CAN EINSTEIN FIELD EQUATION BE GENERALIZED?

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Abstract. This work is based on idea of extending Einstein field equations leading to more general statement about gravity field, this leads to new field equations and new idea about gravity field. Gravity field is now directly connected to matter field. From it follows that matter field in given point of space is responsible for spacetime curvature in that point so matter field can't vanish in any point of space. Field equations now become fourth order equations and include Riemann tensor Ricci tensor and metric tensors with Ricci scalar, same goes for stress-momentum tensor that is now composed out of stress-momentum tensor,metric tensors and its trace.

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Contents

1. Physical need for field equation

1.1. **What is object and it's gravity field?** I will start simple thought experiment, let me assume gravity field of a star that is located in far way distance from very small object who's mass relative to that start is to small to make it count on as gravity source. So I have a gravity source star and gravitating object towards that star that is really far away not infinite far away but enough far away that any communication between those two objects with speed of light will take long enough time. Now I want to evaluate gravity field of that star, star is accelerating in all directions so it's accelerating towards far away object, I will for now assume perspective of gravitating object first so that's why I assume there is no spacetime curvature just star accelerating in all directions. This leads of course problem of size of star if it's accelerating in all directions start radius would be bigger, so I need to account for spacetime or more precise space falling (in simplest case scenario) into star preventing it from changing it's radius thus acting opposite way than it's movement. From far away gravitating object it will look like is accelerating towards it and space around star is accelerating into the star. It means that object will accelerate with space falling into start now from star perspective there it's accelerating and space is falling into it so it stays stationary but far away object from it will accelerate towards it. Those two perspective are not equal, star will state that it is stationary and gravitating object in accelerating towards center of star but gravitating object will state that it's stationary and it's star accelerating towards it. Even adding space falling into star doses not resolve this problem, as I can say that from star perspective that it's stationary and it's gravitating object that accelerate with space into the star so star is no longer stationary that is clear violation of what star from it's reference frame will observe. If those two perspectives are not equal it means acceleration is relative and spacetime curvature can't be based on acceleration. As object, both objects can say that they are stationary and it's other object in motion relative to them. Adding spacetime curvature to picture will not help as saying that it's really star that is accelerating and it's the gravitating object that is accelerating with space falling into the star will break what is observed. On the other hand if both object are accelerating towards each other and star is accelerating in all directions so is gravitating object there is no real change in radius of those objects as those effects will cancel out. There is another problem, star can't affect by instant object that is so far away as it would take time for light beam to reach it and as I assumed object is enough far away that this it will take enough time for light beam to reach it that gravity effects can't be felt by instant. It means that star is just accelerating in all directions and so is gravitating object from opposite perspectives, so acceleration in this case is relative and if gravity field would depend on acceleration it could be said that both objects are cause of motion of another that would be clear violation of how gravity field works. As gravity field should depend on objects energy and energy of star is far bigger than small gravitating object far away from it.

1.2. **Gravity field equals matter field.** Gravity field can't be relative as it's true cause of motion of objects. But in truth how motion and spacetime curvature relate? If I go back to example stated before, only solution to this problem is that both objects are really accelerating and their motion is not relative. But how can acceleration be transferred by instant to object that is located in space? It could be not, if gravity field makes both object accelerate star is accelerating faster than gravitating object, when they "meet" in some location in space they will both accelerate with same rate. It means that to cause object acceleration from any distance there has to be acceleration present in all space that is not relative and does not depend on frame of reference. But if there is acceleration present in space there is energy in that space as energy is true cause of gravity field thus motion. So star will be present in all space as it's present where we observe it but change is that more distance from observed position of that star energy of that star is more spread out that will lead to less acceleration and weaker gravity field. It means that if gravity field is not relative there is need for non-vanishing matter field. In each point of space there is acceleration that is equal to energy content of that part of space, this could be explained by two ways, gravity field has energy or gravity field and matter field is same. If gravity field has energy still that would mean that acceleration of any object in space does not depend on matter field directly but on effects of matter field on spacetime. But if it's so how to notice any change between gravity field and matter field? If matter field causes changes in gravity field and gravity field carries energy equal to those changes why to assume that those two phenomena are not same thing? If gravity field carries energy and matter carries energy , matter will change gravity field by its energy and gravity field will posses that energy there is no way to separate those two, I can assume that there is only gravity field carrying energy and it's cause of acceleration and from it follows spacetime curvature or I can assume that there is only matter field that carries energy and it's cause of spacetime curvature. This hint is what this hypothesis is based on. There is no measurable way to say is matter field and gravity field two separate phenomenons. As from it follows that both can be source of acceleration and both can be source of spacetime curvature that acts on that acceleration. This leads to simple conclusion, gravity field is equal to matter field and all effects of gravity field can be equivalent of matter field that will lead to spacetime curvature. Spacetime curvature will always counteract effects of motion of gravity field, as star expands in all directions space around it is falling into it with same acceleration, but it does not only happen in space around star but in whole space. It means that matter field is present in all space and from it follows energy is present in all space. If so matter field can't vanish as it would lead to gravity field vanishing.

