem of divergences does not exist in principle because in FM there are no infinities. We emphasize that **Statement** is not only our wish, but a fact proven mathematically in [1]. Therefore, those mathematicians and physicists who insist on their position that SM is more general (fundamental) than FM must either give arguments that **Definition** is not justified or show that the proof in [1] is erroneous. However, in numerous discussions with me, those mathematicians and physicists have presented various arguments that, in their opinion, emphasize the correctness of their position. Typical arguments are:

- a) Formally, you have no divergences, but you introduce the cutoff p which is a huge number. Therefore, in cases when infinities arose in the standard theory, you will get a huge number p which is practically infinite.

- b) The argument of the famous mathematician Yu. I. Manin was this: in your theory there is only one parameter p and it is not clear why this parameter is this and not another. He said that he preferred the approach with adeles when there are many characteristics which are on equal footing.

- c) An argument that has some similarities with b) is this: when you say that God only invented finite sets of numbers and everything else (infinitesimals,

infinitely large etc.) was invented by people, do you think that he "invented" a biggest (finite) p?

I will discuss these arguments below. But first I would like to discuss

2 Analogy between SR and FM

Before the creation of SR, it was believed that NM was the most general (fundamental) mechanics. There are no restrictions on the magnitude of speed there which can be in the interval $[0, \infty)$. However, in SR, the speed cannot exceed c.

The fact that there is a speed limit greatly changes the standard philosophy of NM. For example, in NM it seems unnatural that the speed of $0.99c$ is possible, but $1.01c$ is not. For this and other reasons, it took a very long time for SR to be accepted by the majority of physicists.

Let's consider a simple model example when in our reference frame some observer moves with speed v_1 and in the reference frame of this observer some particle moves in the same direction with speed v_2. Then, according to the rules of NM, the speed of the particle in our reference frame will be $V = v_1 + v_2$. So, even if $v_1 < c$ and $v_2 < c$ then, in NM, a situation is possible when $V > c$ and this may suggest that the statement of SR about the speed limit is not self-consistent. However, the result of SR in such a situation is not $V = v_1 + v_2$ but

$$V = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \tag{3}$$

and this value cannot exceed c. In particular, if $v_1 = v_2 = 0.6c$ then V is not equal to $1.2c$ as one might think from naive considerations, but $V \approx 0.882c$, and if $v_1 = v_2 = 0.99c$ then V is not equal to $1.98c$ but $V \approx 0.9999495c$ The lesson of this example is that it is not always correct to make judgments proceeding from "common sense".

Here there is an analogy with FM: for example, if a and b are such natural numbers that $a < p$, $b < p$ and in SM there may be a situation when $(a + b) > p$, then in FM such a situation cannot exist because always $(a + b) \ (mod \ p) < p$.

It is now generally accepted that SR is confirmed experimentally to a greater extent than NM. Also, as noted above, it follows from **Definition** that NM is a degenerate case of SR since SR can reproduce any fact of NM with some choice of c, while NM cannot reproduce those facts of SR in which it is essential that c is finite and not infinite. Thus, SR does not disprove NM, but shows that it works with good accuracy when speeds are much less than c. There is an analogy here with the fact that FM does not refute SM, but shows that the latter is a good approximation to reality only in situations where the numbers in a given problem are much less than p.

In complete *logical* analogy with the objections to FM in points (a-c) in Sec. 1, one can put forward similar objections to SR, but now the role of p will be played by c. Therefore, I think that, *for being completely consistent, if we reject FM,*

we must also reject SR, and if we accept SR then, by the same logic, we must also accept that FM is more general (fundamental) than SM.

As follows from the above results, it is not necessary to apply SR in everyday life when speeds are much less than c because in this case NM works with a very high accuracy. Analogously, for describing almost all phenomena at the macroscopic level, there is no need to apply QT. For example, there is no need to describe the motion of the Moon by the Schrödinger equation. In principle this is possible but results in unnecessary complications. At the same time, microscopic phenomena can be correctly described only in the framework of QT.

3   Basic facts about finite mathematics

SM starts from the infinite set of natural numbers but FM can involve only a finite number of elements. FM starts from the ring $R_p = (0, 1, 2, ...p - 1)$ where addition, subtraction and multiplication are defined as usual but modulo p. In our opinion the notation Z/p for R_p is not adequate because it may give a wrong impression that FM starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number.

