CAN EINSTEIN FIELD EQUATION BE GENERALIZED?

TOMASZ KOBIERZYCKI

Abstract. In this short paper I did show a simple model of extending Einstein field equations.

Date: December 27, 2024.

Contents

1. Is faster than speed of light motion possible?

Lorentz transformations apply for velocity that is less than speed of light. But if velocity is bigger than it they no longer apply in their normal form. There is a simple way to make them stick to light cone but it would need to make them in switch time and space. Method is pretty simple I modify Lorentz factor by a simple function, where i is imaginary unit:

$$\sigma(v) = \begin{cases} 1 & \text{if} \quad v < c \\ -i & \text{if} \quad v > c \end{cases} \tag{1.1}$$

So now Lorentz factor is equal to:

$$\gamma = \frac{\sigma(v)}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.2}$$

Now let me test this idea, for simplest form of Lorentz transformation, I will assume only movement in one x direction so coordinate transformation takes form:

$$t' = \frac{\sigma(v)}{\sqrt{1 - \frac{v^2}{c^2}}} \left(t - \frac{vx}{c^2}\right) \tag{1.3}$$

$$x' = \frac{\sigma(v)}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt) \tag{1.4}$$

$$y' = y \tag{1.5}$$

$$z' = z \tag{1.6}$$

Now let me assume that velocity of object is infinite, by taking a limit I will arrive at:

$$t' = \lim_{v \to \infty} \frac{\sigma(v)}{\sqrt{1 - \frac{v^2}{c^2}}} \left(t - \frac{vx}{c^2}\right) = \lim_{v \to \infty} \frac{-i}{\sqrt{1 - \frac{v^2}{c^2}}} \left(t - \frac{vx}{c^2}\right) = \frac{x}{c} \tag{1.7}$$

$$x' = \lim_{v \to \infty} \frac{\sigma(v)}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt) = \lim_{v \to \infty} \frac{-i}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt) = ct \tag{1.8}$$

It means that time and space did switch roles. This approach leaves out imaginary part of that transformation making all transformations real. It means to cover all possible space of light cones I need to take into account light cone with space and times switched places. But what about motion backwards in time? it seems that I covered all possible velocities form $-\infty$ to $+\infty$ but in truth there is a imaginary one left.

2. Light cone full space

Now I can do more general approach to problem from previous subsection. I will write those transformation but forward in time ones and backwards in time new ones:

$$t'_{+} = \frac{\sigma(v)}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right) \tag{2.1}$$

$$x'_{+} = \frac{\sigma(v)}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) \tag{2.2}$$

$$t'_{-} = \frac{-\sigma(v)}{\sqrt{1+\frac{v^2}{c^2}}}\left(t + \frac{ivx}{c^2}\right) \tag{2.3}$$

$$x'_{-} = \frac{-\sigma(v)}{\sqrt{1+\frac{v^2}{c^2}}}(x + ivt) \tag{2.4}$$

First two represent speeds that are real and second two that are imaginary. Now objects can travel with always not speed of light but not only faster but faster forward in time and backwards in time, so whole possible space of light cones is covered. Now I can see that energy in all those regions is never imaginary, it will be negative but never imaginary. I can write energy for those regions:

$$E_{+} = \frac{\sigma(v)\, m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2.5}$$

$$E_{-} = \frac{-\sigma(v)\, m_0 c^2}{\sqrt{1+\frac{v^2}{c^2}}} \tag{2.6}$$

Now normal matter that moves slower than light will have either positive energy or negative energy, same for matter that moves faster than light. Negative energy represents matter moving backwards in time in both cases. Now only thing to check is how spacetime interval will react to those transformation. In cases of slower than speed of light it will behave normally, but for faster than speed of light it will give negative distance squared that is normal result from relativity. Now question can be why to bother to extend those into four possible light cones direction? It's really simple from fact that is possible and could give more complete view of spacetime itself.