1.3. **Light cone in motion.** Light cone is defined by a casual connection region, it connects past and future for given observer at given point. But what happens to light cone if gravity field changes in given point? Normally local action can propagate with faster than speed of light but here it's spacetime itself that is changing. Stated before, gravity field and matter field are equal from it follows that change in gravity field thus change in matter field will change curvature of spacetime. Going back to example with star and far away gravitating object, if star changes it's position it will for example move towards gravitating object that change will shift location of all space. As star is source of gravity field that extends in all space, changing it's motion will result in changing whole gravity field. That change will be felt by an instant as whole space will change if star moves forward in space, that will result in space going back in all locations. More distance from star it will lead to less space movement as energy of star will be spread out in bigger volume. It means that gravity interactions with matter field being equal to gravity field in case of source field extended in space will lead to change in gravity field by instant. If source of gravity field changes , curvature of spacetime will change by instant as it's fully depend on matter field. As in stationary example where star expands in all directions thus space contracts in all directions, when I do add movement to it space in all points will move opposite way to it. How it's possible if speed of light has finite speed? Answer is that there is not a propagation of any kind of field here, there is change in field. That change will affect directly curvature but to put into more detailed perspective I need to assume that light cones in whole space will change it's location as space in all points in space moves. It means that locally causal structure is preserved but it does not apply to gravity field itself as it leads to spacetime curvature. All observers will agree that there was change in gravity field so in spacetime curvature. It means that matter field can't locally propagate faster than speed of light but gravity field will itself change by instant matter field changes. Light cones in all space if I assume weak enough gravity field will change it's position from fact that they are located in space that will now move with matter field. It means that gravity field being equal to matter field and then matter field being equal to curvature of space lead so instant change in gravity field where there is change in matter field. There is no information in reality that travels between field itself as it's field that changes , from it follows that field can lead to non-local effects but in truth there is no information propagation but synchronized change in matter field- whole matter field changes so does curvature of spacetime. It means that speed of gravity field change is equal to speed of matter field change and if I change matter field I will change spacetime itself so there is no delay in changing matter field and gravity field thus curvature of spacetime.

1.4. **Einstein tensor.** Einstein tensor is basis of General Relativity, it's derived from contracting Bianchi Identity [2] that will lead to it's form:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{1.1}$$

From Bianchi identities comes naturally that this tensor does vanish with respect to covariant derivative that can be written [3]:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{1.2}$$

This need comes naturally from matter field conservation on the right side of equations that is given by stress-momentum tensor $T^{\mu\nu}$ that will vanish when taking covariant derivative that would give matter field conservation. But as stated before if I make matter field vanish I will arrive at non zero curvature of spacetime or saying it other way, there are vacuum solutions to those equations where there is no matter field present. This comes from fact that Ricci tensor is used in equations and from it follows that spacetime in vacuum is just Ricci flat and Riemann tensor is not vanishing so spacetime outside a massive object is still curved. This it's not valid from previous subsections reasoning. If matter field vanishes so does gravity field and thus spacetime curvature. It means that field equation that is valid from this model point of view has to deal not with Ricci Tensor but with Riemann tensor as it needs to link spacetime curvature directly to matter field. So Einstein field equations [4] need to be extended to Riemann tensor and constrain on Ricci tensor is that if it's equal to zero so is Riemann tensor and from it follows that I will arrive at flat spacetime. I need to link gravity field with Riemann tensor then if there is no matter field present Ricci tensor does vanish but so does first Riemann tensor so it leads to flat spacetime. This strong need is source of this idea, matter field vanishing will lead to gravity field vanishing. But it's not easy to arrive at equations like those, first I will write normal Einstein field equations:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} \tag{1.3}$$

As stated before I will need to change this equation to fourth order equation. It will be done in simplest possible way, instead of contraction Bianchi Identity twice I will contract it only once. It will lead to nice expression of simplest conserved fourth order tensor, but still im left with right side of equation so stress-momentum tensor part. That will lead to fourth order tensor that when doing contractions of both side of equations I will arrive on Einstein field equations or trace reversed field equations and in last two cases zero. To sum it up, I will extend Einstein field equations by first using Bianchi identity that provides simplest fourth order tensor that does vanish when taking covariant derivative then match right side of equation - stress momentum tensor to lead to Einstein field equations or Trace reversed Einstein field equations when contracted or to zero in last two cases. So I will arrive at equations that link gravity field directly to matter field.