In the set of natural numbers, only addition and multiplication are always possible. In order to make addition invertable negative integers are introduced. They do not have a direct physical meaning (e.g., the phrase "this computer has -100 bits of memory" is meaningless) and their only goal is to get the ring of integers Z. In contrast to this situation, R_p is the ring without adding new elements and the number p is called the characteristic of this ring. For example, if $p = 5$ then 3+1=4 as usual but 3·2=1, 4·3=2, 4·4=1 and 3+2=0. Therefore -2=3 and -4=1. Moreover, if p is prime then R_p becomes the Galois field F_p where all the four operations are possible. For example, 1/2=3, 1/4=4 etc.

One might say that those examples have nothing to do with reality since 3+2 always equals 5 and not zero. However, since operations in R_p are modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm (p - 1)/2)\}$ if p is odd and as a set $\{0, \pm 1, \pm 2, ..., \pm (p/2 - 1), p/2\}$ if p is even. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p. Then for elements $a \in R_p$ such that $|f(a)| \ll p$, addition, subtraction and multiplication are the same as in Z. In other words, for such elements we do not notice the existence of p.

One might say that nevertheless the set F_p cannot be used in physics since $1/2 = (p + 1)/2$, i.e., a very large number when p is large. However, as explained in [1], since quantum states are projective then, even in SQT, quantum states can be described with any desired accuracy by using only integers and therefore the concepts of rational and real numbers play only an auxiliary role.

If elements of Z are depicted as integer points on the x axis of the xy plane then, if p is odd, the elements of R_p can be depicted as points of the circumference in Figure 1 and analogously if p is even. This picture is natural from the following
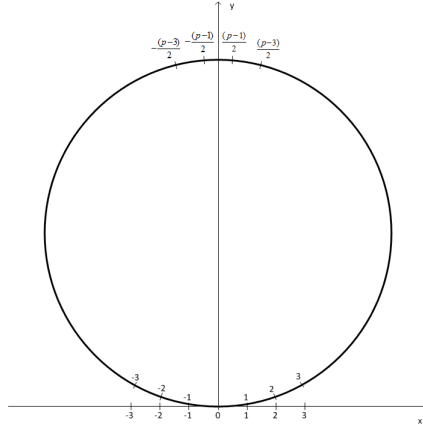
Figure 1: Relation between R_p and Z

considerations. As explained in textbooks, both R_p and Z are cyclic groups with respect to addition. However, R_p has a higher symmetry because it has a property which we call *strong cyclicity*: if we take any element $a \in R_p$ and sequentially add 1 then after p steps we will exhaust the whole set R_p by analogy with the property that if we move along a circumference in the same direction then sooner or later we will arrive at the initial point. At the same time, if we take an element $a \in Z$ then the set Z can be exhausted only if we first successively add +1 to a and then -1 to a or *vice versa* and those operations should be performed an infinite number of times. As noted in [1], in QT based on FM, strong cyclicity plays an important role. In particular, it explains why one IR of the symmetry algebra describes a particle and its antiparticle simultaneously.

The above construction has a known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of curvature. Analogously one might think that the set of numbers describing physics in our universe has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

4   Proof that the ring Z is the limit of the ring R_p when $p \to \infty$

In this section, following Sec. 6.3 of [1], we prove that, as follows from **Definition**,

Statement 1: The ring R_p is more general than the ring Z and the latter is a degenerate case of the former in the formal limit $p \to \infty$.

Note that in the *technique* of SM, infinity is understood only as a limit (i.e., as *potential infinity*) but *the basis* of SM does involve *actual infinity*: SM starts from the infinite ring of integers Z and, even in standard textbooks on mathematics,

it is not even posed a problem whether Z can be treated as a limit of finite rings. The problem of actual infinity is discussed in a vast literature, and, in SM, Z is treated as actual and not potential infinity, i.e., there is no rigorous definition of Z as a limit of finite rings. Moreover, classical set theory considers infinite sets with different cardinalities.

As explained in [1], *Statement 1* is the basic stage in proving **Statement**, i.e., that FM is more general than SM. In particular, as explained in detail in [1], since we treat mathematics in the approach ii) in Sec. 1, this means that QT based on FM is more general (fundamental) than SQT.

Therefore *Statement 1* should not be based on the results of SM. In particular, it should not be based on properties of the ring Z derived in SM. The statement should be proved by analogy with standard proof that a sequence of natural numbers (a_n) goes to infinity if $\forall M > 0 \; \exists n_0$ such that $a_n \geq M \; \forall n \geq n_0$. In particular, the proof should involve only potential infinity but not actual one.