3. Light signal interpretation of Relativity

How does light signal relate to spacetime? From previous subsection I can answer this question. What transformation does apply when moving with speed of light? Objects that move slower than speed of light can't stand still in time, object moving faster than light can't stand still in space- combined I get answer for light particles. They both can't stand still in space and time, from their perspective they act exactly as a clock and ruler of reference system. It means that Lorentz transformations for photons are just:

$$t'_+ = x'_+ = \frac{\sigma(v)}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right) = \frac{\sigma(v)}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) \tag{3.1}$$

$$t'_- = x'_- = \frac{-\sigma(v)}{\sqrt{1-\frac{v^2}{c^2}}}\left(t + \frac{ivx}{c^2}\right) = \frac{-\sigma(v)}{\sqrt{1+\frac{v^2}{c^2}}}(x + ivt) \tag{3.2}$$

Photon transformations are exactly dependent on frame of reference. They will always state that units of time and space are equal from their perspective both axis of space and time are same axis. So its mapping all point of space onto all space of time to always preserve exactly relation between space and time being equal. For flat spacetime it's just:

$$c^2dt^2 = dx^2 + dy^2 + dz^2 \tag{3.3}$$

Now this idea is core of this interpretation of Relativity, light signals will always move same units in space and time no matter what is velocity of observer or geometry of spacetime. To generalize it to any coordinate system or any spacetime I just need to expand space and time components of metric:

$$g_{\mu\nu}dx^\mu dx^\nu = 0 \tag{3.4}$$

$$g_{00}dx^0dx^0 + g_{0a}dx^0dx^a + g_{a0}dx^adx^0 + g_{ab}dx^adx^b = 0 \tag{3.5}$$

$$g_{00}dx^0dx^0 + 2g_{0a}dx^0dx^a + g_{ab}dx^adx^b = 0 \tag{3.6}$$

It means that invariant property of spacetime is light cone structure that is affected by matter field. Like in case of flat spacetime in general case light cones define how events change so all casual structure. Spacetime is defined by matter field and matter field changes how light cones propagate, each change of it is defined by change in curvature of spacetime. Light frame of reference is always defined as all events in space merging with time parts. This means that in general light particles can't be stationary both in space and time thus they define space and time change. In general case light cones forming any possible geometry that is defined by matter field.

4. Generalizing Einstein

I will assume that contraction of generalized Einstein tensor will lead to Einstein tensor [1] . First I will expand Riemann [2] tensor itself, then add extra term to preserve contraction properties and finally change constant number of Weyl tensor, so for n dimensions this tensor takes from:

$$G_{\alpha\mu\beta\nu} = \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right)$$

$$-\frac{1}{n-2}\left(R_{\alpha\nu}g_{\mu\beta} + R_{\mu\beta}g_{\alpha\nu}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)R \tag{4.1}$$

Contraction of this tensor leads to Einstein tensor [3] , and this tensor just like Einstein tensor is conserved [4]:

$$G^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{4.2}$$

$$\nabla_{\alpha}G^{\alpha}{}_{\mu\beta\nu} = 0 \tag{4.3}$$

Following from this tensor I can right side of equations, that will be generalized Energy momentum tensor. I will assume that it's not conservative in ordinary sense, it follows Ricci tensor [5]:

$$\kappa\nabla_{\nu}K^{\nu}{}_{\mu} = \frac{1}{2}\kappa\nabla_{\mu}K = \nabla_{\nu}R^{\nu}{}_{\mu} = \frac{1}{2}\nabla_{\mu}R \tag{4.4}$$

From it follows that:

$$\nabla_{\mu}\left(K^{\mu\nu} - \frac{1}{2}g^{\mu\nu}K\right) = 0 \tag{4.5}$$

$$\nabla_{\mu}K^{\mu\nu} = \frac{1}{2}\nabla_{\mu}g^{\mu\nu}K \tag{4.6}$$

This gives new fourth order equation, that can be reduced to two equations. Forth order equation is:

$$G_{\alpha\mu\beta\nu} = \kappa K_{\alpha\mu\beta\nu} \tag{4.7}$$

$$K_{\alpha\mu\beta\nu} = \frac{1}{n-3}\hat{K}_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(K_{\alpha\beta}g_{\mu\nu} + K_{\mu\nu}g_{\alpha\beta}\right)$$

$$-\frac{1}{n-2}\left(K_{\alpha\nu}g_{\mu\beta} + K_{\mu\beta}g_{\alpha\nu}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)K \tag{4.8}$$

Conservation law for this stress energy momentum tensor is:

$$\nabla_{\mu}K^{\mu\nu} = \frac{1}{2}\nabla_{\mu}g^{\mu\nu}K \tag{4.9}$$

That in flat spacetime turns into:

$$\partial_{\mu}K^{\mu\nu} = \frac{1}{2}\partial_{\mu}\eta^{\mu\nu}K \tag{4.10}$$