1.5. **Simplest form of extended Einstein tensor.** I will assume most general form of extending Einstein tensor. It's not easy task so let's starting by writing a field equation in form of:

$$aR^{\nu}_{\ \mu\kappa\eta} + b\left(\delta^{\nu}_{\kappa}R_{\mu\eta} - \delta^{\nu}_{\eta}R_{\mu\kappa}\right) + c\left(g_{\mu\eta}R^{\nu}_{\kappa} - g_{\mu\kappa}R^{\nu}_{\eta}\right) + dR\left(\delta^{\nu}_{\eta}g_{\mu\kappa} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) \quad (1.4)$$

Where a, b, c, d are constants. I used this form of this tensor as each term is connected to Riemann tensor covariant derivative, I can use Bianachi identities for Riemann tensor covariant derivative that will lead to:

$$R^{\nu}_{\ \mu\kappa\eta} - \nabla_{\kappa}R_{\mu\eta} + \nabla_{\eta}R_{\mu\kappa} = 0 \quad (1.5)$$

$$R^{\nu}_{\ \mu\kappa\eta} = \nabla_{\kappa}R_{\mu\eta} - \nabla_{\eta}R_{\mu\kappa} \quad (1.6)$$

Now term with a is equal to just it, same goes with term with b and c and term wih d is equal to twice of it. Before I will determine constants I need to prove my assumption. It's pretty simple for b as if I take covariant derivative with respect to ν I will arrive at:

$$\nabla_{\nu}\left(\delta^{\nu}_{\kappa}R_{\mu\eta} - \delta^{\nu}_{\eta}R_{\mu\kappa}\right) = \nabla_{\kappa}R_{\mu\eta} - \nabla_{\eta}R_{\mu\kappa} \quad (1.7)$$

Now for term with c is simple to but now I need to use a trick first write it and contract then use contracted Bianachi identity:

$$\nabla_{\nu}\left(g_{\mu\eta}R^{\nu}_{\kappa} - g_{\mu\kappa}R^{\nu}_{\eta}\right) = \frac{1}{2}\left(\nabla_{\kappa}g_{\mu\eta}R - \nabla_{\eta}g_{\mu\kappa}R\right) = \nabla_{\kappa}R_{\mu\eta} - \nabla_{\eta}R_{\mu\kappa} \quad (1.8)$$

And finally with last equation I will just use same trick as before so I will arrive at:

$$\nabla_{\nu}R\left(\delta^{\nu}_{\eta}g_{\mu\kappa} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) = \left(\nabla_{\kappa}g_{\mu\eta}R - \nabla_{\eta}g_{\mu\kappa}R\right) = 2\left(\nabla_{\kappa}R_{\mu\eta} - \nabla_{\eta}R_{\mu\kappa}\right) \quad (1.9)$$

Now only thing left is to solve for constants. It will be system with four unknowns and five equations. First is need that if I sum all terms will arrive at zero so total covariant derivative is equal to zero. Rest of equations will deal with contractions of equations and need it will lead to Einstein field equations so contraction is also conserved. I can write those five equations:

$$a + b + c + 2d = 0 \quad (1.10)$$

$$a + 3b - c = 1 \quad (1.11)$$

$$c + 3d = -1/2 \quad (1.12)$$

$$a + 3b + 3c + 12d = -1 \quad (1.13)$$

$$b = c \quad (1.14)$$

I can use Wolfram Alpha to solve those equations fast and will arrive at [5]:

$$a = -1 \quad b = c = 1 \quad d = -\frac{1}{2} \quad (1.15)$$

$$(1.16)$$

Now I can write field equation with all constants in it:

$$-R^{\nu}_{\ \mu\kappa\eta} + \left(\delta^{\nu}_{\kappa}R_{\mu\eta} - \delta^{\nu}_{\eta}R_{\mu\kappa}\right) + \left(g_{\mu\eta}R^{\nu}_{\kappa} - g_{\mu\kappa}R^{\nu}_{\eta}\right) - \frac{1}{2}R\left(\delta^{\nu}_{\kappa}g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) \quad (1.17)$$

1.6. **Energy momentum tensor and new field equations.** Now only thing left is find right side of equation so matter field part. Idea is very simple I search for tensor that contractions lead to stress-momentum tensor or minus stress-momentum tensor and finally zero to follow Riemann tensor contraction properties. I will do same thing as before write all combinations of tensor from previous subsection but with stress-momentum tensor instead of Ricci tensor:

$$a\left(\delta^\nu_\kappa T_{\mu\eta} - \delta^\nu_\eta T_{\mu\kappa}\right) + b\left(g_{\mu\eta}T^\nu_\kappa - g_{\mu\kappa}T^\nu_\eta\right) + cT\left(\delta^\nu_\kappa g_{\mu\eta} - g_{\mu\kappa}\delta^\nu_\eta\right) \tag{1.18}$$