The meaning of *Statement 1* is that for any $p_0 > 0$ there exists a set S belonging to all R_p with $p \geq p_0$ and a natural number n such that for any $m \leq n$ the result of any m operations of summation, subtraction or multiplication of elements from S is the same as in R_p for any $p \geq p_0$ and that cardinality of S and the number n formally go to infinity when $p_0 \to \infty$. This means that for the set S and number n there is no manifestation of operations modulo p, i.e., the results of any $m \leq n$ operations of elements from S are formally the same in R_p and Z.

This implies that for experiments involving only such sets S and numbers n it is not possible to conclude whether the experiments are described by a theory involving R_p with a large p or by a theory involving Z.

In the literature, we did not succeed in finding *a direct proof* of *Statement 1*. As noted e.g., in [1], the fact that Z can be treated as a limit of R_p when $p \to \infty$ follows from a construction called ultraproducts. However, theory of ultraproducts is essentially based on classical results involving actual infinity, in particular, on Łoś' theorem involving the axiom of choice. Therefore theory of ultraproducts cannot be used in proving that FM is more general than SM.

We now describe our proof of *Statement 1*. We define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. Let n be a natural number and $U(n)$ be a set of elements $a \in R_p$ such that $|f(a)|^n \leq h(p)$. Then $\forall m \leq n$ the result of any m operations of addition, subtraction or multiplication of elements $a \in U(n)$ is the same as for the corresponding elements $f(a)$ in Z, i.e., in this case operations modulo p are not explicitly manifested.

Let $g(p)$ and $G(p)$ be functions of p with the range in the set of natural numbers such that the set $U(g(p))$ contains at least the elements $\{0, \pm 1, \pm 2, ..., \pm G(p)\}$. In what follows, $M > 0$ is a natural number. If there is a sequence of natural numbers (a_n) then standard definition that $(a_n) \to \infty$ is that $\forall M \; \exists n_0$ such that $a_n \geq M \; \forall n \geq n_0$. By analogy with this definition we will now prove

Proposition: There exist functions $g(p)$ and $G(p)$ such that $\forall M \; \exists p_0$ such that $g(p) \geq M$ and $G(p) \geq 2^M \; \forall p \geq p_0$.

Proof: $\forall p > 0$ there exists a unique natural n such that $2^{n^2} \leq h(p) <$

$2^{(n+1)^2}$. Define $g(p) = n$ and $G(p) = 2^n$. Then $\forall M \ \exists p_0$ such that $h(p_0) \geq 2^{M^2}$. Then $\forall p \geq p_0$ the conditions of *Proposition* are satisfied.

Proposition implies that the ring Z is the limit of the ring R_p when $p \to \infty$, and the result of any finite number of additions, subtractions and multiplications in Z can be reproduced in R_p if p is chosen to be sufficiently large. On the contrary, when the limit $p \to \infty$ is already taken then one cannot return back from Z to R_p, and in Z it is not possible to reproduce all results in R_p because in Z there are no operations modulo a number. According to **Definition**, this means that *Statement 1* is valid, i.e., that *the ring R_p is more general than Z, and Z is the degenerate case of R_p.*

When p is very large then $U(g(p))$ is a relatively small part of R_p, and, in general, the results in Z and R_p are the same only in $U(g(p))$. This is analogous to the fact mentioned in Sec. 2 that the results of NM and SR are the same only in relatively small cases when velocities are much less than c. However, when the radius of the circumference in Figure 1 becomes infinitely large then a relatively small vicinity of zero in R_p becomes the infinite set Z when $p \to \infty$. This example demonstrates that, *even from pure mathematical point of view, the concept of infinity cannot be fundamental because, as soon as we involve infinity and replace R_p by Z, we automatically obtain a degenerate theory because in Z there are no operations modulo a number.*

In quantum theory based on finite mathematics (FQT), states are elements of linear spaces over R_p. One might get the impression that SQT is a more general theory than FQT because in SM, Z is generalized to the case of rational and real numbers. However, as noted above (see also [1]), since even in SQT, the states are projective, it is sufficient to use only integers for describing experimental data with any desired accuracy.