Those fourth order equation can be expressed as two equations:

$$R_{\mu\nu} = \kappa K_{\mu\nu} \quad C_{\alpha\mu\beta\nu} = \kappa\hat{K}_{\alpha\mu\beta\nu} \tag{4.11}$$

5. Meaning of field equations

Field equation split into two equations one is responsible for matter field and one for gravity field. First equation states that Ricci tensor is equal to energy stress tensor times Einstein constant, they are both not conserved that leads to important result for matter field, I can write for flat spacetime conservation law for matter as set of four equations:

$$\partial_1 K^{10} + \partial_2 K^{20} + \partial_3 K^{30} = -\frac{1}{2}\partial_0 (K^{00} + K^{11} + K^{22} + K^{33}) \tag{5.1}$$

$$\partial_0 K^{01} + \partial_2 K^{21} + \partial_3 K^{31} = -\frac{1}{2}\partial_1 (K^{00} + K^{11} - K^{22} - K^{33}) \tag{5.2}$$

$$\partial_0 K^{02} + \partial_1 K^{12} + \partial_3 K^{32} = -\frac{1}{2}\partial_2 (K^{00} + K^{22} - K^{11} - K^{33}) \tag{5.3}$$

$$\partial_0 K^{03} + \partial_1 K^{13} + \partial_2 K^{23} = -\frac{1}{2}\partial_3 (K^{00} + K^{33} - K^{11} - K^{22}) \tag{5.4}$$

I will focus on simplest case of those equations where there is only matter density and pressure so only diagonal elements of stress energy tensor is not equal to zero:

$$\partial_0 (K^{11} + K^{22} + K^{33}) = -\partial_0 K^{00} \tag{5.5}$$

$$\partial_1 (K^{22} + K^{33} - K^{11}) = \partial_1 K^{00} \tag{5.6}$$

$$\partial_2 (K^{11} + K^{33} - K^{22}) = \partial_2 K^{00} \tag{5.7}$$

$$\partial_3 (K^{11} + K^{22} - K^{33}) = \partial_3 K^{00} \tag{5.8}$$

Now additionally I will assume pressure is same in every direction so it will lead to:

$$\partial_0 P = -\frac{1}{3}\partial_0 \rho \tag{5.9}$$

$$\partial_a P = \partial_a \rho \tag{5.10}$$

First equations states that if matter density increases with time pressure has to decrease. For example for collapsing matter if mass density will increase it means that pressure in all directions will decrease for this matter type field. Next third equations state that If I do move in given direction of space change in pressure in that direction is equal to change of matter density. For example if I move in x direction and pressure does increase it means that energy density increases, if I have temporally static object like a star it will mean that closer to core of a star more energy density is there is more pressure it has. This type of matter and way that is conserved does not fulfill normal conservation laws. For example a collapsing object will not increase it's total pressure but it will decrease that can be thought as transfer of energy from pressure to matter density. Still field equation for matter only field is just, but it's not normal stress momentum tensor but one defined in previous subsection:

$$R_{\mu\nu} = \kappa T_{\mu\nu} \tag{5.11}$$

6. Ricci tensor connection with matter field

Same equation is fulfilled by Ricci tensor, first i skip flat spacetime and write it in curved spacetime:

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla_\mu g^{\mu\nu} R \tag{6.1}$$

$$\nabla_0 R^{00} + \nabla_1 R^{10} + \nabla_2 R^{20} + \nabla_3 R^{30} = \frac{1}{2}\left(g^{00}\nabla_0 R + g^{10}\nabla_1 R + g^{20}\nabla_2 R + g^{30}\nabla_3 R\right) \tag{6.2}$$

$$\nabla_0 R^{01} + \nabla_1 R^{11} + \nabla_2 R^{21} + \nabla_3 R^{31} = \frac{1}{2}\left(g^{10}\nabla_0 R + g^{11}\nabla_1 R + g^{12}\nabla_2 R + g^{13}\nabla_3 R\right) \tag{6.3}$$

$$\nabla_0 R^{02} + \nabla_1 R^{12} + \nabla_2 R^{22} + \nabla_3 R^{32} = \frac{1}{2}\left(g^{20}\nabla_0 R + g^{21}\nabla_1 R + g^{22}\nabla_2 R + g^{23}\nabla_3 R\right) \tag{6.4}$$

$$\nabla_0 R^{03} + \nabla_1 R^{13} + \nabla_2 R^{23} + \nabla_3 R^{33} = \frac{1}{2}\left(g^{30}\nabla_0 R + g^{31}\nabla_1 R + g^{32}\nabla_2 R + g^{33}\nabla_3 R\right) \tag{6.5}$$