If I do contract this tensor it will lead to:

$$a\left(\delta^\nu_\nu T_{\mu\eta} - \delta^\nu_\eta T_{\mu\nu}\right) + b\left(g_{\mu\eta}T^\nu_\nu - g_{\mu\nu}T^\nu_\eta\right) + cT\left(\delta^\nu_\nu g_{\mu\eta} - g_{\mu\nu}\delta^\nu_\eta\right) \tag{1.19}$$

$$3aT_{\mu\eta} + bg_{\mu\eta}T - bT_{\mu\eta} + 3cTg_{\mu\eta} \tag{1.20}$$

This contraction will have to give me just stress-momentum tensor so I can write it:

$$3aT_{\mu\eta} + bg_{\mu\eta}T - bT_{\mu\eta} + 3cTg_{\mu\eta} = T_{\mu\eta} \tag{1.21}$$

I will assume that $a = b$ for consistency with previous equations so im left with:

$$3aT_{\mu\eta} + ag_{\mu\eta}T - aT_{\mu\eta} + 3cTg_{\mu\eta} = T_{\mu\eta} \tag{1.22}$$

$$2aT_{\mu\eta} + ag_{\mu\eta}T + 3cTg_{\mu\eta} = T_{\mu\eta} \tag{1.23}$$

This leads to two equations:

$$2a = 1 \tag{1.24}$$

$$a + 3c = 0 \tag{1.25}$$

Solutions to those are trivial and easy to find , $a = \frac{1}{2}$ from it follows that $c = -\frac{1}{6}$, so final form of this tensor is equal to:

$$\frac{1}{2}\left(\delta^\nu_\kappa T_{\mu\eta} - \delta^\nu_\eta T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^\nu_\kappa - g_{\mu\kappa}T^\nu_\eta\right) - \frac{1}{6}T\left(\delta^\nu_\kappa g_{\mu\eta} - g_{\mu\kappa}\delta^\nu_\eta\right) \tag{1.26}$$

Now finally I can construct a field equation, I have two tensors that I will write in full form then in compact form:

$$\begin{aligned} &-R^\nu_{\ \mu\kappa\eta} + \left(\delta^\nu_\kappa R_{\mu\eta} - \delta^\nu_\eta R_{\mu\kappa}\right) + \left(g_{\mu\eta}R^\nu_\kappa - g_{\mu\kappa}R^\nu_\eta\right) - \frac{1}{2}R\left(\delta^\nu_\kappa g_{\mu\eta} - g_{\mu\kappa}\delta^\nu_\eta\right) \\ &= \kappa\left[\frac{1}{2}\left(\delta^\nu_\kappa T_{\mu\eta} - \delta^\nu_\eta T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^\nu_\kappa - g_{\mu\kappa}T^\nu_\eta\right) - \frac{1}{6}T\left(\delta^\nu_\kappa g_{\mu\eta} - g_{\mu\kappa}\delta^\nu_\eta\right)\right] \end{aligned} \tag{1.27}$$

$$G^\nu_{\ \mu\kappa\eta} = \kappa T^\nu_{\ \mu\kappa\eta} \tag{1.28}$$

Those equations fulfill all that was stated before. In next section I will show that indeed those equations are what they should be to fulfill this model statement about gravity field. They are not unique in any way, but i presented simplest method of deriving them.

2. Meaning of field equations

2.1. **Matter field vanishing.** I can easy prove that when matter field does vanish so does right side of equations vanish then Riemann tensor will be equal to zero:

$$-R^{\nu}_{\ \mu\kappa\eta} + \left(\delta^{\nu}_{\kappa} R_{\mu\eta} - \delta^{\nu}_{\eta} R_{\mu\kappa}\right) + \left(g_{\mu\eta} R^{\nu}_{\kappa} - g_{\mu\kappa} R^{\nu}_{\eta}\right) - \frac{1}{2} R \left(\delta^{\nu}_{\kappa} g_{\mu\eta} - g_{\mu\kappa} \delta^{\nu}_{\eta}\right) = 0 \tag{2.1}$$

First I will contract this equation that will lead to Einstein field equations, from it I can deduce that both Ricci tensor and scalar are equal to zero so from it follows that Riemann tensor [7] does vanish with them:

$$G^{\nu}_{\ \mu\nu\eta} = R_{\mu\eta} - \frac{1}{2} R g_{\mu\eta} = 0 \tag{2.2}$$

$$g^{\eta\mu} R_{\mu\eta} - \frac{1}{2} R g^{\eta\mu} g_{\mu\eta} = 0 \tag{2.3}$$

$$R - 2R = 0 \tag{2.4}$$

$$R = 0 \rightarrow R_{\mu\eta} = 0 \tag{2.5}$$

$$-R^{\nu}_{\ \mu\kappa\eta} = 0 \tag{2.6}$$

$$R^{\nu}_{\ \mu\kappa\eta} = 0 \tag{2.7}$$

So from it follows that if matter field vanishes I will not only get Ricci flat spacetime but flat spacetime. At it follows that both Riemann tensor and Ricci tensor vanishes. This gives exactly expected result as matter field will generate spacetime curvature.