5   Problems with describing nature by standard mathematics

Standard education develops a belief that SM is the most fundamental mathematics, while FM is something inferior what is used only in special applications. Historically, it happened so because more than 300 years ago Newton and Leibniz proposed the calculus of infinitesimals, and, since that time, a titanic work has been done on foundation of SM. As noted in Sec. 1, this problem has not been solved till the present time, but for most physicists and many mathematicians the most important thing is not whether a rigorous foundation exists but that in many cases SM works with a very high accuracy.

The idea of infinitesimals was in the spirit of existed experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and, even in the 19th century, people did not know about elementary particles. But now we know that when we reach the level of elementary particles then standard

division loses its usual meaning and in nature there are no arbitrarily small parts and no continuity.

For example, typical energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with different particles. If it were possible to break the electron into parts, then it would have been noticed long ago.

Another example is that if we draw a line on a sheet of paper and look at this line by a microscope then we will see that the line is strongly discontinuous because it consists of atoms. That is why standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected, standard macroscopic theory can work well only in this approximation etc.

Differential equations work well in approximations where it is not necessary to take into account that matter consists of atoms. However, it seems unnatural that SQT is based on SM. Even the name "quantum theory" reflects a belief that nature is quantized, i.e., discrete, and this name has arisen because in QT some quantities have discrete spectrum (e.g., the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom etc.). But this discrete spectrum has appeared in the framework of SM.

I asked physicists and mathematicians whether, in their opinion, the indivisibility of the electron shows that in nature there are no infinitesimals and standard division does not work always. Some mathematicians say that sooner or later the electron will be divided. On the other hand, as a rule, physicists agree that the electron is indivisible and in nature there are no infinitesimals. They say that, for example, dx/dt should be understood as $\Delta x/\Delta t$ where Δx and Δt are small but not infinitesimal. I ask them: but you work with dx/dt, not $\Delta x/\Delta t$. They reply that since mathematics with derivatives works well then there is no need to philosophize and develop something else (and they are not familiar with finite mathematics).

In view of efforts to describe discrete nature by continuous mathematics, my friend told me the following joke: "*A group of monkeys is ordered to reach the Moon. For solving this problem each monkey climbs a tree. The monkey who has reached the highest point believes that he has made the greatest progress and is closer to the goal than the other monkeys*". Is it reasonable to treat this joke as a hint on some aspects of the modern science? Indeed, people invented continuity and infinitesimals which do not exist in nature, created problems for themselves and now apply titanic efforts for solving those problems. As follows from the results of Sec. 4 (see also [1]), SM is a degenerate case of FM.

The founders of QT and scientists who essentially contributed to it were highly educated. But they used only SM, and even now FM is not a part of standard education for physicists. The development of QT has shown that the theory contains anomalies and divergences. Most physicists considering those problems, work in the framework of SM and do not acknowledge that they arise just because this mathematics is applied.

Several famous physicists (including the Nobel Prize laureates Gross, Nambu

and Schwinger) discussed approaches when QT involves FM (see e.g., [4]). A detailed discussion of these approaches has been given in the book [5] where they are characterized as hybrid quantum systems. The reason is that here coordinates and/or momenta belong to a finite ring or field but wave functions are elements of standard complex Hilbert spaces. Then the problem of foundation of QT is related to the problem of foundation of SM. On the other hand, in [6, 7, 8], we have proposed an approach called finite quantum theory (FQT) where not only physical quantities but also wave functions involve finite rings or fields. As explained in [1] FQT is more general (fundamental) than SQT.

6 Why finite mathematics is more natural than classical one

In view of the above discussion, the following problem arises: is it justified to use mathematics with infinitesimals for describing nature in which infinitesimals do not exist? There is no doubt that the technique of SM is very powerful and in many cases describes physical phenomena with a very high accuracy. However, a problem arises whether there are phenomena which cannot be correctly described by mathematics involving infinitesimals.

Some facts of SM seem to be unnatural. For example, $tg(x)$ is one-to-one reflection of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, i.e., the impression might arise that the both intervals have the same numbers of elements although the first interval is a nontrivial part of the second one. Another example is the Hilbert paradox with an infinite hotel. But mathematicians even treat those facts as pretty ones. For example, Hilbert said: "No one shall expel us from the paradise that Cantor has created for us".