Now If I assume only diagonal parts of metric, Ricci and energy tensors I will arrive at:

$$\nabla_0 R^{00} = g^{00}\nabla_0\left(g_{11}R^{11} + g_{22}R^{22} + g_{33}R^{33}\right) \tag{6.6}$$

$$\nabla_1 R^{11} - g^{11}\nabla_1\left(g_{22}R^{22} + g_{33}R^{33}\right) = g_{00}g^{11}\nabla_1 R^{00} \tag{6.7}$$

$$\nabla_2 R^{22} - g^{22}\nabla_2\left(g_{11}R^{11} + g_{33}R^{33}\right) = g_{00}g^{22}\nabla_2 R^{00} \tag{6.8}$$

$$\nabla_3 R^{33} - g^{33}\nabla_3\left(g_{11}R^{11} + g_{22}R^{22}\right) = g_{00}g^{33}\nabla_3 R^{00} \tag{6.9}$$

Those equations correspond to same equations for energy tensor:

$$\nabla_0\rho = -3g^{00}\nabla_0 P \tag{6.10}$$

$$\nabla_1 P - g^{11}\nabla_1\left(g_{22} + g_{33}\right)P = g_{00}g^{11}\nabla_1\rho \tag{6.11}$$

$$\nabla_2 P - g^{22}\nabla_2\left(g_{11} + g_{33}\right)P = g_{00}g^{22}\nabla_2\rho \tag{6.12}$$

$$\nabla_3 P - g^{33}\nabla_3\left(g_{11} + g_{22}\right)P = g_{00}g^{33}\nabla_3\rho \tag{6.13}$$

I can multiply both side of those equations by metric tensor part:

$$g_{11}\nabla_1 P - \left(g_{22} + g_{33}\right)\nabla_1 P = g_{00}\nabla_1\rho \tag{6.14}$$

$$g_{22}\nabla_2 P - \left(g_{11} + g_{33}\right)\nabla_2 P = g_{00}\nabla_2\rho \tag{6.15}$$

$$g_{33}\nabla_3 P - \left(g_{11} + g_{22}\right)\nabla_3 P = g_{00}\nabla_3\rho \tag{6.16}$$

So finally I will get, using fact that each component of pressure is equal:

$$g_{aa}\nabla_a P = g_{00}\nabla_a\rho \tag{6.17}$$

So combing it with Ricci equations, that gives exactly same result as before:

$$\nabla_0 K^{00} = -3g^{00}\nabla_0 P = \nabla_0 R^{00} = -3g^{00}\nabla_0 R^{bb} \tag{6.18}$$

$$g_{aa}\nabla_a P = g_{00}\nabla_a\rho = g_{aa}\nabla_a R^{bb} = g_{00}\nabla_a R^{00} \tag{6.19}$$

7. Gravity field in field equations

Second equation to that field equation can be reduced is responsible for a gravity field , so a vacuum. It connects Weyl tensor with vacuum stress momentum tensor:

$$C_{\alpha\mu\beta\nu} = \kappa\hat{K}_{\alpha\mu\beta\nu} \tag{7.1}$$

It is traceless as Weyl tensor, if first equation defines a matter field second one defines a vacuum and from it follows that this tensor says all about gravity interaction between matter field. As said before it's first property is that it's traceless for all indexes:

$$\hat{K}^{\alpha}{}_{\mu\alpha\nu} = 0 \tag{7.2}$$

Next property is that it's conservation equation so covariant derivative is equal to Weyl tensor:

$$\nabla_{\alpha}C^{\alpha}{}_{\mu\beta\nu} = \kappa\nabla_{\alpha}\hat{K}^{\alpha}{}_{\mu\beta\nu} \tag{7.3}$$

In General Relativity there is no a gravity energy momentum tensor. Here there is a gravity energy momentum tensor. If I write full stress momentum tensor it's conserved:

$$\nabla_{\alpha}K^{\alpha}{}_{\mu\beta\nu} = 0 \tag{7.4}$$

It can be split into two parts:

$$K_{\alpha\mu\beta\nu} = \frac{1}{n-3}\hat{K}_{\alpha\mu\beta\nu} + \tilde{K}_{\alpha\mu\beta\nu} \tag{7.5}$$