2.2. **Trace reversed field equations.** I can trace reverse field equations, first I will start by writing field equation and using fact that Ricci scalar equals negative stress-momentum tensor times Einstein constant:

$$-R^{\nu}_{\ \mu\kappa\eta} + \left(\delta^{\nu}_{\kappa} R_{\mu\eta} - \delta^{\nu}_{\eta} R_{\mu\kappa}\right) + \left(g_{\mu\eta} R^{\nu}_{\kappa} - g_{\mu\kappa} R^{\nu}_{\eta}\right) + \frac{1}{2}\kappa T \left(\delta^{\nu}_{\kappa} g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) \quad (2.8)$$

Then i will use Einstein field equation and trace reverse it so I can get equation to plug in for Ricci tensor:

$$\begin{aligned}-R^{\nu}_{\ \mu\kappa\eta} &+ \kappa\left(\delta^{\nu}_{\kappa}\left(T_{\mu\eta} - \frac{1}{2}g_{\mu\eta}T\right) - \delta^{\nu}_{\eta}\left(T_{\mu\kappa} - \frac{1}{2}g_{\mu\kappa}T\right)\right) \\ &+\kappa\left(g_{\mu\eta}\left(T^{\nu}_{\kappa} - \frac{1}{2}\delta^{\nu}_{\kappa}T\right) - g_{\mu\kappa}\left(T^{\nu}_{\eta} - \frac{1}{2}\delta^{\nu}_{\eta}T\right)\right) \\ &+\frac{1}{2}\kappa T\left(\delta^{\nu}_{\kappa} g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) \end{aligned} \quad (2.9)$$

$$\begin{aligned}-R^{\nu}_{\ \mu\kappa\eta} &+ \kappa\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \frac{1}{2}\delta^{\nu}_{\kappa}g_{\mu\eta}T - \delta^{\nu}_{\eta}T_{\mu\kappa} + \frac{1}{2}\delta^{\nu}_{\eta}g_{\mu\kappa}T\right) \\ &+\kappa\left(g_{\mu\eta}T^{\nu}_{\kappa} - \frac{1}{2}\delta^{\nu}_{\kappa}g_{\mu\eta}T - g_{\mu\kappa}T^{\nu}_{\eta} - \frac{1}{2}\delta^{\nu}_{\eta}g_{\mu\kappa}T\right) \\ &+\frac{1}{2}\kappa T\left(\delta^{\nu}_{\kappa} g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right) \end{aligned} \quad (2.10)$$

$$-R^{\nu}_{\ \mu\kappa\eta} + \kappa\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \delta^{\nu}_{\eta}T_{\mu\kappa}\right) + \kappa\left(g_{\mu\eta}T^{\nu}_{\kappa} - g_{\mu\kappa}T^{\nu}_{\eta}\right) - \frac{1}{2}\kappa T g_{\mu\kappa}\delta^{\nu}_{\eta} \quad (2.11)$$

Now I can move stress-momentum tensor part to right side of equation:

$$\begin{aligned}&-R^{\nu}_{\ \mu\kappa\eta} \\ &= -\kappa\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \delta^{\nu}_{\eta}T_{\mu\kappa}\right) - \kappa\left(g_{\mu\eta}T^{\nu}_{\kappa} - g_{\mu\kappa}T^{\nu}_{\eta}\right) + \frac{1}{2}\kappa T g_{\mu\kappa}\delta^{\nu}_{\eta} \\ &+\kappa\left[\frac{1}{2}\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \delta^{\nu}_{\eta}T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^{\nu}_{\kappa} - g_{\mu\kappa}T^{\nu}_{\eta}\right) - \frac{1}{6}T\left(\delta^{\nu}_{\kappa}g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right)\right] \end{aligned} \quad (2.12)$$

$$\begin{aligned}-R^{\nu}_{\ \mu\kappa\eta} &= -\kappa\left[\frac{1}{2}\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \delta^{\nu}_{\eta}T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^{\nu}_{\kappa} - g_{\mu\kappa}T^{\nu}_{\eta}\right) - \frac{1}{3}T\left(\delta^{\nu}_{\kappa}g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right)\right] \\ R^{\nu}_{\ \mu\kappa\eta} &= \kappa\left[\frac{1}{2}\left(\delta^{\nu}_{\kappa}T_{\mu\eta} - \delta^{\nu}_{\eta}T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^{\nu}_{\kappa} - g_{\mu\kappa}T^{\nu}_{\eta}\right) - \frac{1}{3}T\left(\delta^{\nu}_{\kappa}g_{\mu\eta} - g_{\mu\kappa}\delta^{\nu}_{\eta}\right)\right] \end{aligned} \quad (2.13)$$

Those are trace reversed field equations, that clear state that if stress-momentum tensor does vanish so does spacetime curvature as Riemann tensor is directly connected to matter field thus stress-momentum tensor and if it's equal to zero then I will arrive at flat spacetime. If I contact those I will arrive at trace reversed Einstein field equation as it should be but restriction is that if Ricci tensor does vanish so does Riemann tensor and spacetime has to be flat.