From the point of view of Hilbert's approach to mathematics (see Sec. 1) it is not important whether the above statements are natural or not, since the goal of the approach is to find a complete and consistent set of axioms. In the framework of this approach, the problem of foundation of SM has been investigated by many great mathematicians (e.g., Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others). Their philosophy was based on macroscopic experience in which the concepts of infinitesimals, continuity and standard division are natural. However, as noted above, those concepts contradict the existence of elementary particles and are not natural in QT. The illusion of continuity arises when one neglects the discrete structure of matter.

The fact that in Hilbert's approach there exist foundational problems follows, in particular, from Gödel's incompleteness theorems which state that no system of axioms can ensure that all facts about natural numbers can be proved, and the system of axioms in SM cannot demonstrate its own consistency. The theorems are written in highly technical terms of mathematical logics. As noted in Sec. 1, in this paper we do not consider Hilbert's approach to mathematics. However, simple arguments in [1] show that, if mathematics is treated as a tool for describing nature, then foundational problems of SM follow from simple arguments described below.

In the 20s of the 20th century, the Viennese circle of philosophers under

the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g., [9, 10]). However, this principle does not work if SM is treated as a tool for describing nature. For example, in Hilbert's approach one of axioms is that $a + b = b + a$ for all natural numbers a and b, and a question whether this is true or false does not arise. However, if mathematics is treated as a tool for describing nature, it cannot be determined whether this statement is true or false.

As noted by Grayling [11], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. So, from the point of view of SM and physics, verification principle is too strong.

Popper proposed the concept of falsificationism [12]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.* In particular, the statement that $a + b = b + a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a + b \neq b + a$.

According to the philosophy of quantum theory, *in contrast to Hilbert's approach to mathematics*, there should be no statements accepted without proof and based on belief in their correctness (i.e., axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. This philosophy is the result of the fact that quantum theory describes phenomena which, from the point of view of "common sense", seem meaningless but they have been experimentally verified. So, the philosophy of QT is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of QT and supported Einstein in his dispute with Bohr.

From the point of view of verificationism and the philosophy of QT, SM is not well defined not only because it contains an infinite number of numbers. Consider, for example, whether the rules of standard arithmetic can be justified.

We can verify that 10+10=20 and 100+100=200, but can we verify that, say $10^{100000} + 10^{100000} = 2 \cdot 10^{100000}$? One might think that this is obvious, and in Hilbert's approach this follows from main axioms. But, if mathematics is treated as a tool for describing nature then this is only a belief based on extrapolating our everyday experience to numbers where it is not clear whether the experience still works.

In Sec. 2 we discussed that our life experience works well at speeds that are much less than c, and this experience cannot be extrapolated to situations where speeds are comparable to c. Likewise, our experience with the numbers we deal with in everyday life cannot be extrapolated to situations where the numbers are much greater.

According to verificationism and principles of quantum theory, the statement that $10^{100000} + 10^{100000} = 2 \cdot 10^{100000}$ is true or false depends on whether this statement can be verified. Is there a computer which can verify this statement? Any

computing device can operate only with a finite number of resources and can perform calculations only modulo some number p. If our universe is finite and contains only N elementary particles, then there is no way to verify that $N + N = 2N$. So, if, for example, our universe is finite, then in principle it is not possible to verify that standard rules of arithmetic are valid for any numbers.

That is why the statements in Eq. (1) are ambiguous because they does not contain information on the computing device which verifies those statements. For example, let us pose a problem whether 10+20 equals 30. If our computing devise is such that $p = 40$ then the experiment will confirm that 10+20=30 while if $p = 25$ then we will get that 10+20=5.

So, the statements that 10+20=30 and even that $2 \cdot 2 = 4$ are ambiguous because they do not contain information on how they should be verified. On the other hand, the statements

$$10 + 20 = 30 \, (mod \, 40), \;\; 10 + 20 = 5 \, (mod \, 25),$$

$$2 \cdot 2 = 4 \, (mod \, 5), \;\; 2 \cdot 2 = 2 \, (mod \, 2)$$

are well defined because they do contain such an information. So only operations modulo a number are well defined.

We believe the following observation is very important: although SM (including its constructive version) is a part of our everyday life, people typically do not realize that *SM is implicitly based on the assumption that one can have any desired number of resources.* So, SM is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite number of resources. Typically, people do not realize that *standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p \to \infty$.* For example, if (a, b, c, p) are natural numbers then Eqs. (1) are implicitly treated as

$$\lim_{p \to \infty} [(a + b) \, (mod \, p)] = c, \;\; \lim_{p \to \infty} [(a \cdot b) \, (mod \, p)] = c, \; etc.$$

As a rule, every limit in mathematics is thoroughly investigated but, in the case of standard operations with natural numbers, it is not even mentioned that those operations are limits of operations modulo p. In real life such limits even might not exist if, for example, the universe contains a finite number of elementary particles.