Where:

$$\tilde{K}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(K_{\alpha\beta}g_{\mu\nu} + K_{\mu\nu}g_{\alpha\beta}\right)$$
$$-\frac{1}{n-2}\left(K_{\alpha\nu}g_{\mu\beta} + K_{\mu\beta}g_{\alpha\nu}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)K \tag{7.6}$$

First part of that tensor is responsible for gravity field and second one is responsible for matter field. Same goes with extended Einstein tensor:

$$G_{\alpha\mu\beta\nu} = \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \tilde{R}_{\alpha\mu\beta\nu} \tag{7.7}$$

$$\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} + R_{\mu\nu}g_{\alpha\beta}\right)$$
$$-\frac{1}{n-2}\left(R_{\alpha\nu}g_{\mu\beta} + R_{\mu\beta}g_{\alpha\nu}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)R \tag{7.8}$$

First part is responsible for gravity field curvature so Weyl tensor and second one for matter field curvature so Ricci tensor.

8. Gravity field conservation equations

I can write general conservation laws for both tensors as:

$$\nabla_\alpha G^\alpha{}_{\mu\beta\nu} = \nabla_\alpha K^\alpha{}_{\mu\beta\nu} = 0 \tag{8.1}$$

If one part of field equations vanishes so either matter field part of gravity matter field part, this equations no longer holds true. In order to hold true at each point of spacetime, there needs be equal amount of both matter field and gravity field in that point of spacetime then:

$$\frac{\kappa}{n-3}\hat{K}_{\alpha\mu\beta\nu}(\mathbf{x}) = \kappa\tilde{K}_{\alpha\mu\beta\nu}(\mathbf{x}) = \frac{1}{n-3}C_{\alpha\mu\beta\nu}(\mathbf{x}) = \tilde{R}_{\alpha\mu\beta\nu}(\mathbf{x}) \tag{8.2}$$

Where I did denote that it's a tensor field so it depends on each point of spacetime. This comes from fact that If I set part of whole tensor field to zero it's will be vanish:

$$\nabla_\alpha G^\alpha{}_{\mu\beta\nu} = \frac{1}{n-3}\nabla_\alpha C^\alpha{}_{\mu\beta\nu} \neq 0 \tag{8.3}$$

$$\nabla_\alpha G^\alpha{}_{\mu\beta\nu} = \nabla_\alpha \tilde{R}^\alpha{}_{\mu\beta\nu} \neq 0 \tag{8.4}$$

$$\nabla_\alpha K^\alpha{}_{\mu\beta\nu} = \frac{1}{n-3}\nabla_\alpha \hat{K}^\alpha{}_{\mu\beta\nu} \neq 0 \tag{8.5}$$

$$\nabla_\alpha K^\alpha{}_{\mu\beta\nu} = \nabla_\alpha \tilde{K}^\alpha{}_{\mu\beta\nu} \neq 0 \tag{8.6}$$

It means that in order for this tensor to vanish at each point of spacetime I need to assume that matter field and gravity field matter does not vanish at each point of spacetime. That mean that at each point of spacetime there is always matter field present equal to gravity field matter. From it follows that both shape and volume of spacetime changes at each point of spacetime. Where there is seemingly vacuum present there has to be matter field equal to that vacuum energy. This is fulfill by first equation:

$$\frac{\kappa}{n-3}\hat{K}_{\alpha\mu\beta\nu}(\mathbf{x}) = \kappa\tilde{K}_{\alpha\mu\beta\nu}(\mathbf{x}) \tag{8.7}$$

And there is spacetime curvature there present equal to it, that changes both volume and shape that creates second equation:

$$\frac{1}{n-3}C_{\alpha\mu\beta\nu}(\mathbf{x}) = \tilde{R}_{\alpha\mu\beta\nu}(\mathbf{x}) \tag{8.8}$$

From it follows that both are equal at each point and they never vanish so equation of conservation is always fulfilled:

$$\nabla_\alpha G^\alpha{}_{\mu\beta\nu}(\mathbf{x}) = \nabla_\alpha K^\alpha{}_{\mu\beta\nu}(\mathbf{x}) = 0 \tag{8.9}$$

References

[1] https://mathworld.wolfram.com/EinsteinTensor.html

[2] https://mathworld.wolfram.com/RiemannTensor.html

[3] https://mathworld.wolfram.com/EinsteinTensor.html

[4] https://mathworld.wolfram.com/EinsteinFieldEquations.html

[5] https://mathworld.wolfram.com/RicciCurvatureTensor.html

Email address: kobierzyckitomasz@gmail.com