New Dirichlet series expansion with recursive coefficient formula

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Abstract

Assuming the Dirichlet series of $q(x)$ is known, we derive a recursive formula for the Dirichlet-series coefficients of $\sqrt{q^2(x)+\alpha}$, $\alpha \in \mathbb{C}$.

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Motivation, method and results

If two Dirichlet-series representable functions f and g satisfy $g = 1/f$ then an elementary recurrent relation exists between their coefficients (2). Searching for similar relations, we derive a recurrent formula for a different dependence, namely $g = \sqrt{f^2(x)+\alpha}$, $\alpha \in \mathbb{C}$.

In what follows we present formal manipulations of general Dirichlet series and assume the existence of a common domain of convergence for all of them.

Let $q(x)$ be a function of a complex variable with known Dirichlet series $Q(x)$ whose coefficients are represented by the arithmetic function[1] q_n. We search for two functions $a(x)$ and $b(x)$ such that

$$b(x) = 1/a(x), \quad b(x) = \omega a(x) + q(x), \quad \omega \in \mathbb{C}\setminus\{0\}. \tag{1}$$

The first equation implies that the arithmetic functions a_n and b_n are inverse with respect to the Dirichlet convolution and b_n can be computed from a_n using the well-known recursive formula

$$b_1 = \frac{1}{a_1}, \quad b_{n>1} = -\frac{1}{a_1}\sum_{d=1,\, d|n}^{n-1} a_{\frac{n}{d}} b_d, \quad a_1, b_1 \neq 0. \tag{2}$$

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[1] We use the index notation to differentiate between the function of a complex variable and an arithmetic function. The symbol q_n denotes, depending on the context, the function itself or its value at n.

With q_n known, conditions (1) imply two unique solutions for (a_n, b_n). Indeed, one substitutes $b_n = \omega a_n + q_n$ to the left-hand side (LHS) of the second equation in (2), separates the first (i.e. $d = 1$) term of the sum on the right-hand side (RHS), and solves for a_n. One gets the recursive formula

$$\begin{aligned} a_1 &= \frac{\pm\sqrt{q_1^2 + 4\omega} - q_1}{2\omega}, \\ a_{n>1} &= -\frac{a_1}{1+\omega a_1^2}\left[q_n a_1 + \sum_{d=2,\, d|n}^{n-1} a_{\frac{n}{d}}\left(\omega a_d + q_d\right)\right]. \end{aligned} \tag{3}$$

Let us emphasize that this result gives us the coefficients of the Dirichlet series $A(x) = \sum_n a_n/n^x$ of the function $a(x)$. We are also able to get the analytic form of $a(x)$. One replaces $b(x)$ on the LHS of the first equation in (1) by the second equation and gets $a^2(x) + a(x)\, q(x) - 1 = 0$. The solution is

$$a(x) = \frac{\pm\sqrt{q^2(x) + 4\omega} - q(x)}{2\omega} = \sum_{n=1}^{\infty} \frac{a_n}{n^x}, \tag{4}$$

which represents our main result: we have a new function $a(x)$ expressed analytically through $q(x)$ and also expressed through its Dirichlet series. The sign in (4) needs to be adjusted accordingly to the sign of a_1 in (3). The result can be further modified

$$\pm\sqrt{q^2(x) + 4\omega} = \sum_{n=1}^{\infty} \frac{2\omega a_n + q_n}{n^x}, \tag{5}$$

where q_n are known by assumption and a_n are given by (3).

One can notice that by considering $q \equiv q_2(x) = \pm\sqrt{q_1^2(x) + 4\kappa}$ in (5), one gets the Dirichlet series for $\pm\sqrt{q_1^2(x) + 4\theta}$, $\theta = \kappa + \omega$. Thus, if one denotes by $\mathcal{I}$ the set of all functions of a complex argument which can be represented by the Dirichlet series in a given domain and by $\mathcal{F}$ the relation

$$\begin{aligned} \mathcal{F} :&\mathcal{I} \times \mathbb{C} \longrightarrow \mathcal{I}, \\ &(q(x), \alpha) \longrightarrow \pm\sqrt{q^2(x) + \alpha}, \end{aligned}$$

then an algebraic structure appears

$$\mathcal{F}\left(\mathcal{F}(q, \alpha), \beta\right) = \mathcal{F}(q, \alpha + \beta).$$

By consequence, a modification of the coefficient formula (3) expressed in terms of $u_n \equiv 2\omega a_n + q_n$ (see (5)) also respects this structure, which might not be seen at the first glance.