So, we see that the question of what 10+20 is equal to is not a question of what some theory says, but a question of how an experiment will be set up to test what this value is equal to. In one experiment the result may be 30, in another 5 and there is no theory that says that one experiment is more preferable than another.

Now let's discuss the question of what p can be equal to in the theory describing modern physics. Recently, an increasing number of works have appeared in the literature that say that the universe works like a computer (see, for example, [13]). From this point of view, the value of p is determined by the state of the universe at a given stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the

universe will be different at different stages. Therefore, by analogy with the discussion of what 10+20 is equal to, we can say that p is not a number that is determined by some fundamental theory, but a number that depends on the state of the universe at a given stage.

We do not say that p changes over time for the following reasons. The problem of time is one of the most fundamental problems of quantum theory. Every physical quantity should be described by a self-adjoined operator but, as noted by Pauli, the existence of the time operator is a problem (see e.g., the discussion in [1]). One of the principles of physics is that the definition of a physical quantity is a description how this quantity should be measured, and *it is not correct to say that some quantity exists but cannot be measured*. The present definition of a second is the time during which 9,192,631,770 transitions in a cesium-133 atom occur. The time cannot be measured with absolute accuracy because the number of transitions is finite. Then one second is defined with the accuracy $10^{-15}s$, and, e.g., [14] describes efforts to measure time with the accuracy $10^{-19}s$. However, it is not clear how to define time in early stages of the universe when atoms did not exist. So, treating time t as a continuous quantity belonging to $(-\infty, +\infty)$ can be only an approximation which works at some conditions. In [1] it has been discussed a conjecture that standard classical time t manifests itself because the value of p changes, i.e., t is a function of p. We do not say that p changes over time because classical time t cannot be present in quantum theory; we say that we feel changing time because p changes. As shown in [15], with such an approach, the known problem of baryon asymmetry of the universe (see the subsequent section) does not arise.

7   Examples when finite mathematics can solve problems which standard mathematics cannot

In [1] we discussed phenomena where it is important that p is finite. They cannot be described in SQT, by analogy with the fact that NM cannot describe cases where it is important that c is finite. Below we describe several such phenomena.

Example 1: gravity. Theoretically, any result of CT should follow from QT in semiclassical approximation. However, the Newton gravitational law cannot be derived in QFT because the theory is not renormalizable. But the law can be derived from FQT in semiclassical approximation [1]. Then the gravitational constant G is not taken from the outside but depends on p as $1/ln(p)$. By comparing this result with the experimental value, one gets that $ln(p)$ is of the order of 10^{80} or more, and therefore p is a huge number of the order of $exp(10^{80})$ or more. One might think that since p is so huge then in practice p can be treated as an infinite number. However, since $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Example 2: Dirac vacuum energy problem. In quantum electrodynamics, the vacuum energy should be zero, but in SQT the sum for this energy

diverges, and this problem was posed by Dirac. To get the zero value, the artificial requirement that the operators should be written in the normal order is imposed, but this requirement does not follow from the construction of the theory. In Sec. 8.8 of [1], I take the standard expression for this sum and explicitly calculate it in FM without any assumptions. Then since the calculations are modulo p, I get zero as it should be.

Example 3: equality of masses of particles and their antiparticles. This is an example demonstrating the power of finite mathematics. A discussion in [1] shows that in QT, an elementary particle and its antiparticle should be considered only from the point of view of IRs of the symmetry algebra. In SQT, the algebras are such that their IRs contain either only positive or only negative energies. In the first case the objects are called particles and in the second one – antiparticles. Then the energies of antiparticles become positive after second quantization.

In SQT, the spectrum of positive energies contains the values $(m_1, m_1 + 1, m_1+2, \cdots \infty)$, and for negative energies — the values $(-m_2, -m_2-1, -m_2-2, \cdots -\infty)$, where $m_1 > 0$, $m_2 > 0$, m_1 is called the mass of a particle and m_2 is called the mass of the corresponding antiparticle. Experimentally $m_1 = m_2$ but in SQT, IRs with positive and negative energies are fully independent of each other. It is claimed that $m_1 = m_2$ because local covariant equations are CPT invariant. However, as explained in [1], the argument x in local quantized fields does not have a physical meaning because it is not associated with any operator. So, in fact, SQT cannot explain why $m_1 = m_2$.