2.3. **Geodesic deviation as equation of motion in field.** Geodesic deviation [8] is simplest interpretation of Riemann tensor, from this fact I can assume that motion of field needs to follow Geodesic deviation equation , where I can use trace reversed field equations to arrive at pure Riemann tensor. Normally to calculate trajectories of particles i will use geodesic equation [9], but from fact that I should define matter field equally as gravity field geodesic deviation suits this purpose better. I can write geodesic deviation equation as:

$$\frac{D^2\xi^\nu}{D\lambda^2} = -R^\nu_{\ \mu\kappa\eta}\frac{dx^\eta}{d\lambda}\frac{dx^\mu}{d\lambda}\xi^\kappa \tag{2.14}$$

Now this term is equal trace reversed field equation:

$$\frac{D^2\xi^\nu}{D\lambda^2} = -R^\nu_{\ \mu\kappa\eta}\frac{dx^\eta}{d\lambda}\frac{dx^\mu}{d\lambda}\xi^\kappa$$
$$= -\kappa\left[\frac{1}{2}\left(\delta^\nu_\kappa T_{\mu\eta} - \delta^\nu_\eta T_{\mu\kappa}\right) + \frac{1}{2}\left(g_{\mu\eta}T^\nu_\kappa - g_{\mu\kappa}T^\nu_\eta\right) - \frac{1}{3}T\left(\delta^\nu_\kappa g_{\mu\eta} - g_{\mu\kappa}\delta^\nu_\eta\right)\right]\frac{dx^\eta}{d\lambda}\frac{dx^\mu}{d\lambda}\xi^\kappa \tag{2.15}$$

This means that gravity field does not have to act only as attraction that makes two geodesic closer to each other, if term on right side changes it's sign it can mean that geodesic will get more far away from each other. That's why it's important to understand trace reversed field equations in more detail. Let there will be dust particle with no pressure and no velocity in any direction then this equation will turn into only time-time components of four velocity:

$$-\kappa\left[\frac{1}{2}\left(\delta^\nu_\kappa T_{00} - \delta^\nu_0 T_{0\kappa}\right) + \frac{1}{2}\left(g_{00}T^\nu_\kappa - g_{0\kappa}T^\nu_0\right) - \frac{1}{3}T\left(\delta^\nu_\kappa g_{00} - g_{0\kappa}\delta^\nu_0\right)\right]\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda}\xi^\kappa \tag{2.16}$$

If im I will assume that care only about time direction of geodesic this equation will change into even simpler form:

$$-\kappa\left[\frac{1}{2}\left(\delta^0_\kappa T_{00} - T_{0\kappa}\right) + \frac{1}{2}\left(g_{00}T^0_\kappa - g_{0\kappa}T^0_0\right) - \frac{1}{3}T\left(\delta^0_\kappa g_{00} - g_{0\kappa}\right)\right]\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda}\xi^\kappa \tag{2.17}$$

That can be expanded into four equations that sum up,where first term vanishes:

$$\frac{D^2\xi^0}{D\lambda^2} = -\kappa\sum_{i=1}^{3}\left[-\frac{1}{2}g_{0i}T^0_0 + \frac{1}{3}Tg_{0i}\right]\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda}\xi^i \tag{2.18}$$

This means that metric has to have non-diagonal components to not make this vanish. On the other hand I can do same calculation but for space components:

$$\frac{D^2\xi^i}{D\lambda^2} = -\kappa\left[\frac{1}{2}\left(T_{00}\right) - \frac{1}{3}Tg_{00}\right]\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda}\xi^i \tag{2.19}$$

This time even for vanishing non-diagonal component I will arrive at change in geodesic. This is simplest work out example of those equations.