Consider now what happens in FQT. For definiteness, we consider the case when p is odd, and the case when p is even can be considered analogously. One starts constructing the IR with the value m_1, and, by acting on the states by raising operators, one gets the values $m_1 + 1, m_1 + 2, \cdots$. However, now we are moving not along the straight line but along the circle in Figure 1. When we reach the value $(p - 1)/2$, the next value is $-(p - 1)/2$, i.e., one can say that by adding 1 to a large positive number $(p - 1)/2$ one gets a large negative number $-(p - 1)/2$. By continuing this process, one gets the numbers $-(p - 1)/2 + 1 = -(p - 3)/2$, $-(p-3)/2+1 = -(p-5)/2$ etc. The explicit calculation [1] shows that the procedure ends when the value $-m_1$ is reached.

Therefore, FM gives a clear proof that $m_1 = m_2$ and shows that, instead of two independent IRs in SM, one gets only one IR describing both, a particle, and its antiparticle. The case described by SM is degenerate because, in the formal limit $p \to \infty$, one IR in FM splits into two IRs in SM. So, when $p \to \infty$ we get symmetry breaking. This example is a beautiful illustration of Dyson's idea [16] that theory A is more general than theory B if B can be obtained from A by contraction. The example is fully in the spirit of this idea because it shows that SM can be obtained from FM by contraction of the symmetry in the formal limit $p \to \infty$. This example also shows that standard concept of particle-antiparticle is only approximate and is approximately valid only when p is very large. Consequently, constructing complete QT based on FM should be based on new principles.

Example 4: the problem of baryon asymmetry of the universe.

This problem is formulated as follows. According to the modern particle and cosmological theories, the numbers of baryons and antibaryons in the early stages of the universe were the same. Then, since the baryon number is the conserved quantum number, those numbers should be the same at the present stage. However, at this stage the number of baryons is much greater than the number of antibaryons.

For understanding this problem, one should understand the concept of particle-antiparticle. In SQT this concept takes place because IRs describing particles and antiparticles are such that energies in them can be either only positive or only negative but cannot have both signs. However, as explained in **Example 3**, IRs in FQT necessarily contain both, positive and negative energies, and in the formal limit $p \to \infty$, one IR in FQT splits into two IRs in SQT with positive and negative energies.

As noted above, the number p is different at different stages of the universe. As noted in **Example 1**, at the present stage of the universe this number is huge, and therefore the concepts of particles and antiparticles have a physical meaning. However, arguments given in [1] indicate that in early stages of the universe the value of p was much less than now. Then, in general, each object described by IR, is a superposition of a particle and antiparticle (in SQT such a situation is prohibited), and the electric charge and baryon quantum number are not conserved. Therefore, in early stages of the universe, SQT does not work, and the statement that at such stages the numbers of baryons and antibaryons were the same, does not have a physical meaning. Therefore, the problem of baryon asymmetry of the universe does not arise.

Example 5: As argued in Sec. 6.8 of [1], the ultimate QT will be based on a ring, not a field, i.e., only addition, subtraction and multiplication are fundamental mathematical operations, while division is not.

The above examples demonstrate that there are phenomena which can be explained only in FM because for them it is important that p is finite and not infinitely large. So, we have an analogy with the case that SR can explain phenomena where c is finite while NM cannot explain such phenomena.

8 Answers to arguments (a-c) in Sec. 1

As noted in Sec. 1, a fundamental problem in SQT is the problem of divergences. To get around this problem, physicists usually do the following. In integrals over the absolute values of momenta, the upper limit of integration is taken not ∞ as it should be, but a certain value L called the Pauli-Villars cutoff. Then all integrals formally become finite, but they depend on the nonphysical very large quantity L. In renormalizable theories, various contributions to the S-matrix can be arranged in such a way that the contributions with L cancel, but in non-renormalizable theories it is not possible to get rid of L.

The idea of argument a) is such that, by analogy with SQT, where there are divergent integrals that are cut off by the value of L, in FQT there are formally no divergences but there are quantities depending on the enormous value p. However, this analogy doesn't work for several reasons.