2.4. **Dark matter or massive gravity field?** From fact that matter does not vanish in all space I can create simplest model of dark matter. I will use equation for total mass in gravity field equal to [10], where its defined by integral:

$$m(r) = 4\pi \int_0^r \rho r^2 dr \tag{2.20}$$

I will change bottom limit of this integral from zero to r_0 to avoid problems with infinity:

$$m(r) = 4\pi \int_{r_0}^r \rho r^2 dr \tag{2.21}$$

Now I can plug matter density equal to just standard formula $\frac{3m}{4\pi r^3}$ that will give me integral equal to:

$$m(r) = \int_{r_0}^r \frac{3m}{r} dr \tag{2.22}$$

That solutions to it are just logarithm functions:

$$m(r) = 3m \log\left(\frac{r}{r_0}\right) + m_0 \tag{2.23}$$

Where I did assume that rest mass is m_0 and body has fixed radius r_0 so it does not blow up to infinity when radius goes to zero. This simple calculations shows total mass of body is not equal to it's rest mass but its bigger than it. It could in principle explain dark matter not as additional mass but mass generated itself by matter field. In general case I will need to solve integral of this kind for mass density function:

$$m(r) = 4\pi \int_{r_0}^r \rho(r) r^2 dr \tag{2.24}$$

Where I need to take into consideration that gravity field is equal to matter field so from it follows that it has mass, so there are no gaps in space where there is no mass present. For general density function it can be challenging. But I can just use previous formula and sum for many fixed masses with some radius that will lead to:

$$m(r_1, ..., r_i) = 3\sum_{i=1}^n m_i \log\left(\frac{r_i}{r_{i0}}\right) + m_{i0} \tag{2.25}$$

From fact that when radius gets infinite I will arrive at infinite amount of mass that is simply consequence of fact that gravity field is extended in whole space, it means that mass of gravity field thus matter field is infinite in this simple case. Still this simple case gives good example of how dark matter would manifest in gravity field. It would mean for example for a very large mass objects like a black holes that rest mass of it is smaller than what is observed as from it follows that event horizon of black hole depends on it's mass and in case of a black hole change between it and normal model would be not possible but in a case of weaker fields it will manifest as additional mass- that could explain dark matter.

2.5. **Simplest case solutions for matter field with constant density.** I can take trace reversed field equations and from it find simplest solutions to field equations for matter with constant density ρ. I will start by writing fully covariant form of those equations:

$$R_{\nu\mu\kappa\eta} = \kappa \left[\frac{1}{2} \left(g_{\nu\kappa} T_{\mu\eta} - g_{\nu\eta} T_{\mu\kappa} \right) + \frac{1}{2} \left(g_{\mu\eta} T_{\nu\kappa} - g_{\mu\kappa} T_{\nu\eta} \right) - \frac{1}{3} T \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) \right] \tag{2.26}$$

Now I will assume that Riemann tensor is equal to constant times tensor with metric tensors [11] so it will be equal to:

$$R_{\nu\mu\kappa\eta} = C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) \tag{2.27}$$

From it follows form of Ricci tensor:

$$g^{\nu\kappa} R_{\nu\mu\kappa\eta} = 3C g_{\mu\eta} \tag{2.28}$$

$$R_{\mu\eta} = 3C g_{\mu\eta} \tag{2.29}$$

And Ricci scalar:

$$g^{\eta\mu} R_{\mu\eta} = 3C g^{\eta\mu} g_{\mu\eta} \tag{2.30}$$

$$R = 12C \tag{2.31}$$

Now energy momentum tensor can be calculated simply from Einstein field equations:

$$3C g_{\mu\eta} - 6C g_{\mu\eta} = -3C g_{\mu\eta} \tag{2.32}$$

$$T_{\mu\eta} = -\frac{3C}{\kappa} g_{\mu\eta} \tag{2.33}$$

Now I can plug everything into field equations:

$$C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) =$$

$$\frac{1}{2} \left(-3C g_{\nu\kappa} g_{\mu\eta} + 3C g_{\nu\eta} g_{\mu\kappa} \right) + \frac{1}{2} \left(-3C g_{\mu\eta} g_{\nu\kappa} + 3C g_{\mu\kappa} g_{\nu\eta} \right) + 4C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right)$$

$$C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) = \left[-3C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) + 4C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) \right]$$

$$C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) = C \left(g_{\nu\kappa} g_{\mu\eta} - g_{\mu\kappa} g_{\nu\eta} \right) \tag{2.34}$$

That constant will be equal to $C = \frac{\kappa\rho}{3}$, where ρ is constant matter density. From it follows I can finally write metric of this spacetime:

$$ds^2 = \left(1 - \frac{\kappa\rho r^2}{3} \right) c^2 dt^2 - \left(1 - \frac{\kappa\rho r^2}{3} \right)^{-1} dr^2 - d\Omega^2 \tag{2.35}$$

This metric will turn into flat spacetime if matter field density is equal to zero, so it fulfills that basic assumption. Still its static spacetime with constant matter density so it's not most important solution to study but it shows that solutions to those equations are possible.

2.6. **Singularities relation with stress-momentum tensor.** From trace reversed equation I can easy calculate Kretschmann scalar [12], I can combine covariant and contra-variant Riemann tensors. This leads to pretty long calculation of actual scalar that will depend only on stress-momentum tensor. This calculation goes as follows:

$$K=\kappa^2\left[\frac{1}{2}\left(g_{\nu\kappa}T_{\mu\eta}-g_{\nu\eta}T_{\mu\kappa}\right)+\frac{1}{2}\left(g_{\mu\eta}T_{\nu\kappa}-g_{\mu\kappa}T_{\nu\eta}\right)-\frac{1}{3}T\left(g_{\nu\kappa}g_{\mu\eta}-g_{\mu\kappa}g_{\nu\eta}\right)\right]$$
$$\times\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right] \tag{2.36}$$

Where combining all terms I will arrive at six lines of terms that will give all terms in that equation:

$$K=\frac{1}{2}g_{\nu\kappa}T_{\mu\eta}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right]$$
$$-\frac{1}{2}g_{\nu\eta}T_{\mu\kappa}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right]$$
$$+\frac{1}{2}g_{\mu\eta}T_{\nu\kappa}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right]$$
$$-\frac{1}{2}g_{\mu\kappa}T_{\nu\eta}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right]$$
$$-\frac{1}{3}Tg_{\nu\kappa}g_{\mu\eta}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right]$$
$$+\frac{1}{3}Tg_{\mu\kappa}g_{\nu\eta}\left[\frac{1}{2}\left(g^{\nu\kappa}T^{\mu\eta}-g^{\nu\eta}T^{\mu\kappa}\right)+\frac{1}{2}\left(g^{\mu\eta}T^{\nu\kappa}-g^{\mu\kappa}T^{\nu\eta}\right)-\frac{1}{3}T\left(g^{\nu\kappa}g^{\mu\eta}-g^{\mu\kappa}g^{\nu\eta}\right)\right] \tag{2.37}$$

That after summing all those terms will lead to pretty simple result:

$$K=\kappa^2\left(aT_{\alpha\beta}T^{\alpha\beta}+bT^2\right) \tag{2.38}$$

Where those a and b are constants. It comes form fact that all terms will converge to first expression or second, I did not calculate those constants but still it will lead to this equation. I can calculate this term for any stress-momentum tensor and if there is no singularity in that tensor this term will be finite. It means that singularities in this model are result of stress-momentum tensor containing them, if stress-momentum tensor is singularity free it wont show in spacetime curvature as expected from need to for matter field to determine curvature of spacetime.

3. Summary

In this work I did show that Einstein field equations can be generalized to new model that leads to new idea about gravity field. It's big departure from Einstein field equations as this model connects spacetime curvature with matter field directly. Matter no longer acts on spacetime but it's equal to spacetime curvature. It means that additionally mass of gravity field is bigger than mass of visible matter field- as matter field never vanishes and it's mass is extended in whole space it would mean additional mass, that additional mass could be explanation of observed dark matter. I did calculate simplest model of fixed radius object with rest mass on its surface that given for one body logarithm function of mass plus rest mass. Even in this simplest case total mass of gravity field is infinite. This would give new rotation curve of galaxy as normal rotation curve would be equal to [13]:

$$v = \sqrt{\frac{Gm\left(r\right)}{r}} \tag{3.1}$$

and if I add additional mass it would change to

$$v\left(r\right) = \sqrt{\frac{Gm_0 + 3Gm_0 \log\left(\frac{r}{r_0}\right)}{r}} \tag{3.2}$$

that for many masses would lead to sum of rest masses plus additional term where there is distance from each mass r_i:

$$v\left(r_1, ..., r_i\right) = \sqrt{\sum_{i=1}^{n} \frac{Gm_{i0} + 3Gm_{i0} \log\left(\frac{r_i}{r_{i0}}\right)}{r_i}} \tag{3.3}$$

that gives simplest model of dark matter and rotation curves. It's easy to calculate does field equation contains singularities as I can directly calculate Kretschmann scalar that will lead to simple expression with two terms that are both contractions of stress-momentum tensor. It means that singularities are consequence in this model of stress-momentum tensor defined to include one as it naturally follows from connection between curvature and matter field. From idea that gravity field depends directly on matter field thus from it follows spacetime curvature gravity interactions are done by instant there is no delay in gravity itself as if matter field changes whole gravity field does change with it. Still gravitational radiation so gravity waves will be bound be speed of light limit. As those two are not same , gravity is direct consequence of spacetime curvature and gravitational waves are just energy emitted by gravity field. Another property of those equations is that if I take trace reversed field equations that connect directly Riemann tensor with stress-momentum tensor ,metric tensor and trace of stress momentum tensor I will arrive at equal amount of unknowns only in four dimensional spacetime, that would mean that this model works only in four dimensional spacetime as for more dimensions there is not equal number of unknowns on both side of equation.

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