In Sec. 2 we noted that, from our experience in NM, we think that some of the arguments are based on common sense. But these arguments only work at speeds which are much less than c and often fail at speeds comparable to c. Likewise, some arguments which, from our experience in SM, seem to come from common sense, usually work in FM only for numbers much less than p, and often fail for numbers comparable to p.

In FM there are no strict concepts of positive and negative and the concepts of $>$ and $<$. These concepts approximately work for numbers that are much less than p and are in some neighborhood of zero on Figure 1.

In SM, when we add two positive numbers, we always get a positive number that is greater than the original arguments. However, since in FM calculations are carried out modulo p, situations are possible when we add two numbers that, from the point of view of "common sense", seem positive we get a number that, from the point of view of "common sense", seem negative. For example, in finite mathematics, $(p-1)/2 + 1 = -(p-1)/2$, i.e., adding two numbers which in Figure 1 are in the right half-plane, we get a number that in this figure is in the left half-plane.

In **Example 2** in Sec. 7, we described an example when in SQT, as a result of adding many positive values, a divergent expression is obtained, while in FQT the result is 0 because the calculations are carried out modulo p. Thus, argument a) does not always work in FQT.

The argument b) is unacceptable even because the theory with adeles is not finite and therefore automatically has foundational problems. The arguments b) and c) that it is not clear from what considerations p is chosen is not a refutation of FQT for the following reason. As explained in Sec. 6, the value of p is not a fundamental parameter that follows from some theory: this value is determined by the state of the universe at the given stage of its development, and at different stages the values of p are different.

To conclude this section, we note the following. One of the objections to FQT is that the authors of these objections interpret p as the greatest possible number in nature and invoke the argument attributed to Euclid that there can be no greatest number in nature because if p is such a number then $(p+1) > p$. Similarly, one can say that c cannot be the greatest possible speed because $1.001c > c$. As explained above, these arguments arise because our experience at speeds which are much less than c and numbers which are much less than p is extrapolated to situations when speeds are comparable to c or numbers are comparable to p.

9 Conclusion

The purpose of this paper is to explain at the simplest possible level why FM is more general (fundamental) than SM. As noted in Sec. 5, the belief of most mathematicians and physicists that SM is the most fundamental arose for historical reasons. However, as explained in Sec. 4, simple mathematical arguments show that SM (involving the concept of infinities) is a degenerate case of FM: SM arises from FM in the degenerate

case when operations modulo a number are discarded.

We call FQT a quantum theory based on FM. It is determined by a parameter p which is the characteristic of a ring in finite mathematics describing physics. We note that in FQT there are no infinities and that is why divergences are absent in principle. Probabilistic interpretation of FQT is only approximate: it applies only to states described by numbers which are much less than p.

In Sec. 6 we have given arguments that p is not a fundamental quantity that is determined by some theory, but depends on the state of the universe at a given stage. Therefore, p is different at different stages of the universe.

The question of why p is this and not another is similar to the question of why the values of $(c, \hbar)$ are such and not others. As explained in [1, 15], currently they are such simply because people want to measure c in m/s and $\hbar$ in $kg \cdot m^2/s$, and it is natural to expect that these values at different stages of the universe are different.

As noted in Sec. 7, at the present stage of the universe, p is an enormous quantity of the order of $exp(10^{80})$. Therefore, at present, SM almost always works with very high accuracy. At the same time, in [1, 15] and Sec. 7 we argue that in early stages of the universe, p was much less than now. Therefore, at these stages, the finitude of mathematics played a much greater role than it does now. As a result, the problem of baryon asymmetry of the universe does not arise.

The famous Kronecker's expression is: "God made the natural numbers, all else is the work of man". However, in view of the above discussion, I propose to reformulate this expression as: "God made only finite sets of natural numbers, all else is the work of man". For illustration, consider a case when some experiment is conducted N times, the first event happens n_1 times, the second one — n_2 times etc. such that $n_1 + n_2 + ... = N$. Then the experiment is fully described by a finite set of natural numbers. But people introduce rational numbers $w_i = w_i(N) = n_i/N$, introduce the concept of limit and define probabilities as limits of the quantities $w_i(N)$ when $N \to \infty$.

The above discussion shows that FM is not only more general (fundamental) than SM but, in addition, in FM there are no foundational problems because every statement can be explicitly verified by a finite number of steps. The conclusion from the above consideration can be formulated as:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

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